# Trading under Uncertainty about other Market Participants

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**Abstract**

I present an asymmetric information model of financial markets that features rational, but uninformed, hedge fund managers who trade against informed and noise traders. Managers are uncertain not only about fundamentals, but also about the proportion of informed to noise traders in the market and use prices to update their beliefs about these uncertainties. Extreme news leads to an increase in both types of uncertainty, while it decreases price informativeness. Prices react asymmetrically to positive and negative news, with higher expected returns at times of increased uncertainty about market composition. The model generates a price-volume relationship that is consistent with established stylized facts. I then extend to a three-period model and study the dynamics of expected returns and volatility.

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# 1 Introduction

Throughout the history of financial markets there have been numerous occasions where investors were taken by surprise by an extreme price movement. In many of these cases, these large price shocks were described as puzzling and could not be rationalized by many professional investors. The Black Monday, the Flash Crash of 2010, or the more recent Bitcoin boom are a few such examples. At the same time, many hedge funds have been paying closer attention to such events by expending more and more resources to track the fluctuation in market sentiment and learn whether trades represent information or noise. Why is it usually more difficult to interpret extreme shocks? What type of information do they convey? And why is uncertainty increasing during these times?

Most theoretical models in finance literature assume that traders know the degree of rationality of other investors in the market. In this paper, instead, I take the perspective of sophisticated investors ("hedge funds") who are uncertain about the proportion of informed - compared to noise - traders in the market. By studying their investment behaviour and the subsequent learning under this assumption, I contribute to financial research in a number of ways. First, I find that uncertainty about the market composition increases when there is an extreme market outcome (for example, a crash). Second, this increased uncertainty leads to higher uncertainty about fundamentals and higher risk premia. As a result, there is an asymmetry in the price reaction to positive and negative shocks. Third, I establish that the variation in market composition constitutes a type of risk, unrelated to fundamentals, for which investors demand higher expected returns. Moreover, I show that during a crash, traders rely less on cashflow news to update their expectations about future payoffs. Finally, the model generates a price-volume relationship that fits well with the relevant stylized facts established in empirical literature and summarized in Karpoff (1987).

The model consists of three types of agents; there are Hedge Fund managers, Informed investors, and Noise traders. Hedge Funds act as rational uninformed investors who are using the information in prices to make their portfolio choices. Informed traders can be thought of as insiders who hold information about payoffs and trade based on it, and Noise traders are the irrational investors in the market, who trade either because of wrong information or because of sentiment shocks. Importantly, managers are uncertain, not only about the fundamentals, but also about the proportion, of informed (to noise) traders that exist in the market. This makes their inference problem much more challenging since they have to use the signal they are getting through price to learn both about the fundamentals and about the composition of the rest of the traders. The main intuition behind this model is that hedge funds learn whether the rest of the traders are (more likely to be) homogeneous or not, by observing the size of the price signal. When the size is high, then the probability that the rest of the traders are of the same type is also high, and thus their uncertainty regarding the number of informed traders increases. One of the greatest challenges of any model with additional uncertainties is to keep it tractable. This is achieved by assuming that investors have a mean variance utility and by considering noise traders who receive signals that are independent, but identically distributed, to those of informed traders, so that their demand functions have the same functional form.

The first main result of this model is that market crashes (and booms) make hedge funds more uncertain about both the market composition and the fundamentals. This is because such extreme outcomes are actually very informative about the belief dispersion of investors in the market; in the limit, they can only occur when all investors behave in the same way. That is, it is much more likely to observe a crash when investors are either all informed or all noise. However, these two cases lead to very different interpretations of the price movement; in the first case, its informativeness is the highest possible, while in the latter it should be completely ignored. Thus, fund managers become less confident about how to interpret the price and their uncertainty about fundamentals increases. Therefore, we find that during a crash (or a boom) the risk premium part of the price increases.

Another important result is that expected returns are increasing in the uncertainty about the proportion of informed traders. For example, a market with fewer sources of information is naturally perceived to have a lower degree of belief dispersion and is associated with a higher variation in the ratio of informed-to-noise traders. As described above, managers who try to interpret the information that is contained in prices are less confident about their interpretation. This constitutes a type of risk, which they anticipate and hence, when the market is dominated by hedge funds, this uncertainty is translated into a higher expected return on the asset. Furthermore, I analyze the effect of this uncertainty on the sensitivity of price to signal. We find there is an asymmetry in the sensitivity of prices to positive and negative shocks, which becomes more pronounced when the uncertainty about the proportion of informed traders becomes the largest. This is because prices consist of an expectation part that moves prices in the direction of the news, and a risk premium part that is affected by the uncertainty about the dividend.

Since this last part is increasing in the size of the news, the price reacts more with negative news than it does with positive news. Moreover, the expected risk premium is larger during these times, therefore leading to a higher expected asymmetry. In a similar way, I show that price instability, defined as the expected price's reaction to a sentiment shock, is higher during a crash.

I run numerical simulations of the model to study the behavior of trading volume. The patterns of volume and price very closely fit established stylized facts. In particular, volume increases when the absolute price increases, because, during that time, the disagreement between fund managers and the rest of the traders is the largest. For example, when there is a crash, managers are reluctant to significantly change their expectation about fundamentals. At the same time, the group of informed and noise traders is very likely to be homogeneous during that period, dominated either by informed or by noise traders; in either case, these traders will then hold a very different opinion, compared to Hedge Funds, about the expected payoff. Thus, there will be a large trading volume. More interestingly, simulations also show that there is a positive correlation between volume and price. This naturally arises in the model because of the abovementioned asymmetry between positive and negative news: that is, a large positive price is associated with an even larger signal than the corresponding negative price. Both of the above effects are amplified when the uncertainty about the proportion of informed traders increases.

Finally, I extend the model to a dynamic setting to discuss resulting implications for the expectation and volatility of returns. Under our assumption, trading in consecutive periods is connected via the updating of beliefs about the mass of informed traders. We then find that a crash (or boom) in period one is associated with higher expected returns, and lower volatility, in period two. This is because sophisticated investors conclude that the rest of the market is dominated either by informed or by noise traders, and so they expect to be less confident about the interpretation of any signal they observe.

Since the work of Radner (1968), it has been understood that uncertainty about payoffs acts in a very different way than uncertainty about other investors' behavior; more recently, there have been attempts by various authors to create such rational expectations models to study the effect of uncertainty about some market parameters. It is on this strand of literature that I build up by extending these ideas, recognizing the relevant uncertainties and introducing general distributional assumptions about the mass of informed traders, that will help us analyze, in a more realistic setting, the corresponding implications. For instance, Romer (1992) and Avery & Zemsky (1998) provide two such models, which can generate price crashes and herding, respectively, caused by uncertainty regarding other traders. Easley et al. (2013) study how expected returns are affected in an economy in which ambiguity-averse traders are uncertain about each other's risk aversion. Finally, some other papers that analyze higher dimensions of uncertainty are those of Yuan (2005), Cao & Ye (2016) and Cao et al. (2002). The abovementioned papers use different models to analyse how non-payoff uncertainties affect prices, while a common characteristic in this literature is that models are often non-tractable. Instead, our focus is on a type of uncertainty that increases during market crashes and can help us explain return dynamics during these times.

The most similar model to ours is that of Banerjee & Green (2015) (BG henceforth), where investors are uncertain whether informed or noise traders are present in the market (but not both). The key novelty of our paper, compared to BG, is that we allow both of these traders to co-exist in the market. This generalization creates the following fundamental difference: the equilibrium price conveys information both about fundamentals and about the composition of traders in the market. In particular, an extreme price movement, in our model, makes Hedge Funds believe that they are trading against either *all Informed* or *all Noise* traders, while a more moderate price will shift their beliefs towards thinking that they face a mixed population of traders. However, in BG there can be no such updating. This allows us to get many results about the way uncertainty about market composition affects expected returns, price informativeness and the slope of the price-volume relationship. Overall, while BG study the role of the first moment of the distribution of the proportion of Informed traders, I study the effect of the second moment and I show how learning about it affects equilibrium results.

Another important relevant paper is that of Gao et al. (2013). The authors consider a Grossman-Stiglitz model, in which the proportion of informed to uninformed traders is unknown; they focus on jumps that may appear due to the multiple equilibria that arise, and they find that there can be complementarity in information acquisition. In contrast, we study the uncertainty in the proportion of informed to noise traders. This type of uncertainty generates very different predictions about return dynamics and our modeling assumptions lead to a unique equilibrium, which is also more tractable.

Methodologically, this paper contributes to the growing literature on non-linear equilibria, in a CARA-normal setting. Building on the Grossman & Stiglitz (1980) paper of asymmetric information, many recent papers, such as Breon-Drish (2015), have shown that by relaxing assumptions about the distribution of dividends such equilibria may exist. In our paper, however, as in Banerjee & Green (2015) this non-linearity arises because of the assumption of uncertainty about a market parameter, while the payoffs remain normally distributed.Moreover, our modeling assumptions resemble that of Mendel & Shleifer (2012). In their paper, they use the same three types of agents and information structure to study the effect of noise traders in the market, even when their mass is negligible. They find that the Outsiders (rational uninformed agents) can rationally amplify the impact of a sentiment shock, leading to prices that significantly diverge from fundamental values. Importantly, their focus is on the price stability and specifically on its behavior as the mass of noise or informed traders changes. In this paper, while we also emphasize the importance of noise trading, our focus is on the behavior of sophisticated investors when they are uncertain about these masses.

Finally, regarding the empirical literature it is worth noting two relevant papers. First, Easley et al. (2002) provides an empirical measure of the probability of information-based trading (PIN) in a market. By estimating PIN through their microstructure model, the authors conclude that informed trading positively predicts returns. In my model, I emphasize that uncertainty about this probability also matters; this issue is discussed in more detail in Section 6. Second, Sadka (2006) provides evidence that the variable component of liquidity risk can explain the momentum and PEAD returns. He further interprets this variable component as representing the *"unexpected variation in the ratio of informed to noise traders"*. My paper presents a theoretical model in which there is uncertainty about this ratio and suggests that this uncertainty is important for the return dynamics and, more specifically, can indeed lead to higher expected returns.

The rest of this paper is organized as follows. In the following section I present the model and its main assumptions. In Section 3 I analyze the resulting equilibrium quantities in the static model. In Section 4, I explore the implications of assuming various distributions for the prior belief about the proportion of informed traders and I analyze some numerical simulations. Section 5 describes a dynamic extension of the model, under some simplifying assumptions. Finally, Section 6 contains a discussion of the paper, while, in Section 7 a conclusion is given.

# 2 The Fundamentals of the Model

## 2.1 Agents

There are three types of agents in the economy, each endowed with an initial wealth $W$. There is a mass 1 of rational uninformed agents ($H$) who are trying to learn from prices, and there is also a mass $m \in [0, 1]$ of informed (I) agents who observe an informative signal about fundamentals at each period, and a mass of $1 - m$ of noise traders (N) who think they are informed and trade in signals that are actually uncorrelated with the fundamentals.[1] The fact that the mass of $H$ is the same as the mass of $I$ and $N$ combined, is just for simplicity and does not drive any results; later, we consider the general case where the mass of hedge funds is $Q$ and the total mass of $I$ and $N$ is $2 - Q$ and we talk about the cases $Q \to 0$ and $Q \to 2$. Our *most important assumption* is that $m$ is not a known parameter, but instead it is a random variable in $[0, 1]$.

## 2.2 Timing

The benchmark model is a static model with two periods. During Period 1, trading takes place, while in Period 2, dividends are paid and uncertainty is resolved. In Section 5, we extend this model to a dynamic version with multiple periods.

## 2.3 Assets

There are two assets in the market, a risk-free asset, with a return normalized to 1, and a risky asset that pays a dividend $d \sim N(0, \sigma^2)$ and is found in supply $Z$, which is a known constant.

[1] In Appendix B, we discuss some slight alterations of the model, in which we have different types of agents, and which lead to some alternative results and interpretations.

### 2.4 Utilities

All agents have mean variance utilities[2] and are price-takers. Traders maximize the utility of their terminal wealth. More specifically, each trader solves the following maximization problem:

$$\max_x \mathrm{E}[W + x(d-p)] - \frac{\alpha}{2}\mathrm{Var}[W + x(d-p)]$$

where $\alpha$ represents the degree of risk-aversion.

### 2.5 Information Structure

Informed and Noise traders behave similarly. They both receive a signal, which they both think is the only source of (payoff-relevant) information in the market. $I$'s signal is:

$$s_I = d + \varepsilon_I$$

where $\varepsilon_I$ is Normally distributed with mean 0 and volatility $\sigma_\varepsilon$. The informativeness of I's signal is given by the signal-to-noise ratio: $\lambda = \dfrac{\sigma^2}{\sigma^2 + \sigma_\varepsilon^2}$.
On the other hand, the signal of Noise traders is:

$$s_N = u + \varepsilon_N,$$

where $u \sim N(0, \sigma^2)$ is independent and identically distributed to $d$, and $\varepsilon_N$ is independent and identically distributed to $\varepsilon_I$. Thus, the perceived informativeness of the noise traders is also equal to $\lambda$. Hedge fund managers are not aware of informed-to-noise traders in the market and have, at $t = 0$, a prior distribution $f(m)$ about $m$. Our model nests the Banerjee and Green model in the case where $f(m)$ is such that $f(1) = \pi_0$ and $f(0) = 1-\pi_0$.

Our specification for Noise traders is different to that in Grossman & Stiglitz (1980) or De Long et al. (1990), where $N$ have a random inelastic demand. In contrast, our approach to modeling noise traders can be also found in Black (1986) or - in a very similar form - in Mendel & Shleifer (2012). The main advantage of this approach is that it delivers a much more tractable model, which better serves the intuition behind this the paper. Uncertainty about $m$ matters, because it alters the perceived homogeneity in the market,

[2] Note that this is *not equivalent* to using CARA utility, as $p$ will be non-linear in the signal and hence will not be normal in equilibrium.

even if the demand functions of the two groups of traders are indistinguishable (from $H$'s point of view). In particular, this assumption, makes all the equilibrium quantities just a function of $ms_I + (1-m)s_N$, and, as we will see in the next section, allow us to find an equilibrium that is *mixed-signal revealing*. It is for the same reason that we assume that the total mass of $I$ and $N$ is known. Otherwise, it would be much more difficult to find an equilibrium.

In this model, we can see that proportion uncertainty is closely related to the belief dispersion in the market. Indeed, Informed and Noise traders form heterogeneous beliefs about the asset's payoff once they receive their signals. From the perspective of the fund managers, who do not know $m$, this *belief dispersion* between $I$ and $N$ traders can be measured by the inverse of the correlation of two random signals in the market. When this correlation is high, this means that it is very likely for all traders to hold the same information and thus the belief dispersion is low, and vice versa. More formally, if $i, j$ are i.i.d. Bernoulli random variables that take the values $I$ and $N$ with probabilities $m, (1-m)$ respectively, then belief dispersion is $\frac{1}{corr(s_i, s_j)}$. This is a function of the second moment of $m$, since it depends on the probability that both $i$ and $j$ are of the same type, which is equal to $m^2+(1-m)^2$. Therefore we get the following corollary:

**Corollary 1.** *Belief dispersion in the market*[3] *is decreasing in the variance of* $m$, $var[m]$, *as long as* $E[m]$ *is a constant.*

Hence, we can see how the uncertainty about the mass $m$ of $I$ traders is related to the belief dispersion between $I$ and $N$ traders. Thereafter, we will use $var[m]^{-1}$ as a measure of this asymmetry in the market.[4]

In the following section, I solve for the equilibrium price in the static model.

[3] We only consider the belief dispersion between two random traders who are either I or N, because those traders obtain their own signals.

[4] We will be able to do that, as we will only focus on the case where $E[m] = \frac{1}{2}$.

# 3 The Equilibrium

## 3.1 Two-period model

Solving the maximization problem for $\theta = I, N$ we get:

$$x_\theta = \frac{\mathrm{E}_\theta[d] - p}{\alpha \mathrm{Var}_\theta[d]} = \frac{\lambda S_\theta - p}{\alpha\sigma^2(1-\lambda)}$$

That is, informed and noise traders behave in the same way (but receive different signals) and do not try to use the price to learn any further information about $d$. On the other hand, $H$ try to learn about $d$ by observing the price and residual demand (i.e. $Z - x_H$). When updating their belief about the fundamental, the above two quantities give them a "mixed" signal that has some information because of $S_I$, but is also contaminated by noise ($S_N$).

Using the market clearing condition, we have $x_H + mx_I + (1-m)x_N = Z$. Therefore:

$$x_H + \frac{\lambda \overbrace{(ms_I + (1-m)s_N)}^{\tilde{s}} - p}{\alpha\sigma^2(1-\lambda)} = Z$$

I write $\tilde{s} = ms_I + (1-m)s_N$. Note that the price and the residual demand can *reveal to $H$ the mixed signal $\tilde{s}$*. This would not be true if the demand of noise traders was simply a random variable $z$ (instead of being a function of price) and, therefore, we would not be in a position to find the equilibrium.

In order to find hedge funds' demand, $x_H$, we need to find the expectation and variance of $d$ from their perspective after they observe the abovementioned mixed signal. Henceforth, I may write $E_H[\cdot]$, $Var_H[\cdot]$ for $E[\cdot|\tilde{s}], Var[\cdot|\tilde{s}]$ respectively, and I use these notations interchangeably. Using the law of iterated expectations, as well as the formula for the conditional expectation of normally distributed variables, we get:

$$\mathrm{E}[d|\tilde{s}] = \mathrm{E}\left[\frac{m}{m^2 + (1-m)^2}|\tilde{s}\right]\lambda\tilde{s}, \tag{3.1}$$

To simplify notation, I set $L(m) := \frac{m}{m^2+(1-m)^2)}$. The term that multiplies $\tilde{s}$ in (3.1) is the expected $\frac{cov(d,\tilde{s})}{var(\tilde{s})}$, which we interpret as the *informativeness* of the signal.

We can now observe that the expectation of $d$ as perceived by the Hedge Funds depends on the conditional probability density function of $m$ given the mixed signal, $f_{m|\tilde{s}}(m|\tilde{s})$. In the next section, I use a prior three-point distribution, $f(m)$, to give the intuition of the results, but I also prove the validity of main results under any symmetric distribution.

Similarly, we can find the perceived variance of $d$ from the perspective of $H$. For that, we will need to use the *law of total variance*. We have:

$$Var[d|\tilde{s}] = \underbrace{\mathrm{E}\left[\mathrm{Var}[d|\tilde{s},m]|\tilde{s}\right]}_{\mathrm{E}[\sigma^2(1-\lambda m L(m))|\tilde{s}]} + \underbrace{\mathrm{Var}\left[\mathrm{E}[d|\tilde{s},m]|\tilde{s}\right]}_{\mathrm{Var}[\lambda L(m)|\tilde{s}](\tilde{s})^2}$$

To simplify further, we will set $c(\tilde{s}) := \lambda^2 \mathrm{Var}[L(m)|\tilde{s}]$ and we will examine the function $c(\cdot)$ later on. Moreover, in the Appendix I prove that for any symmetric $f(m)$, we have $E[mL(m)|\tilde{s}] = \frac{1}{2}$. Therefore:

$$\mathrm{Var}_H[d] = \sigma^2(1-\frac{\lambda}{2}) + c(\tilde{s})\tilde{s}^2$$

So we observe that both the expectation and variance depend on $m, s_N$ and $s_I$, only through the mixed signal, $\tilde{s}$.

By using the market clearing condition, we can thus get the following proposition:

**Proposition 1.** *In the two-period model, there exists a mixed signal ($\tilde{s}$) revealing equilibrium. The price in this equilibrium is given by:*

$$P = \underbrace{\lambda\tilde{s}\kappa(\tilde{s}) + \lambda\tilde{s}\mathrm{E}\left[L(m)|\tilde{s}\right](1-\kappa(\tilde{s}))}_{\textit{Expectation component}} - \underbrace{\alpha\kappa(\tilde{s})\sigma^2(1-\lambda)Z}_{\textit{Risk premium component}},$$

*where*

$$\kappa(\tilde{s}) = \frac{Var[d|\tilde{s}]}{\sigma^2(1-\lambda) + Var[d|\tilde{s}]}$$

*Proof.* A more general proof, for when the mass of $H$ traders is $Q \in (0,2)$ and the mass of $I$ and $N$ together is $2-Q$, can be found in the Appendix (baseline model corresponds to $Q = 1$). □

In the case where $Q \to 2$, in which Hedge Funds dominate the market, the price takes the simple form:

$$p \approx E[d|\tilde{s}] - \frac{1}{2}\alpha Z Var[d|\tilde{s}].$$

What we can see from the above proposition is that the equilibrium price is not linear in the mixed signal, and, consequently, non-linear in $s_I$. The expectation component is a weighted average of the expectations of each group of agents, and the weight is given by $\kappa(\tilde{s})$. This weight is increasing in $Var_H[d]$, and hence, as I will prove in Section 4, it is also increasing in $|\tilde{s}|$ , whereas in standard models, the posterior variance of fundamentals is independent of the signal, because of the properties of the conditional normal distributions.[5] As in BG, this makes the expectation and risk premium components of the price to not behave in the same way with positive and negative realizations of $s_I$ (or of $s_N$). This creates an asymmetry that makes the derivative of price to $\tilde{s}$ greater for negative realizations of the mixed signal than for positive ones, which I will discuss further in the next section.

[5] That is, if A,B are jointly normal random variables, $var[A|B]$ is not a function of B.

# 4 Results

To explore the equilibrium implications of this model we need to make an assumption about the distribution of $m$. In BG (2015), this is assumed to be a Bernoulli distribution that takes the value 1 with $\pi_0$ and 0 with $1-\pi_0$, but, in such a model, the belief dispersion is always constant and there is no uncertainty or learning about it. In contrast, we will assume that this dispersion is unknown, and we will study how Hedge Funds learn about it and how it affects their demand functions. The simplest way to provide the intuition is to use a simple, three-point distribution, which allows us to talk of different levels of belief dispersion between $I$ and $N$ traders in the market. Then, in Section 4.2, we will generalize the main results to any symmetric distribution in $[0, 1]$.

## 4.1 Three-point distribution

Assume that $m \in \{0, \frac{1}{2}, 1\}$ with $\pi_i = P(m = i)$ for $i \in \{0, 1/2, 1\}$. As I explain later, augmenting the support of $m$ to include values in $(0, 1)$ leads to learning about $m$, by observing $\tilde{s}$. Importantly, what we will see is that the *posterior belief* $\hat{\pi}_i = P(m = i|\tilde{s})$ *is in general not equal to* $\pi_i$. This seems to be self-evident; however, it is not true under the simple assumption of the BG model, and this is what drives many of the interesting results that are different. Using Bayes' rule we find that the posterior distribution satisfies $f_{m,\tilde{s}}(i, \tilde{s}) = \mathrm{P}(m = i) f_{\tilde{s}|m}(\tilde{s}|m = i)$, where the conditional distribution of the right hand side is normal, with mean 0 and variance $(i^2 + (1-i)^2)(\sigma^2 + \sigma_\varepsilon^2)$. Note that $\tilde{s}|(m = 1) = s_I$ and $\tilde{s}|(m = 0) = s_N$, which are identically distributed. Thus, we can define $h_1(x)$ to be the pdf of $\tilde{s}$ under $m = 1$ or $m = 0$ and $h_2(x)$ to be the pdf under$m = \frac{1}{2}$. We have:

$$\hat{\pi}_0 = \mathrm{P}(m = 0|\tilde{s}) = \frac{\pi_0 h_1(\tilde{s})}{(\pi_0 + \pi_1) h_1(\tilde{s}) + \pi_{1/2} h_2(\tilde{s})} \tag{4.1}$$

and similarly for $\pi_{1/2}$ and $\pi_1$. We see, therefore, that the posterior probabilities depend on $s$ through $\frac{h_2(\tilde{s})}{h_1(\tilde{s})}$, which is decreasing in $|\tilde{s}|$. Let us now see what is the implication of this fact, in the symmetric case where $\pi_0 = \pi_1 = \pi$ (and $\pi_{1/2} = 1 - 2\pi$). We have the following corollary:

**Corollary 2.** *Uncertainty about the proportion of Informed traders, $m$, increases as (mixed) news becomes more extreme, i.e., $|\tilde{s}|$ becomes large.*

Extreme news is naturally associated with extreme prices. Hence, this corollary tells us that, during crashes or booms, uncertainty about market composition spikes. This uncertainty has plenty of implications as we see in the results that follow. Technically, the fact that a mixed signal distribution, corresponding to $m = \frac{1}{2}$, is less fat-tailed than a normal distribution, is what causes $\hat{\pi}$ to increase as $|\tilde{s}|$ increases. In other words, a more extreme signal makes $H$ believe that the other group of traders is (more likely) either all Informed or all Noise. Hence, Hedge Funds believe that extreme market outcomes occur when the rest of the traders are more homogeneous, or, in other words, they believe that the *belief dispersion* in the market is low.[6] In fact, although $H$ becomes more uncertain about $m$, he does become more certain about the deviation of $m$ from its mean, $|m - \frac{1}{2}|$. Especially when $|\tilde{s}| \to \infty$, we know that $P(|m - \frac{1}{2}| = \frac{1}{2}) \to 1$; that is, Hedge Funds learn that all the agents (between $I$ and $N$) in the market are of the same type.

The first implication that we will examine is the effect on the informativeness of the signal in extreme times; managers becomes almost certain that the signal is either very informative or not at all. We get the following corollary:

**Corollary 3.** *The Hedge Funds' informativeness of the price signal is decreasing in the size ($|\tilde{s}|$) of the signal.*

*Proof.* As defined in equation (3.1), informativeness is $\lambda \mathrm{E}[L(m)|\tilde{s}]$. Now note that given the formulas we provided for $\hat{\pi}_i$, we can easily see that when $\pi_0 = \pi_1$ then $\hat{\pi}_0 = \hat{\pi}_1 = \hat{\pi}$ (more generally, if the prior distribution is symmetric with respect to $1/2$ then the posterior distribution will also be symmetric). Therefore:Informativeness $= \lambda(1 - \hat{\pi})$. But since $\hat{\pi}$ is increasing in $\tilde{s}^2$, the expected informativeness must be decreasing in $\tilde{s}^2$. $\square$

The main idea that can be conveyed, even with a three-point distribution, is that an extreme signal (either very positive or very negative) shifts the posterior beliefs about the proportion in such a way that it is now much more likely that the other group of traders is either all Noise or all informed ($m = 0$ or $1$). In turn, this causes the informativeness of the price to be decreasing on $|\tilde{s}|$, since managers are now more reluctant to interpret any signal as representing information. More interestingly, simulations show that for very small values of the prior $\pi$, the expectation of fundamentals, $E_H[d]$ is non monotonic on

[6] We cannot use Corollary 1 to directly claim that belief dispersion as defined in that lemma decreases, since the conditional covariance $cov(s_i, s_j|\tilde{s}) = 0$. However, we abuse the term belief dispersion here to refer to the the probability that two traders (randomly drawn) are not the same (which has now decreased).

$|\tilde{s}|$; that is, a very high $\tilde{s}$ may even lead a hedge fund manager to lower his expectation about $d$, since the effect of the reduced informativeness may outweigh the effect of the increased $\tilde{s}$.

In addition to the effect on informativeness, when the news is extreme, the perceived variance of informativeness increases. This is because hedge funds understand that it is more likely that either all other traders are informed or all noise, and hence they are most uncertain about the weight they should put on the signal they observe (in one case, this is very informative, and, in the other, they should completely ignore it).

This, in turn, affects the uncertainty of managers about the fundamentals of the assets and leads us to the following proposition.

**Proposition 2.** *Hedge Funds become more uncertain about the fundamentals, $var[d|\tilde{s}]$, when they observe more extreme news.*

*Proof.* As detailed in the Appendix, we can get that $c(\tilde{s}) = \lambda^2\hat{\pi}(1-\hat{\pi})$ which is increasing in $\tilde{s}^2$. □

That is, $var_H[d]$ and hence the risk premium (and $\kappa(\cdot)$) are increasing in $|\tilde{s}|$. When the magnitude of the mixed signal increases, the posterior variance of $m$ increases; this makes the investors more uncertain about the composition of the market and, hence, about the informativeness of the signal. In turn, this leads to an increase in the uncertainty about fundamentals.[7] Having established the above results, we now have that $\kappa$, and hence the risk premium component of the price, is also increasing in the magnitude of the mixed signal. This means that a very high signal (seemingly positive news) could yield a lower price than an averagely good signal in two ways; first, by implying a lower expectation of fundamentals (from H's perspective), and second, by increasing the uncertainty of fundamentals. Moreover, as we see in the proof of proposition 2 in the Appendix, hedge funds' perceived variance of fundamentals, $Var_H[d]$, is increasing in the posterior belief $\hat{pi}$; therefore, we can conclude that it is also increasing in their perceived variance of the proportion $m$ of informed traders. This then leads us to the following important lemma:

[7] This result is also true in BG model. However, in our setting it is amplified by the uncertainty about $m$, and it then naturally leads to Proposition 3. Moreover, the fact that uncertainty about informativeness, measured by $c(\tilde{s})$, is increasing in $|\tilde{s}|$ is a novel result.

**Lemma 1.** *Hedge funds' expected uncertainty about fundamentals, $E[Var_H[d]]$, is increasing in the prior variance of $m$.*

This lemma is based on Proposition 2 together with the fact that if $var[m_1] > var[m_2]$, then the corresponding signal $\tilde{s}^2_{m_1}$ stochastically dominates $\tilde{s}^2_{m_2}$. As shown in the Appendix, this stochastic dominance leads us to conclude that the expected risk premium part of the price is increasing in $var[m]$. Therefore, we obtain the following proposition:

**Proposition 3.** *When the market is dominated by Hedge Funds, the expected return of the asset, $E[d-p]$, is increasing in the uncertainty about the proportion of informed traders, $var[m]$ and hence in the perceived homogeneity of the market.*

From Corollary 1, we have seen that the variance of $m$ can be interpreted as the inverse of the level of (perceived) heterogeneity in the market. Therefore, one can deduce that the higher this heterogeneity, the lower the expected return of the risky asset. In other words, managers who trade in a homogeneous market want to be compensated for the additional risk that they are taking, given that they do not know whether the price contains any information at all. In the next subsection, I use the propositions and corollaries found above, to discuss the sensitivity of prices to changes in the signals.

### 4.1.1 Price to signal sensitivity

A measure that is extensively studied in the work of Mendel & Shleifer (2012) is that of price instability, defined as the derivative of price to a sentiment shock (corresponding to $\tilde{s}_N$ here). The higher the price instability, the more fragile the price and the more likely it will lead to an abrupt jump (big change) in the price. We will use the results of the previous section, to study a similar measure in the context of this model in which there is a higher degree of uncertainty.

First of all, we have the following corollary, that is a generalization of a corresponding proposition in Banerjee & Green (2015), which indicates an important asymmetry between positive and negative news:

**Corollary 4.** *For any $\tilde{s}_0 > 0$, the derivative of price to the mixed signal is smaller for $\tilde{s}_0$ than it is for $-\tilde{s}_0$. That is:*

$$\frac{dP}{d\tilde{s}}\Big|_{\tilde{s}=\tilde{s}_0} < \frac{dP}{d\tilde{s}}\Big|_{\tilde{s}=-\tilde{s}_0} \tag{4.2}$$

*Moreover, the expected size of this asymmetry is larger when the variance of $m$ is larger.*

When the mixed signal is positive the expectation component of price is (generally) increasing, while the risk premium component makes the price lower than it would be, making the derivative smaller. On the other hand, when $\tilde{s}$ is negative, under a more extreme signal, both the expectation part and the risk premium part move the price to the same direction (making it more negative), thus increasing the sensitivity of the price movement to the change of $\tilde{s}$. Mathematically, we have that:

$$P(\tilde{s}) + P(-\tilde{s}) = -2\alpha Z \kappa(\tilde{s}) \sigma^2 (1 - \lambda) \tag{4.3}$$

which is decreasing for $\tilde{s} > 0$, and hence gives us first part of corollary 4. This result is important, because it shows that, in this setting, negative news can lead to a more extreme drop in price than the corresponding positive news. Finally, by taking expectations in equation (4.3) and then differentiating with respect to $\tilde{s}$, we also get the second part of the corollary, since by Proposition 3 we get that $E[\kappa(\tilde{s})]$ is increasing in $var(m)$.

Moreover, we can study the expected sensitivity of price to a sentiment shock (or to a shock in a Noise trader's signal in our model) and see how it varies for different values of $\tilde{s}$. Mendel & Shleifer (2012) relate this measure to the Hedge Funds' demand, and, more particularly, to whether these traders end up chasing noise or not. In this model, the sign of the sensitivity of $H$'s demand to mixed news changes for different values of $\tilde{s}$, and it turns out that $H$ traders may be chasing noise when $|\tilde{s}|$ gets large, and act in the opposite way when $\tilde{s}$ is close to 0.

**Corollary 5.** *Price instability is higher when $|\tilde{s}|$ goes to infinity than when $\tilde{s}$ goes to 0.*

The proof of the above result can be found in the Appendix. Because of the uncertainty about market composition, when the size of the signal is large, Hedge Funds prefer to stay away from the market and do not trade aggressively. Thus, during these times, the effect of noise trading is amplified, and the price becomes more sensitive to any sentiment shock.

## 4.2 Any Symmetric Distribution

Having described the main intuition using a simple three-point distribution, we will now work with a continuous distribution for $m$ to make our model more rich and realistic. In fact, we will prove the main propositions using any continuous distribution that is symmetric in $[0, 1]$ (w.r.t 1/2). This makes the results of the model robust to a variety of distributional assumptions and corresponds a more realistic setting where $m$ can take any value in $[0, 1]$. First of all, when the pdf of $m$ is symmetrically distributed around $\frac{1}{2}$, we can easily get the following corollary:

**Corollary 6.** *When the prior distribution of the proportion $m$ of Informed traders is symmetric w.r.t $\frac{1}{2}$, the posterior distribution from the perspective of $H$ (conditioning on $\tilde{s}$) is also symmetric. Hence:*

$$\mathrm{E}_H[m] = \frac{1}{2}$$

This means that the expectation about the proportion of informed traders remains unchanged, independently of the observation of $\tilde{s}$. However, that does not mean, that the distribution of $m$, and hence the informativeness of the signal, does not change. In Appendix, I explain how we can compute the joint density of $(d, u, m, \tilde{s})$, from which we can find the conditional distribution of $d|\tilde{s}$ or of $m|\tilde{s}$. In short, we can use Bayes' rule to find the posterior distribution $f_{m|\tilde{s}}(m|\tilde{s})$. We first find the joint distribution $g(m, \tilde{s})$, as: $g(m, \tilde{s}) = g(\tilde{s}|m)f_m(m)$. But $\tilde{s}|m$ is a linear combination of normals, hence it is a normal itself, with mean 0 and variance $C(m) = (m^2 + (1 - m)^2)(\sigma^2 + \sigma_\varepsilon^2)$. Hence, we get:

$$g(m, \tilde{s}) = \frac{1}{\sqrt{2\pi C(m)}} \exp{(-\frac{\tilde{s}^2}{2C(m)})} f_m(m)$$

Finally, this means that $f_{m|\tilde{s}}(m|\tilde{s}) = \dfrac{g(m, \tilde{s})}{g_{\tilde{s}}(\tilde{s})}$, where $g_{\tilde{s}}(\tilde{s})$ is the density function of $\tilde{s}$. Therefore, we see that

$$f(m) = f(1 - m) \Longrightarrow g(m, \tilde{s}) = g(1 - m, \tilde{s}) \Longrightarrow f_{m|\tilde{s}}(m|\tilde{s}) = f_{m|\tilde{s}}(1 - m|\tilde{s}),$$

i.e., the posterior is symmetric, as required. Using the above, we can now prove the following, proposition.

**Corollary 7.** *For any symmetric prior distribution of $m$, the expected informativeness of the signal is decreasing in the size of the signal.*

*Proof.* Proof can be found in the Appendix and is based on the use of Cauchy-Schwarz inequality. $\square$

How can we interpret this result? As also described in the previous section, when fund managers observe a large realization of $\tilde{s}$ they know this is a good sign for the fundamentals (assuming $\tilde{s} > 0$), but also need to estimate how accurate this sign is. The larger the $\tilde{s}$ gets, the more likely it is that $m$ has an extreme value (closer to 0 or 1). However, since (as I prove in the Appendix) the informativeness decreases more as $m \to 0$ than it increases for $m \to 1$, its expected value ends up decreasing in $|\tilde{s}|$ and this is a key result.[8]. As the mixed signal gets larger the signal-to-noise ratio becomes smaller and smaller, which can even lead H's expectation of fundamentals to be decreasing in $|\tilde{s}|$ (in particular when $\pi_0$ is small, simulations show that $E_H[d]$ can be non-monotonic on $|\tilde{s}|$). All in all, Corollary 7 shows us how the uncertainty about other traders affects the expectation part of the price.

We can also prove the following result, which generalizes Proposition 2, and is one of the main results of this paper.

**Proposition 4.** *For any symmetric prior distribution of $m$, Hedge Funds' perceived variance of fundamentals is increasing in $|\tilde{s}|$. Therefore, the informativeness of the signal, measured by $var[d|\tilde{s}]^{-1}$ is decreasing in $|\tilde{s}|$.*

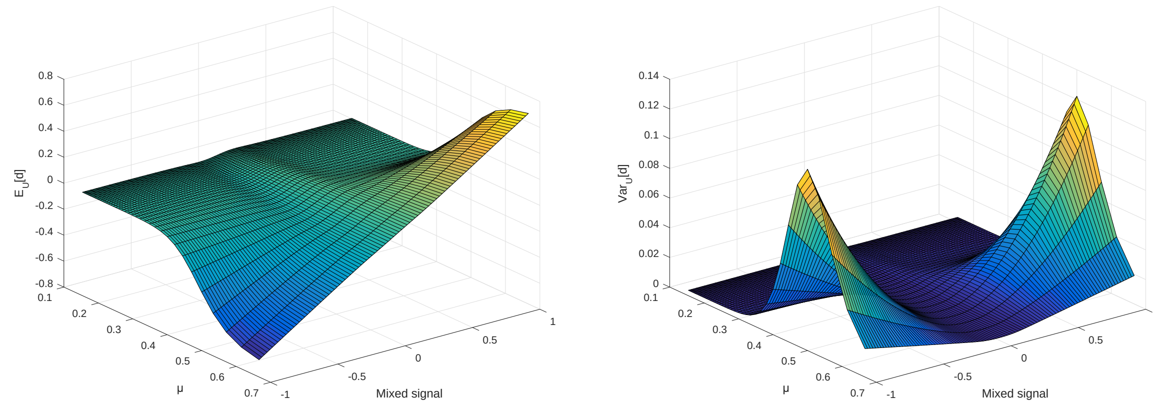
The importance of this proposition lies in establishing the fact that the risk premium part of the price is increasing in $|\tilde{s}|$, for any prior symmetric distribution (thus covering most of the known *uninformative priors* that are assumed in cases of parameters for which we have no information).

Finally, one more corollary we can obtain concerning $\mathrm{E}_H[d]$, is the following:

**Corollary 8.** *For any symmetric prior distribution of $m$, H's perceived expectation of dividends satisfies the following inequality:*

$$\frac{1}{2}\lambda|\tilde{s}| \leq |\mathrm{E}[d|\tilde{s}]| \leq \lambda|\tilde{s}|$$

[8]The case $\tilde{s} < 0$ is totally symmetric, since $\mathrm{E}[d|\tilde{s}] = -\mathrm{E}[d| - \tilde{s}]$.




(a) $\mathrm{E}_H[d]$ w.r.t. $\tilde{s}$ and $\mu$

(b) $\mathrm{Var}_H[d]$ w.r.t. $\tilde{s}$ and $\mu$

Figure 4.1: Hedge Funds' perceived expectation and variance of dividends, as a function of the mixed signal $\tilde{s}$ and of the prior expectation of $m$ ($\mu = E[m]$).

This corollary helps us to get a grasp of the magnitude of the posterior expectation of the fundamentals from the perspective of Hedge Funds. For example, we can easily see that the left hand side of the inequality implies that the expectation is strictly smaller in size than $\lambda|\tilde{s}|$. This means that, even when $s_I = s_N$, the expectation of the Hedge Funds will be different than the expectation of the other agents.

## 4.3 Simulations

One very general distribution in $[0, 1]$ that we can select in order to run simulations and show our results graphically, is the Beta distribution with parameters, $a, b$.[9] Note that the uniform distribution $U[0, 1]$, is just a special case of Beta with parameters $a = 1, b = 1$, while the so called "uninformative" Jeffrey's prior is also a Beta distribution with $a = 1/2, b = 1/2$. The parameters $a, b$ determine the shape of the distribution: the mean is equal to $\mu = \frac{a}{a+b}(= E[m])$, and the distribution is positively skewed iff $a < b$. Using the Beta(a,b) distribution, we can now make various plots, which can help us in the interpretation of the model.

In particular, Figures 4.1a and 4.1b show H's perceived expectation and variance of $d$, as a function of the mixed signal, as $\mu$ varies (keeping $a = 1$). The parameters we have used are: $Q = 1$, $\sigma = 0.06, \sigma_\varepsilon = 0.04$, as well as $\alpha = 1$ and $Z = 10$.

[9] The density function of Beta(a,b) is $f_m(m) = \dfrac{m^{a-1}(1-m)^{b-1}}{B(a,b)}$. Throughout this section, we focus on the case where $a = 1$, unless otherwise stated.

When $\mu$ is large, the expectation is increasing in the signal $\tilde{s}$, and in the limit as $\mu \to 1$, then $E_H[d] \to \lambda\tilde{s}$. On the other hand, when $\mu$ small is enough (i.e., small expected number of informed agents) the expectation of $d$ stays almost constant at 0, since hedge fund managers do not think that $\tilde{s}$ can be very useful in updating their prior knowledge about payoffs. It is also interesting to note that for $\mu \leq \frac{1}{2}$ it is not necessarily true that the expectation is increasing in the signal. This is because a higher signal can lead $H$ to update their belief about $m$ (downwards), leading them to believe that the mixed signal they observe is uninformative, thus tilting their expectation about dividends closer to their prior expectation, i.e. 0.

To understand this further, we need to examine the two forces acting against each other in the case where $\tilde{s}$ is increasing. On the one hand, this increase causes the expectation to directly increase, as managers know that the mixed signal can be, at least partially, attributed to good news from informed agents. On the other hand, as $\tilde{s}$ increases it becomes more probable that $m$ is closer to either 0 or 1. A heuristic way to see this is to note that $u$ or $d$ (corresponding to $m = 0$ or $m = 1$ respectively) have a larger variance than, for example, $\frac{1}{2}d + \frac{1}{2}u$ (corresponding to $m = 1/2$). Thus, when the prior is that there will be more noise traders, i.e., small $m$ , the additional information that $\tilde{s}$ is large makes $H$ update their information about $m$ so that they believe that $m$ is closer to 0, or equivalently, that the mixed signal is driven by noise traders and is less informative than $H$ previously thought. So, this effect leads $H$ to trust the signal less, and leads to this non-monotonicity of $\mathrm{E}_H[d]$, with respect to $\tilde{s}$. The above observation leads us to a main difference between this model and the BG model; in the present model, price can be *non-monotonic* in the signal, even in the case of risk-neutral uninformed agents, while in BG this non-monotonicity, could only happen in case of large risk aversion or supply $(\alpha Z)$ due to the effect of the risk premium component.

As far as the variance is concerned, we can see in Figure 4.1b that it is increasing in the size of the signal as well as symmetric with respect to 0; in combination with the fact that $\mathrm{E}[d|\tilde{s}] = -\mathrm{E}[d|-\tilde{s}]$, this means that the price will behave asymmetrically to positive and negative news (Corollary 4). Finally, we note that $\mathrm{Var}_H[d]$ is almost constant for $\mu$ small enough. This is because, in that case, fund managers expect very few Informed traders to be in the market and, thus, do not use the signal too much to update their beliefs about $d$; hence, their posterior variance of $d$ is close to the prior one, independently of the observed mixed signal.

Moreover, Figure 4.3 shows the equilibrium price[10] with respect to $\tilde{s}$ for two different values of $E[m]$ (corresponding to $Beta(1,1)$ and $Beta(1,3)$ prior distributions). We verify that price is non-linear and that it exhibits the aforementioned asymmetric reaction to positive/negative news. Finally, we see that these effects are largest when $\mu = 0.5$. Instead, a small $\mu$ means that more agents are probably noise, making the signal less important. In that case, we can see more clearly that the price can even decrease with a higher signal (in the neighborhood around $\tilde{s} = 0$), both because of the effect of risk premium and because of the decreased informativeness of the signal.

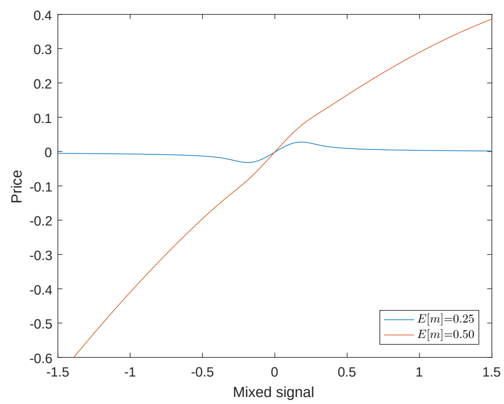


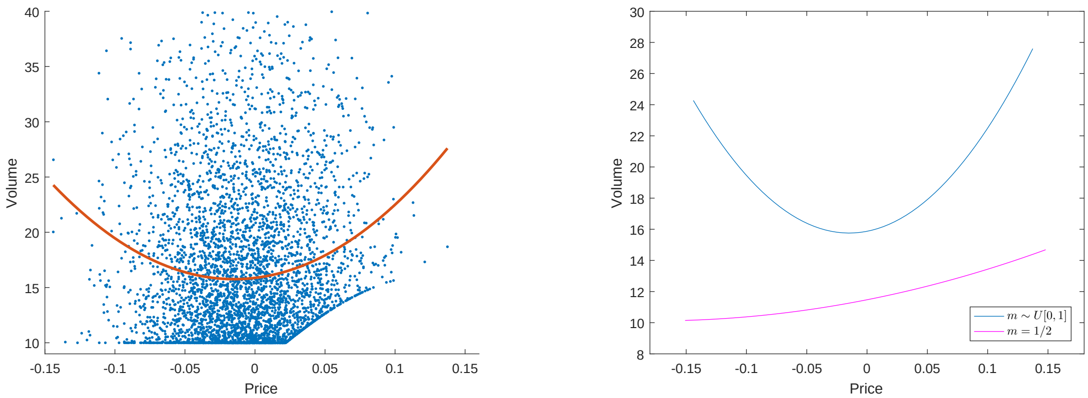

Figure 4.2: Equilibrium price as a function of $\tilde{s}$ for $E[m] = 0.50$ and $E[m] = 0.25$

### 4.3.1 Volume-price relationship

We will see from the simulations that follow, the volume-price relationship that arises from simulated data of our model fits very well with the empirical facts found in many studies about volume. This is in spite of the fact that our model was not constructed with the intention of matching these specific empirical results. In particular, the survey of Karpoff (1987) establishes the following stylized facts. The first, that can be consistently found in many empirical papers, is that volume and the absolute change in price ($|\Delta p|$) are positively related. The second is that there is a positive correlation between volume and price change *per se*; that is, volume is higher when there is a positive price changes, than when there is a negative one. I verify that this model provides suggestive evidence in favor of both of these predictions,[11] I study how results are affected by the composition

[10] Prices are negative, because of the parameter values that have been chosen but could be shifted up by a constant $\bar{d}$, simply by assuming that prior mean of $d$ (and $u$) is $\bar{d}$.

[11] We will think of price, in the context of our static model, as corresponding to the *price change* in empirical studies.




(a) Fitting a quadratic polynomial (b) The role of composition uncertainty

Figure 4.3: Volume-price relationship

uncertainty and I explain the intuition behind them.

Table 4.1: Estimation results : Volume on squared price.

Baseline model: Volume $\sim$ (price)$^2$ + price + constant.

| **Variable** | **Coefficient** | **(Std. Err.)** |
|---|---|---|
| (Price)$^2$ | **510.65**$^{**}$ | (52.108) |
| Price | 15.197$^{**}$ | (2.979) |
| Intercept | 15.87$^{**}$ | (0.12837) |

Significance levels : $\dagger$ : 10% $*$ : 5% $**$ : 1%

I use the baseline model, with $m \sim U[0,1]$, to simulate a dataset of price and trading volume[12] and I run two main regressions on this data. Table 4.1 shows the results of a regression of trading volume on a quadratic function of price.[13] What we observe is that the coefficient on the quadratic term, (price)$^2$, is positive and significant. In other words, higher volume arises when the absolute price is larger. This means that if we fit a quadratic polynomial of price into the simulated data, the relationship between price and volume is U-shaped, as shown in Figure 4.3. This is a result arising in many "Differences of Opinion" (DO) models, such as Harris & Raviv (1993), because disagreement increases in periods where signals are larger (simply because an informed traders' signal is more informative than that of the uninformed).

Although this result in our model has a similar flavour with abovementioned models, we also offer a new insight. More specifically, when price is large in absolute value, the

[12]Trading volume is computed using the equation: Volume $= \frac{1}{2}(|x_H| + m|x_I| + (1-m)|x_N| + |Z|)$.

[13]The regression of volume on absolute price gives qualitatively similar results, but we prefer this specification to emphasize that the coefficient of the linear term is also positive.

uncertainty of Hedge Funds increases (since $|\tilde{s}|$ is large) and $H$'s expectation about payoffs becomes less sensitive to cashflow news, as proved in Corollary 3. At the same time, it becomes highly likely that the group of $I$ and $N$ is very homogeneous (Corollary 2), and they all hold beliefs in which they have greatly updated their expectation about $d$. This leads to high disagreement between $H$ and the (average of the) rest of the agents, which we can measure using the difference in their expected payoffs; that is

$$|E[d|\tilde{s}] - \lambda\tilde{s}| = \lambda|\tilde{s}| \left(1 - E[L(m)|\tilde{s}]\right), \tag{4.4}$$

which is increasing in $|\tilde{s}|$.[14] This is, in turn, translated into high trading volume, which is, in fact, even greater when the prior uncertainty about $m$ is higher. Indeed, when the composition uncertainty is higher the expected disagreement increases since $|\tilde{s}|$ is likely to be higher and $E[L(m)|\tilde{s}]$ is lower; as a result, the expected volume is higher during these times. As we discuss in Section 5, this could have implications for predicting the magnitude of the regression coefficients; in particular, a crash today would make $var[m]$ increase and hence would predict a higher trading volume tomorrow as well as a greater slope of volume on $(\text{price})^2$ because of the $E[d|\tilde{s}]$ term in equation (4.4).

To study the effect of $var[m]$, I have also run the same simulations for different distributions of $m$, including the case where $m = \frac{1}{2}$ (no composition uncertainty). The coefficients of the squared price term in these regressions are smaller and in the extreme case, in which Hedge Funds are certain that $m = 1/2$, this coefficient is non-significant. This is presented in Figure 4.3(b), where we see that the quadratic curve fitting simulated data for $m = \frac{1}{2}$, almost becomes a line. Indeed, in that case, the difference in beliefs of $H$ with the (average of the) rest of the traders is 0 since $E[d|\tilde{s}, m = \frac{1}{2}] = \lambda\tilde{s}$, thus disagreement in that case does not depend on $|\tilde{s}|$. That is, we get the prediction that the coefficient on the price-squared term is increasing on the uncertainty about $m$. Finally, further simulations show that the expected volume is also increasing in $var[m]$, as explained in the end of the last paragraph.

Moreover, an econometrician might want to test the effect of price *per se* on trading volume. For this reason, I run a simple linear regression on the simulated data. What we can clearly see from table 4.2 is that there is a positive and significant (on the 2% level) relationship between volume and price. This is directly related to the asymmetry

[14] Note that $x_H = \frac{E[d|s] - \lambda\tilde{s}}{\alpha(\sigma^2(1-\lambda) + var[d|\tilde{s}])} + Z(1 - \kappa(\tilde{s}))$. As $\tilde{s} \gg 0$ this is likely to be very negative, while when $\tilde{s} \ll 0$ this is likely to be much larger than $Z$; this is because the increase in $|E[d|\tilde{s}] - \lambda\tilde{s}|$ dominates the increase in $var[d|s]$.

Table 4.2: Estimation results : Volume on price.

Baseline model: Volume $\sim$ price + constant.

| Variable | Coefficient | (Std. Err.) |
|---|---|---|
| Price | **6.5247*** | (2.8715) |
| Intercept | 16.546** | (0.10931) |
| Significance levels : $\dagger$ : 10% | $*$ : 5% | $**$ : 1% |

in positive and negative shocks, established in Corollary 4. Indeed, a large positive price is associated with an even larger positive signal. Hence, the intuition described above leads (on average) to an even higher volume for $p > 0$ than for $-p$, since, in this case, the uncertainty about $m$ and the corresponding disagreement is larger. Given that the -expected- asymmetry in positive/negative news is higher in times of greater composition uncertainty (Corollary 4), the model predicts that the slope of this regression is also higher during these times.

Therefore, we can see that composition uncertainty can even generate patterns of volume and price that arise in data and for which theoretical justifications are still not very concrete.

# 5 Extension: Dynamic Model

We will now extend our model to a dynamic version to study some of its predictions when there are more than two periods. In this case, learning about $m$ from each period, will yield the future returns partially predictable. We will assume that the market is dominated by Hedge Funds; that is their mass $Q$ is close to 2 (see proof of Proposition 1). As far as the prior distribution of $m$ is concerned, we will keep the simple assumption of a three-point symmetric distribution, as in Section 4.1.

Due to the non-normality of the equilibrium price $p$, it would be very difficult to solve the problem of long-term maximization, where agents maximize consumption over their terminal wealth. Instead, we will assume that agents trade two independent short-dated assets maturing at dates 2 and 3, respectively, with the corresponding dividends denoted by $d_2$ and $d_3$.[15] We also assume that they only observe the realized dividends after they have traded, so they cannot use them for updating their beliefs. The only thing connecting the trading of the two periods is the updating on $m$, because of the observed mixed signals. Moreover, all agents are myopic. First, we have the following characterization of the equilibrium price in each period, as in Proposition 1.

**Corollary 9.** *When the market is dominated by Hedge Funds, the equilibrium price, $p_i$ in each period $i$ is equal to*[16]

$$p \to E[d_{i+1}|\tilde{s}^{(i)}] - \frac{1}{2}\alpha Z\, var[d_{i+1}|\tilde{s}^{(i)}] \tag{5.1}$$

*where $d_i$ is the cash-flow news for the risky asset in the $i$-Period, $\tilde{s}_i$ is the realization of the mixed signal at Period $i$ and $\tilde{s}^{(i)} = \{\tilde{s}_1, \tilde{s}_2, ..., \tilde{s}_i\}$, i.e. it is the history of mixed signal realizations up to time $i$.*

This gives us a very simple characterization of the price, similar to that obtained in our baseline model. Let us now write $E_i[\cdot], var_i[\cdot]$ to denote $E[\cdot|\tilde{s}^{(i)}], var[\cdot|\tilde{s}^{(i)}]$, re-

[15]Alternatively, we could assume $d_i$, $i = 2, 3$ are independent and identically distributed cash-flow news that arrive in the market at period $i$, such that the final dividend of the asset is $d = d_2 + d_3$. Traders in Period $i$ would choose their investment to maximize their utility as if $d_{i+1}$ was actually their next Period's dividend. This interpretation is preferred when we think of the implications in the stock market.

[16]Note that we have made the simplifying assumption that at the beginning of the 2nd period, agents optimize by assuming they will just get $d_3$ in the future. Hence when forming their expectations and variances at $t = 2$, they do not take $d_2$ into account.

spectively, as perceived by Hedge Funds. Thus, for instance, $E_1[d_3 - p_2]$ denotes the expectation from the perspective of Hedge Funds of the return they will receive from Period 2 to Period 3, conditioning on the information they have at period 1, $\tilde{s}_1$. Under the above assumptions, we can now establish the following interesting result:

**Proposition 5.** *Assume that market is dominated by Hedge Funds. Then, the expectation of the future return of the asset, as of Period 1, $E_1[d_3-p_2]$, is increasing in the uncertainty about the proportion of informed traders, and hence also in the size of the mixed signal of the first Period, $|\tilde{s}_1|$.*

*Proof.* First of all note that from equation 5.1 we have that:

$$E_1[d_3 - p_2] = E_1[d_3] - \mathrm{E}_1\left[\mathrm{E}[d_3|\tilde{s}^{(2)}]\right] + \frac{\alpha Z}{2}\mathrm{E}_1\left[\mathrm{var}[d_3|\tilde{s}^{(2)}]\right] \tag{5.2}$$

$$= \frac{\alpha Z}{2}\mathrm{E}_1\left[\mathrm{var}[d_3|\tilde{s}^{(2)}]\right] \tag{5.3}$$

because of the law of iterated expectations (note that we take the expectation at the end of Period 1 when $H$ has already observed $\tilde{s}_1$). But, using Corollary 2, we know that $E_1\left[\mathrm{var}[d_3|\tilde{s}^{(2)}]\right]$ is increasing on the perceived variance of $m$ (that is, $\mathrm{var}[m|\tilde{s}_1]$). Since $\mathrm{var}[m|\tilde{s}_1]$ is increasing on $|\tilde{s}_1|$, the above proposition follows. □

It is worth discussing the above proposition further. It states that the more extreme the price in the first period is (e.g. during a crash) the higher H's expected return about the next period, as agents want to be compensated for the risk they are taking; this additional risk comes from their uncertainty about whether they are trading against signal or noise (increasing variance of $m$). Note that the signal of the first period gives no information at all about the dividends of the next period ($d_3$). Instead, it is the learning about $m$ that happens in the first period, through $\tilde{s}_1$, that carries information about the return of the next period and makes it (partially) predictable.[17] Empirically, this observation implies that during periods that are extreme in terms of the mixed news that arrives in the market (including real and fake news, or market sentiment), we should see that the future expected return of the risky asset becomes higher.

Finally, we would like to examine what happens to the volatility of prices in the second period, in terms of the first period's mixed signal. We can see that this relation

[17] Importantly in a model, such as that of BG(2015), where $H$ cannot update their perceived distribution of $m$ through the mixed signals, this channel cannot exist.

will depend on our parameters. In particular, we have the following Corollary:

**Corollary 10.** *For risk considerations ($\alpha Z$) close to* 0*, we have:*

*(i) If the market is dominated by Hedge Funds, then $var_1[p_2]$ is decreasing in $var_1[m]$ and hence in $|\tilde{s}_1|$.*

*(ii) If the market is dominated by $I$ and $N$ agents, then $var_1[p_2]$ is increasing in $var_1[m]$ and hence in $|\tilde{s}_1|$.*

Note that the first case corresponds to $Q \to 2$, while the second to $Q \to 0$. If there are only $I$ and $N$ agents in the market, a high mixed signal in the first period implies that it is more likely that $m = 0$ or $m = 1$. This, in turn, makes the variance of $\tilde{s}_2$ higher, and thus leads to higher variance of $p_2$. On the other hand, if $H$ traders are setting the price in this market, when they observe a higher signal in the first period, they understand that they should expect a high conditional variance in the next period (see Lemma 1), and adjust their future expectation of dividends, $E[d|\tilde{s}^(2)]$, so that their perceived variance of the price becomes decreasing in $|\tilde{s}_1|$.

It would also be interesting to see what the limiting behavior (and learning) would be in this economy after many Periods. Note that Hedge Funds would then be able to condition on a history of realizations $\tilde{s}^{(n)} = \{\tilde{s}_1, \tilde{s}_2, ..., \tilde{s}_n\}$. We are interested in finding the posterior distribution $f_{m|\tilde{s}^{(n)}}(m|\tilde{s}^{(n)})$. Note that, as shown in Corollary 7, if the prior distribution $f(m)$ is symmetric then the posterior distribution would remain symmetric, after any number of periods. However, as $n \to \infty$, we can get the following proposition.

**Proposition 6.** *As the number of periods tends to infinity, Hedge Funds learn $|m - \frac{1}{2}|$. Thus, the posterior distribution of $m$ converges to a symmetric two-point distribution.*

*Proof.* Can be found in the Appendix. The main idea is that by the Law of Large numbers we can find the empirical variance of $\tilde{s}$ and then equate it with the actual variance of $\tilde{s}$ (conditional on $m$). □

The above proposition implies that, in the long-run, managers can learn the true distance of $m$ from $\frac{1}{2}$, but since the posterior distribution is always symmetric they can never distinguish between $m$ or $1 - m$. If we call $var[m|\tilde{s}^{(n)}]$ the long-run uncertainty about $m$, then this uncertainty is higher when $|m - \frac{1}{2}|$ is larger. This is because, as

$n \to \infty$, $var[m|\tilde{s}^{(n)}] \to \frac{1}{2}((m^*)^2 + (1-m^*)^2 - \frac{1}{2})$, where $m^*$ is the realization of the random variable m. As in Lemma 1, we can thus show (see Appendix) that the expected returns are higher when this long run uncertainty, or equivalently, $|m^* - \frac{1}{2}|$ is higher. Finally, using $(corr(s_i, s_j))^{-1}$, which is decreasing in $m^2 + (1-m)^2$, as the measure of belief dispersion in the market (*see Corollary 1*), we conclude that higher belief dispersion leads, in the long run, to lower expected returns.

# 6 Discussion

There are two key main points that the model in this paper makes. The first is that a mixed signal that traders observe can change their perceived variance about the proportion of informed traders in the market. In particular, the higher the size of this signal, the higher the uncertainty about this proportion. The second major point is that this variance is a measure of the (perceived) belief dispersion in the market and affects the informativeness of the prices, it creates an asymmetric reaction to positive and negative news, and leads to predictability about future expected returns. We would thus like to interpret the $var[m]$ as a measure of the asymmetric information in the market. By doing so, we could have a way of empirically testing some of the predictions of this model. In particular, there have been empirical studies that use the probability of informed trading (PIN), constructed by Easley et al. (2002), as a measure of the asymmetric information. This paper claims that, controlling for the expectation of PIN, what should also matter is the variance of PIN, or the uncertainty about its value.

In particular, according to our model, one should empirically expect to find that a higher variance of PIN leads to higher expected returns. Moreover, to study the effect of this variance on informativeness one could look into the sensitivity of investment decisions to stock prices (which is a proxy of informativeness used in papers such as Bond et al. (2012)), to see whether times with more uncertain PIN, are associated with lower such informativeness.

A main empirical challenge would be to try find a good proxy for the mixed signal ($\tilde{s}$) used in this paper. I believe that some indices of market sentiment can be used as a proxy for $\tilde{s}$; indeed, a survey-based market sentiment index can contain both information and noise, and hence it could be an appropriate proxy for $\tilde{s}$. Similarly, another proxy we could use for the mixed signal would be the mutual fund flows. Both these proxies have been used as proxies for pure noise (investor sentiment), but have been criticized exactly because they are subject to confounding variables (related to fundamental information).

# 7 Conclusion

In this paper I describe a model of asymmetric information in which the proportion of Informed traders to Noise traders is unknown to the Hedge Fund managers who trade in the market. I study how traders learn about this additional uncertainty and I examine the resulting equilibrium quantities. Moreover, I relate this uncertainty to the perceived heterogeneity of beliefs in the market. More specifically, it is shown that Hedge Funds become more uncertain about fundamentals when they observe extreme news (for example, during a market crash), as they becomes less confident in inferring information from the prices. In addition, in this setting, the expected returns are decreasing in the perceived heterogeneity in the market, and there is an asymmetric price reaction in positive and negative news.

I illustrate the intuition of the key findings by assuming at first a three-point prior distribution for the proportion of informed traders, and I extend the results to the case of any symmetric continuous distribution. Furthermore, I find that this model is consistent with the empirical stylized facts concerning the volume-price relationship and I thus offer a possible theoretical explanation for these findings. Overall, the focus of this paper lies on understanding how traders learn about fundamentals, while also learning about their market environment given the signal that equilibrium quantities convey. As shown in the dynamic extension of the model, this setting carries many implications about the information quality of prices and the resulting volatility in the market. Finally, the fact that the equilibrium price does not fully reveal the signal of informed agents, provides a very useful model to work on and makes a dynamic version of the model, in which agents learn from dividends or from stale information, very interesting to investigate further.

# 8 References

Avery, C. & Zemsky, P. (1998), 'Multidimensional uncertainty and herd behavior in financial markets', *American economic review* pp. 724–748.

Banerjee, S. (2011), 'Learning from prices and the dispersion in beliefs', *The Review of Financial Studies* **24**(9), 3025–3068.

Banerjee, S. & Green, B. (2015), 'Signal or noise? uncertainty and learning about whether other traders are informed', *Journal of Financial Economics* **117**(2), 398–423.

Black, F. (1986), 'Noise', *The journal of finance* **41**(3), 528–543.

Blume, L., Easley, D. & O'hara, M. (1994), 'Market statistics and technical analysis: The role of volume', *The Journal of Finance* **49**(1), 153–181.

Bond, P., Edmans, A. & Goldstein, I. (2012), 'The real effects of financial markets', *Annu. Rev. Financ. Econ.* **4**(1), 339–360.

Breon-Drish, B. (2015), 'On existence and uniqueness of equilibrium in a class of noisy rational expectations models', *The Review of Economic Studies* **82**(3), 868–921.

Cao, H. H., Coval, J. D. & Hirshleifer, D. (2002), 'Sidelined investors, trading-generated news, and security returns', *The Review of Financial Studies* **15**(2), 615–648.

Cao, H. H. & Ye, D. (2016), 'Transaction risk, derivative assets, and equilibrium', *Quarterly Journal of Finance* **6**(01), 1650001.

Chen, Q., Goldstein, I. & Jiang, W. (2006), 'Price informativeness and investment sensitivity to stock price', *The Review of Financial Studies* **20**(3), 619–650.

David, A. & Veronesi, P. (2013), 'What ties return volatilities to price valuations and fundamentals?', *Journal of Political Economy* **121**(4), 682–746.

De Long, J. B., Shleifer, A., Summers, L. H. & Waldmann, R. J. (1990), 'Noise trader risk in financial markets', *Journal of political Economy* **98**(4), 703–738.

Easley, D., Hvidkjaer, S. & O'hara, M. (2002), 'Is information risk a determinant of asset returns?', *The journal of finance* **57**(5), 2185–2221.

Easley, D., O'Hara, M. & Yang, L. (2013), 'Opaque trading, disclosure, and asset prices: Implications for hedge fund regulation', *The Review of Financial Studies* **27**(4), 1190–1237.

Gao, F., Song, F. & Wang, J. (2013), 'Rational expectations equilibrium with uncertain proportion of informed traders', *Journal of Financial Markets* **16**(3), 387–413.

Grossman, S. J. & Stiglitz, J. E. (1980), 'On the impossibility of informationally efficient markets', *The American economic review* **70**(3), 393–408.

Harris, M. & Raviv, A. (1993), 'Differences of opinion make a horse race', *The Review of Financial Studies* **6**(3), 473–506.

Harrison, J. M. & Kreps, D. M. (1978), 'Speculative investor behavior in a stock market with heterogeneous expectations', *The Quarterly Journal of Economics* **92**(2), 323–336.

Karpoff, J. M. (1987), 'The relation between price changes and trading volume: A survey', *Journal of Financial and quantitative Analysis* **22**(1), 109–126.

Kondor, P. (2012), 'The more we know about the fundamental, the less we agree on the price', *The Review of Economic Studies* **79**(3), 1175–1207.

Kreps, D. M. (1977), 'A note on "fulfilled expectations" equilibria', *Journal of Economic Theory* **14**(1), 32–43.

Kyle, A. S. (1985), 'Continuous auctions and insider trading', *Econometrica: Journal of the Econometric Society* pp. 1315–1335.

Li, T. (2013), 'Insider trading with uncertain informed trading'.

Marmora, P. & Rytchkov, O. (2018), 'Learning about noise', *Journal of Banking & Finance* **89**, 209–224.

Mendel, B. & Shleifer, A. (2012), 'Chasing noise', *Journal of Financial Economics* **104**(2), 303–320.

Pálvölgyi, D. & Venter, G. (2015), 'Multiple equilibria in noisy rational expectations economies'.

Radner, R. (1968), 'Competitive equilibrium under uncertainty', *Econometrica: Journal of the Econometric Society* pp. 31–58.

Romer, D. (1992), Rational asset price movements without news, Technical report, National bureau of economic research.

Rossi, S. & Tinn, K. (2014), 'Man or machine? rational trading without information about fundamentals.'.

Sadka, R. (2006), 'Momentum and post-earnings-announcement drift anomalies: The role of liquidity risk', *Journal of Financial Economics* **80**(2), 309–349.

Spiegel, M. & Subrahmanyam, A. (2000), 'Asymmetric information and news disclosure rules', *Journal of Financial intermediation* **9**(4), 363–403.

Tetlock, P. C. (2011), 'All the news that's fit to reprint: Do investors react to stale information?', *The Review of Financial Studies* **24**(5), 1481–1512.

Yuan, K. (2005), 'Asymmetric price movements and borrowing constraints: A rational expectations equilibrium model of crises, contagion, and confusion', *The Journal of Finance* **60**(1), 379–411.

# 9 Appendix A

**Proof of Proposition 1:**

*Proof.* We will write down a proof for the general case where the mass of Hedge funds is Q, while the total mass of agents remains 2 (for the baseline model Q=1) and the proportion of $I$ to $N$ is still $m : (1-m)$. The first step for this proof is to write down the equilibrium demand function of $H$. We can easily show that under mean variance utility this is:

$$x_H = \frac{E_H[d] - p}{\alpha \mathrm{Var}_H[d]},$$

where as we have described throughout section 2, $H's$ expectation and variance is conditional on the mixed signal $\tilde{s}$ (which in equilibrium is revealed if he observes the price and the residual demand). Therefore, using the market clearing condition, we have:

$$Qx_H + (2-Q)(mx_I + (1-m)x_N) = Q\frac{E_H[d] - p}{\alpha \mathrm{Var}_H[d]} + (2-Q)\frac{\lambda\tilde{s} - p}{\alpha\sigma^2(1-\lambda)} = Z$$

Equivalently

$$p \cdot \left[(2-Q)\mathrm{Var}_H[d] + Q\sigma^2(1-\lambda)\right] = Q\sigma^2(1-\lambda)E_H[d] + (2-Q)\mathrm{Var}_H[d]\lambda\tilde{s} - \alpha Z \mathrm{Var}_H[d]\sigma^2(1-\lambda)$$

.

Finally, defining

$$\kappa(\tilde{s}) = \frac{(2-Q)var[d|\tilde{s}]}{Q\sigma^2(1-\lambda) + (2-Q)var[d|\tilde{s}]},$$

gives us the equilibrium price:

$$P = \lambda\tilde{s}\kappa(\tilde{s}) + \mathrm{E}\left[d|\tilde{s}\right](1-\kappa(\tilde{s})) - \frac{1}{2-Q}\alpha\kappa(\tilde{s})\sigma^2(1-\lambda)Z,$$

□

**Proof of Corollary 1:**

*Proof.* First of all, note that $E[s_i|m] = mE[s_I] + (1-m)E[s_N] = 0$, hence, by the Law of

Total covariance, and since $cov(E[s_i|m], E[s_j|m]) = 0$, we have

$$\begin{aligned} cov(s_i, s_j) = E[cov(s_i, s_j|m)] &= E[E[s_i s_j|m]] \\ &= E[m^2 E[s_I^2] + 2m(1-m)E[s_I s_N] + (1-m)^2 E[s_N]^2] \\ &= E[m^2 + (1-m)^2](\sigma^2 + \sigma_\varepsilon^2) \\ &= (2(var[m] + E[m]^2 - E[m]) + 1)(\sigma^2 + \sigma_\varepsilon^2) \end{aligned}$$

Moreover, $var[s_i] = E[E[s_i^2|m]] = E[mE[s_I^2] + (1-m)E[s_N^2]] = \sigma^2 + \sigma_\varepsilon^2$. Therefore, we get:

$$corr(s_i, s_j) = \frac{cov(s_i, s_j)}{\sqrt{var[s_i]var[s_j]}} = 2(var[m] + E[m]^2 - E[m]) + 1$$

Hence, if $E[m]$ is constant (which is the case when the distribution of beliefs about $m$ is symmetric), then a higher $var[m]$ implies a higher covariance between two random signals $s_i, s_j$ and thus also a lower degree of belief dispersion. □

**Proof of Corollary 2:**

*Proof.* We need to show how the posterior beliefs about $m$ depend on $\tilde{s}^2$. We have:

$$\frac{h_2(\tilde{s})}{h_1(\tilde{s})} = \sqrt{2}\exp\left(-\tilde{s}^2 \cdot \frac{1}{2(\sigma^2 + \sigma_\varepsilon^2)}\right)$$

Therefore $\frac{h_2(\tilde{s})}{h_1(\tilde{s})}$ is decreasing in $\tilde{s}^2$. But from equation (4.1) we can see that $\hat{\pi}_0$ (and similarly $\hat{\pi}_1$) is decreasing in $\frac{h_2(\tilde{s})}{h_1(\tilde{s})}$. Therefore we get that $\hat{\pi}_0$ and $\hat{\pi}_1$ are increasing in $\tilde{s}^2$. In contrast $\hat{\pi}_{1/2} = 1 - \hat{\pi}_0 - \hat{\pi}_1$ and hence it is decreasing in $\tilde{s}^2$. That is to say, a more extreme signal means that is more probable that the other group of traders is either all informed or all noise, while a signal closer to 0 means it is more probable there is a mixture of both. □

**Proof of Proposition 2:**

*Proof.* First of all we have:

$$\begin{aligned}
c(\tilde{s}) &= \lambda^2 \mathrm{Var}[L(m)|\tilde{s}] = \\
&= \lambda^2 \left( E[(L(m))^2|\tilde{s}] - (E[L(m)|\tilde{s}])^2 \right) = \\
&= \lambda^2 \left[ ((1-2\hat{\pi}) \cdot \frac{1/2}{1/2} + \hat{\pi} \cdot 1) - (1-\hat{\pi})^2 \right] = \\
&= \lambda^2 \hat{\pi}(1-\hat{\pi})
\end{aligned}$$

which is increasing in $\tilde{s}^2$ since $\frac{d\hat{\pi}}{d\tilde{s}^2} > 0$ (by Corollary 2) and $\hat{\pi} < 1/2$, as $P(m \in 0, 1) = 2\hat{\pi}$.

Moreover, with a few algebraic manipulations we can get:

$$E[mL(m)|\tilde{s}] = \lambda \left[ (1-2\hat{\pi}) \cdot \frac{1/4}{1/2} + \hat{\pi} \cdot 1 \right] = \frac{1}{2}$$

Therefore H's perceived variance of $d$ must be increasing in $\tilde{s}^2$. $\square$

**Proof of Lemma 1**:

*Proof.* First of all, note that:

$$\mathrm{Var}[m] = \pi + (1-2\pi) \cdot \frac{1}{4} - \frac{1}{4} = \frac{\pi}{2}.$$

Thus, it is sufficient to prove that $E[var[d|\tilde{s}]]$ is increasing in $\pi = P(m = 0)$, as $var[m]$ is increasing in $\pi = P(m = 0)$ (for symmetric 3-point distribution of $m$). Moreover, we know that $var[d|\tilde{s}]$ is increasing in $\tilde{s}^2$. Hence it would be sufficient to prove that $\tilde{s}^2(\pi_1)$ first order stochastically dominates $\tilde{s}^2(\pi_2)$ if $\pi_1 > \pi_2$. Indeed we have, for any $x > 0$:

$$\begin{aligned}
P(\tilde{s}^2 \leq x) =& P(-\sqrt{x} \leq \tilde{s} \leq \sqrt{x}) = \\
=& 1 - 2P(\tilde{s} \leq -\sqrt{x}) = \\
=& 1 - 2[2\pi P(s_I \leq -\sqrt{x}) + (1-2\pi) P(\frac{s_I + s_N}{2} \leq -\sqrt{x})] = \\
=& 1 - 2 \left[ P(\frac{s_I + s_N}{2} \leq -\sqrt{x}) + 2\pi \left[ P(s_I \leq -\sqrt{x}) - P(\frac{s_I + s_N}{2} \leq -\sqrt{x}) \right] \right]
\end{aligned}$$

But now note that:

$$\begin{aligned}
P(s_I \leq -\sqrt{x}) - P(\frac{s_I + s_N}{2} \leq -\sqrt{x}) =& P\left(\frac{s_I}{\sigma} \leq \frac{-\sqrt{x}}{\sigma}\right) - P\left(\frac{s_I + s_N}{\sigma\sqrt{2}} \leq \frac{-\sqrt{x})}{\sigma/\sqrt{2}}\right) \\
=& \Phi(\frac{-\sqrt{x}}{\sigma}) - \Phi(\frac{-\sqrt{x}}{\sigma/\sqrt{2}}) > \\
>& 0
\end{aligned}$$

since $\frac{-\sqrt{x}}{\sigma} > \frac{-\sqrt{x}}{\sigma/\sqrt{2}}$.

Thus, $P(\tilde{s}^2 \leq x)$ is decreasing in $\pi$ and hence this shows that $\tilde{s}^2(\pi_1)$ first order stochastically dominates $\tilde{s}^2(\pi_2)$ when $\pi_1 > \pi_2$, and the proof is completed. □

**Proof of Proposition 3:**

*Proof.* Combining the fact that $\kappa(\tilde{s})$ is increasing in $\tilde{s}^2$ together with the stochastic dominance established in the proof of Lemma 1, we get that $\kappa(\tilde{s})$, and hence the risk premium is increasing in $var[m]$.

Moreover, when the market is dominated by Hedge Funds, $Q \to 2$ and $p = E[d|\tilde{s}] - \frac{1}{2}\alpha Z var[d|\tilde{s}]$. Therefore, by using the law of iterated expectations we get

$$E[d - p] = \frac{1}{2}\alpha Z E[var[d|\tilde{s}]]$$

which is increasing in $var[m]$ by the abovementioned lemma. □

**Proof of Corollary 5:**

*Proof.* Because of the symmetry of $f(m)$ with respect to 1/2, we can get:

$$E[\frac{dp}{ds_N}] = E[(1-m)\frac{dp}{d\tilde{s}}] = \frac{1}{2}E[\frac{dp}{d\tilde{s}}]$$

where the first equality is because of chain rule, while the second uses the abovementioned symmetry. Taking the derivative of price with respect to the mixed signal and setting $I(s) := E[L(m)|\tilde{s}]$, we get:

$$\frac{dp}{d\tilde{s}} = \lambda(\kappa(\tilde{s}) + \tilde{s}\kappa^{'}(\tilde{s}) + I(\tilde{s})(1-\kappa(\tilde{s})) + \tilde{s}I^{'}(\tilde{s})(1-\kappa(\tilde{s})) - \tilde{s}I(\tilde{s})\kappa^{'}(\tilde{s})) - \alpha\kappa^{'}(\tilde{s})\sigma^2(1-\lambda)Z$$

When $|\tilde{s}| \to \infty$, we get:

$$\begin{aligned}
\kappa(\tilde{s}) &\to 1 \\
\kappa'(\tilde{s}) &\to 0 \\
\tilde{s}\kappa'(s) &\to 0 \\
I(\tilde{s}) &\to \frac{1}{2} \\
I'(\tilde{s}) &\to 0 \\
\tilde{s}I'(\tilde{s}) &\to 0
\end{aligned}$$

where the 3rd and the 6th lines hold because $\kappa'(\tilde{s})$ is of order[18] $\tilde{s}^{-2}$ and $I'(\tilde{s})$ is exponentially decreasing. Therefore we have $\frac{dp}{d\tilde{s}} \to \lambda$, as $|\tilde{s}| \to \infty$.

On the other hand we can see how this derivative behaves close to $\tilde{s} = 0$ and because of continuity, it is sufficient to just calculate the derivative at 0. Indeed, we have:

$$\frac{dp}{d\tilde{s}}\Big|_{\tilde{s}=0} = \lambda\left(\kappa(\tilde{s}) + I(\tilde{s})(1-\kappa(\tilde{s}))\right) - \alpha\kappa'(\tilde{s})\sigma^2(1-\lambda)Z < \lambda$$

since $I(\tilde{s}) < 1$, and $1 > \kappa(\tilde{s}) > 0$ (even in the limit as $\tilde{s} \to 0$) and $\kappa'(s) \to 0$ as $\tilde{s} \to 0$, since $\kappa(\tilde{s})$ is increasing in $\tilde{s}^2$ (see Proposition 2), which implies $\kappa'(\tilde{s})$, is positive for $\tilde{s}$ positive, and negative for $\tilde{s}$ negative (and therefore that $\kappa'(0) = 0$ from Darboux theorem). □

**Joint distribution of $d, \tilde{s}$:**

Given a beta prior distribution for $m$, we can use a change of variables transformation to calculate the conditional densities $d|\tilde{s}$ and $m|\tilde{s}$. Using the map $d, u, m, \varepsilon \mapsto d, u, m, \underbrace{d + u + \varepsilon}_{\tilde{s}}$ we get the joint distribution of $d, u, m, \tilde{s}$:

$$g(d, u, m, \tilde{s}) = f_{d,u,m,\varepsilon}(d, u, m, \tilde{s} - md - (1-m)u) \cdot |\det(J)|,$$

where the $J$ is the Jacobian of the inverse map, and it can be easily computed to be an upper triangular matrix with 1's in the main diagonal.

Combining the above with the fact that $d, u, m, \varepsilon$ are independent we get $g(d, u, m, \tilde{s}) =$

[18] In particular, note that $\kappa'(\tilde{s}) = \frac{(var_H[d])'}{(\sigma^2(1-\lambda)+var_H[d])^2}$. As $|\tilde{s}| \to \infty$, the denominator is of order $\tilde{s}^4$ while the numerator is equal to $(\lambda^2\hat{\pi}(1-\hat{\pi})\tilde{s}^2)'$ which is of order smaller than $\tilde{s}^2$ since the derivative of $\hat{\pi}(1-\hat{\pi})$ at infinity, is definitely bounded, since it is an increasing bounded function of $\tilde{s}^2$.

$f_d(d)f_u(u)f_m(m)f_\varepsilon(\tilde{s} - md - (1-m)u)$, where $f_u, f_d, f_\varepsilon$ are normal pdfs and $f_m(m)$ is the pdf of a Beta distribution. Integrating out $u$ and $m$, by completing the square where necessary (to get rid of the integral w.r.t. $u$), we get:

$$
\begin{aligned}
g(d,\tilde{s}) &= \int_0^1 \int_{-\infty}^{\infty} g(d,u,m,\tilde{s})dudm \\
&= \ldots = \\
&= \frac{e^{-\frac{d^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}} \int_0^1 \frac{m^{a-1}(1-m)^{b-1}}{B(a,b)} \frac{1}{\sqrt{2\pi V(m)}} e^{-\frac{(\tilde{s}-md)^2}{2V(m)}} dm.
\end{aligned}
$$

In the same way we can derive the joint distribution of $m, \tilde{s}$ and hence get the conditional density $m|\tilde{s}$ that we need, in order to make simulations for $m \sim Beta(a,b)$. For the latter, we can write alternatively, using Bayes' rule:

$$
g(m,\tilde{s}) = g(\tilde{s}|m)f_m(m)
$$

But $\tilde{s}|m$ is a linear combination of normals, hence it is a normal itself, with mean 0 and variance $C(m) = (m^2 + (1-m)^2)(\sigma^2 + \sigma_\varepsilon^2)$. Hence, we get equation (4.2).

**Proof of Corollary 7**:

*Proof.* Let $V(m) := m^2 + (1-m)^2$. We will show that the informativeness of the signal, $E_H[\frac{m\lambda}{V(m)}]$, is decreasing in $\tilde{s}^2$. We will use the notation: $g_{\tilde{s}^2}(m,s)$ to refer to partial derivative: $\frac{\partial g(m,\tilde{s})}{\partial \tilde{s}^2}$. Using the formula for $g(m,\tilde{s})$, described in equation (4.2), we get: $g_{\tilde{s}^2}(m,\tilde{s}) = -\frac{g(m,\tilde{s})}{2V(m)(\sigma^2+\sigma_\varepsilon^2)}$. We will now use the following auxiliary result:

$$
E_H[\frac{m\lambda}{V(m)}] = \frac{1}{2}E_H[\frac{\lambda}{V(m)}]
$$

This is because $V(m)$ and $f_{m|\tilde{s}}(m|\tilde{s})$ is unchanged under the change of variables $m \mapsto 1-m$ (as long as $m$ has symmetric distribution).

Using the above, together with the Leibniz integral rule, which allows us to interchange an integral with a partial derivative, as long as the integrand is a continuous function, we get:

$$\frac{\partial}{\partial \tilde{s}^2} E_H[\frac{m\lambda}{V(m)}] = \frac{1}{2}\int_0^1 \frac{\lambda}{V(m)} \left[\frac{g_{\tilde{s}^2}(m,\tilde{s})g(\tilde{s}) - g(m,\tilde{s})g_{\tilde{s}^2}(\tilde{s})}{g(\tilde{s})^2}\right] dm$$
$$= \frac{\lambda}{2g(\tilde{s})^2}\int_0^1 \frac{g(m,s)}{V(m)} \left[-\frac{g(\tilde{s})}{2V(m)(\sigma^2+\sigma_\varepsilon^2)} - g_{\tilde{s}^2}(\tilde{s})\right] dm$$

Now noting that $g(\tilde{s}) = \int_0^1 g(m,s)dm$ and using that to obtain $g_{\tilde{s}^2}(\tilde{s}) = -\int_0^1 \frac{g(m,s)}{2V(m)(\sigma^2+\sigma_\varepsilon^2)}dm$ we get, that the sign of the derivative we want, depends on the sign of:

$$-\int_0^1 g(m,s)dm \int_0^1 \frac{g(m,s)}{V^2(m)}dm + \int_0^1 \frac{g(m,s)}{V(m)}dm \int_0^1 \frac{g(m,s)}{V(m)}dm$$

But using the Cauchy-Schwarz inequality[19] for integrals we know that:

$$\left(\int_0^1 \frac{g(m,s)}{V(m)}dm\right)^2 < \int_0^1 g(m,s)dm \int_0^1 \frac{g(m,s)}{V^2(m)}dm$$

Therefore the sign of the derivative we want is strictly negative. In other words $E_H[\frac{m\lambda}{V(m)}]$ is decreasing in $\tilde{s}^2$, thus *completing the proof.*

□

**Proof of Proposition 4**

*Proof.* We want to prove that the perceived variance of $d$ is decreasing in $\tilde{s}^2$. Firstly, since the posterior of $m$ is symmetric, as we showed before, we will have:

$$E_H[\frac{m^2}{V(m)}] = E_H[\frac{(1-m)^2}{V(m)}]$$

But since $V(m) = m^2 + (1-m)^2$ we get the the above expectations are equal to 1/2 (since their sum is equal to 1). Therefore:

$$E[mL(m)|\tilde{s}] = \frac{1}{2},$$

consistent with what we showed before for the case of the three-point distribution (note:

[19]Equality would only hold if $g(m,s)$ and $\frac{g(m,s)}{V^2(m)}$ were proportional, which of course it is not the case.

$mL(m) = \frac{m^2}{V(m)}$). As a result, it would be sufficient to prove that $c(\tilde{s})$ is increasing in $\tilde{s}^2$. Indeed, we have

$$\frac{\partial}{\partial \tilde{s}^2}[\mathrm{Var}_H[\frac{m}{V(m)}]] \propto \int_0^1 \frac{m^2 g(m,\tilde{s})}{V^2(m)}\left[-\frac{g(\tilde{s})}{C(m)} - g_{\tilde{s}^2}(\tilde{s})\right] dm -$$
$$- 2\int_0^1 \frac{m}{V(m)} f_{m|\tilde{s}}(m|\tilde{s}) dm \int_0^1 \frac{m \cdot g(m,\tilde{s})}{V(m)}\left[-\frac{g(\tilde{s})}{C(m)} - g_{\tilde{s}^2}(\tilde{s})\right] dm$$

where the proportionality is with respect to a positive integer.

Now if we let $A(m,\tilde{s}) = \frac{g(m,\tilde{s})}{V(m)}$, and simplify $g_{\tilde{s}^2}(\tilde{s})$, as in the proof of corollary 7, we need to prove that the following is positive:

$$-g(\tilde{s})\int_0^1 \frac{m^2}{V^2(m)} A(m,\tilde{s}) dm + \int_0^1 A(m,\tilde{s}) dm \int_0^1 \frac{m^2}{V^2(m)} g(m,\tilde{s}) dm -$$
$$- \frac{2\int_0^1 mA(m,\tilde{s})dm}{g(\tilde{s})}\left[-g(\tilde{s}) m \int_0^1 \frac{mA(m,\tilde{s})}{V(m)} dm + \int_0^1 A(m,\tilde{s}) dm \int_0^1 \frac{mg(m,\tilde{s})}{V(m)} dm\right]$$

Now we will use our usual transformation of $m \longrightarrow 1-m$ to get the following results:

$$\int_0^1 \frac{m}{V(m)} A(m,\tilde{s}) dm = \frac{1}{2}\int_0^1 \frac{A(m,\tilde{s})}{V(m)} dm$$
$$\int_0^1 \frac{m^2}{V^2(m)} A(m,\tilde{s}) dm = \frac{1}{2}\int_0^1 \frac{A(m,\tilde{s})}{V(m)} dm$$
$$\int_0^1 \frac{m^2}{V^2(m)} g(m,\tilde{s}) dm = \frac{1}{2}\int_0^1 A(m,\tilde{s}) dm$$
$$\int_0^1 mA(m,\tilde{s}) dm = \frac{1}{2}\int_0^1 A(m,\tilde{s}) dm$$

Plugging in the above, and multiplying the result by 2, we see that it is sufficient to prove that the following expression is positive:

$$-g(\tilde{s})\int_0^1 \frac{A(m,\tilde{s})}{V(m)} dm + (\int_0^1 A(m,\tilde{s}) dm)^2 + \int_0^1 A(m,\tilde{s}) dm \int_0^1 \frac{A(m,\tilde{s})}{V(m)} dm - \frac{(\int_0^1 A(m,\tilde{s})dm)^3}{g(\tilde{s})}$$

But the above expression is equal to

$$\left[\frac{(\int_0^1 A(m,\tilde{s})dm)^2}{g(\tilde{s})} - \int_0^1 \frac{A(m,\tilde{s})}{V(m)} dm\right] \cdot \left[g(\tilde{s}) - \int_0^1 A(m,\tilde{s}) dm\right]$$

Finally note that both of the above brackets are less than 0. The first one is less than 0 since by Cauchy-Schwarz inequality:

$$(\int_0^1 A(m,\tilde{s})dm)^2 < (\int_0^1 A(m,\tilde{s})V(m)dm)\int_0^1 \frac{A(m,\tilde{s})}{V(m)}dm$$

while, the second one is negative because $A(m,\tilde{s}) = \frac{g(m,\tilde{s})}{V(m)} \geq g(m,\tilde{s})$, because $V(m) \leq 1$.

Therefore the overall expression is positive, making $c(\tilde{s})$ increasing in $\tilde{s}$ and thus concluding the proof that $\text{Var}_H[d]$ is increasing in $\tilde{s}^2$.

□

**Proof of Corollary 8:**

*Proof.* We firstly observe that: $V(m) \in [\frac{1}{2}, 1]$, as the function $m^2 + (1-m)^2$ defined on the interval $[0,1]$ takes its minimum at $m = 1/2$ and its maximum at $m = 0, 1$. Moreover for $f_m(m)$ symmetric with respect to 0.5, we can easily see that: $f_{m|\tilde{s}}(m|\tilde{s}) = f_{m|\tilde{s}}(1-m|\tilde{s})$ and $V(m) = V(1-m)$. Therefore, using the change of variables $m' = 1 - m$ we get, as before:

$$\int_0^1 \frac{m\lambda}{V(m)} f_{m|\tilde{s}}(m|\tilde{s})dm = \frac{1}{2}\int_0^1 \frac{\lambda}{V(m)} f_{m|\tilde{s}}(m|\tilde{s})dm$$

To prove the inequalities, we now just need to note that:

$$|\text{E}_H[d]| = |\frac{\tilde{s}\lambda}{2}\int_0^1 \frac{1}{V(m)} f_{m|\tilde{s}}(m|\tilde{s})dm|$$

Hence, combined with the abovementioned bounds on $V(m)$ we get the required result. □

**Proof of Corollary 10:**

*Proof.* We write $E_i[\cdot], var_i[\cdot]$ to denote $E[\cdot|\tilde{s}^{(i)}], var[\cdot|\tilde{s}^{(i)}]$ respectively. For the first case, $Q \to 2$ and $p_2 \to E[d_3|\tilde{s}^{(2)}]$. Therefore, using the law of total variance, we get:

$$var_1[p] \approx var_1[E[d_3|\tilde{s}^{(2)}]] = var_1[d_3] - E_1[var[d_3|\tilde{s}^{(2)}]]$$

which is decreasing in $|\tilde{s}_1|$, since $var_1[d_3] = var[d_3]$ and $E_1[var[d_3|\tilde{s}^{(2)}]]$ is increasing in $|\tilde{s}_1|$ as shown in Proposition 3, because conditioning on $\tilde{s}_1$ can only affect the distribution of $\tilde{s}_2$ through $m$ (and higher $|\tilde{s}_1|$ implies higher $var_1[m]$).

For the second case, $Q \to 0$ and $p_2 \to \lambda\tilde{s}_2$. Therefore $var_1[p] \approx \lambda^2 var_1[s_2]$, which is increasing in $|\tilde{s}_1|$, since a higher $|s_1|$ leads to a higher $var[m]$ which then leads to higher $var_1[\tilde{s}_2]$ (remember, the intuition, that the variance of $s_I$ or $s_N$ is higher than the variance of $\frac{s_I+s_N}{2}$). □

**Proof of Proposition 6:**

*Proof.* We have that

$$\tilde{s}_j^2 = (ms_{I,j} + (1-m)s_{N,j})^2 = m^2 s_{I,j}^2 + (1-m)^2 s_{N,j} + 2m(1-m)s_{I,j}s_{N,j}$$

Since $s_{I,j}, s_{N,j}$ for $j = 1, \ldots, n$ can be seen as realizations of a $(2 \times 1)$ random variable, with covariance matrix $\begin{pmatrix} \sigma^2+\sigma_\varepsilon^2 & 0 \\ 0 & \sigma^2+\sigma_\varepsilon^2 \end{pmatrix}$ by the Law of Large numbers, as $n \to \infty$ we get:

$$\sum_{j=1}^{n} \frac{\tilde{s}_j^2}{n} \to E[\tilde{s}_j^2] = (m^2 + (1-m)^2)(\sigma^2 + \sigma_\varepsilon^2)$$

That is, Hedge Funds learn the value of $m^2 + (1-m)^2$. By solving this quadratic one can see that there are always 2 solutions of the form $m*, (1-m*)$ and the managers have no way of distinguishing between the two, as the posterior distribution needs to remain symmetric.

Further to the above, we want to show that in the long run, the expected returns are higher when the long run uncertainty (which is proportional to $m^2 + (1-m)^2$) is higher. For that, we will first show that if $m_1^2 + (1-m_1)^2 > m_2^2 + (1-m_2)^2$ then $\tilde{s}^2(m1)$ first order stochastically dominates $\tilde{s}^2(m_2)$.

Indeed $P(\tilde{s}^2 \leq x) = 1 - 2P(\tilde{s} \leq -\sqrt{x})$. Now since $\tilde{s}$ has a symmetric distribution (when $m$ can take only 2 values, $m^*$ or $1-m^*$ with equal probability, which happens in the long run), we get:

$$\begin{aligned} P(\tilde{s} \leq -\sqrt{x}) =& \frac{1}{2}P(m^* s_I + (1-m^*)s_N \leq -\sqrt{x}) + \frac{1}{2}P(m^* s_N + (1-m^*)s_I \leq -\sqrt{x}) \\ =& P(m^* s_I + (1-m^*)s_N \leq -\sqrt{x}) \\ =& \Phi\left(\frac{-\sqrt{x}}{((m^*)^2 + (1-m^*)^2)(\sigma^2 + \sigma_\varepsilon^2)}\right) \end{aligned}$$

which is increasing in $(m^*)^2 + (1-m^*)^2$. This concludes the proof that $\tilde{s}^2(m_1) \succeq \tilde{s}^2(m_2)$.

Therefore as in Proposition 3, we get that in the long run, the expected returns are increasing in $(m^*)^2 + (1 - m^*)^2$ and hence in $|m^* - \frac{1}{2}|$. $\square$

# 10 Appendix B

In this section of the Appendix we would like to briefly discuss some alternative specifications (or interpretations) of the model, that can lead to results similar to these in the main body of the paper.

## 10.1 Two groups of Informed Traders

Here, we will discuss the case where instead of the Noise traders, the market is comprised of Hedge Funds, and two distinct groups of Informed traders, say A and B, who obtain signals on $d$ and are *cursed*, in the sense that they do not use the price to update their beliefs further. A traders interpret their signal correctly, but are dogmatic so that they do not take into account B's signal. B on the other hand, are dogmatic but also overconfident about their signal; that is they behave as if $\psi = 1$. Their signals are:

$$s_A = d + \varepsilon_A$$
$$s_B = \psi d + \sqrt{1-\psi^2}u + \varepsilon_B$$

where $\psi \in [0, 1]$, $\varepsilon_A, \varepsilon_B$ i.i.d variables $\sim N(0, \sigma_\varepsilon)$ and $u$ is distributed identically to $d$, but $B$ think that their signal is the same as the signal of informed traders. As before, hedge fund managers are uncertain about the ratio of $A$ to $B$ traders. When $\psi = 0$, then $B$ are Noise traders, and we get back to our original model (as $A$ do not even need to be dogmatic; they know they hold all relevant information). An alternative special case is when $\psi = 1$; in that case, $A$ and $B$ are completely symmetric, and can be interpreted as groups of analysts who obtain their own signal through their research and are dogmatic about their signals.

Under this specification, we still have the same price equation, as in Proposition 1. Moreover, the informativeness of the equilibrium quantities is decreasing on the size of the mixed news. This is because the main concept of this paper that extreme outcomes are more likely to occur when traders are more homogeneous still holds. However, Hedge Fund's uncertainty about fundamentals is not monotonically increasing as news get more extreme, contrary to Proposition 2. Indeed, when for instance there is a crash, hedge funds uncertainty about price informativeness is decreasing because they deduce that market consists (with high likelihood) of either all $A$ or all $B$ traders, and in both cases, the informativeness of their signal is the same. In contrast, when there are either all informed or all noise traders, informativeness takes two extreme values (0 and $\lambda$). Therefore all implications that are based on Proposition 2, are no longer true under this specification.

## 10.2 Only I and N in the market

Another specification we could think of, would be to have a market in which only $I$ and $N$ trade with each other. Then instead of hedge funds, we can have some managers (M) who (do not trade[20] but) use the price to infer information about the quality of their firms so that they can make better investment decisions, and we can then view the implications from the perspective of $M$. As before, the managers do not know the proportion $m$ of $I$ to $N$ traders.

In this model, market clearing is simply:

$$m\frac{\lambda s_I - p}{\alpha\sigma^2(1-\lambda)} + (1-m)\frac{\lambda s_N - p}{\alpha\sigma^2(1-\lambda)} = Z \tag{10.1}$$

and hence price would simply take the form

$$p = \lambda\underbrace{(ms_I + (1-m)s_N)}_{\tilde{s}} - \alpha\sigma^2(1-\lambda)Z$$

Then when the manager wants to get information about the firm using the price, he is faced with the same problem as that of the Hedge Funds of our baseline model. In particular, a larger price is associated with a larger $|\tilde{s}|$ and hence leads to a reduced price informativeness. That is an explanation of why extreme circumstances with very large (or very small) prices can be bad for real efficiency. In addition, this model has an extra advantage; since, prices are a strictly increasing function of $\tilde{s}$, there is no need for the assumption that agents condition on both the price and the residual demand to get an equilibrium. Starting from this very simple model, we can see how informativeness changes, when noise traders have sentiment shocks as in Mendel & Shleifer (2012), or when noise traders are just usual liquidity traders (with normal inelastic demand). Also, we could analyse whether the informativeness can be computed analytically, in the more general case where both the mass of I, $m_I$, and mass of N, $m_N$, are independent and

[20] Managers may not trade, for instance, due to regulatory restrictions.

unknown (in contrast with the original model where $m_I + m_N = 1$). To sum up, under this interpretation we could have a simpler and more tractable way to think of the effect of composition uncertainty on real economic decisions.