Agency, Firm Growth, and Managerial Turnover

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Abstract

We study managerial incentive provision under moral hazard in an environment where growth opportunities arrive stochastically over time and taking them requires a change of management. The firm faces a tradeoff between the benefit of always having a manager able to seize new opportunities and the cost of incentive provision. The optimal dynamic contract may grant partial job protection whereby the firm insulates its managers from the risk of growth-induced dismissal and foregoes attractive opportunities when they come after periods of good performance. Moreover, the prospect of growth-induced turnover limits the firm's ability to rely on deferred pay—resulting in more front-loaded compensation. The empirical evidence for the U.S. is broadly supportive of the model's predictions. Firms with better growth prospects experience higher CEO turnover and use more front-loaded compensation.

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Introduction

When ownership and control are separated, firm performance depends crucially on having the right managers at the helm and incentivising them properly. Over time, changes in business conditions may call for a change of top management to seize new opportunities or overcome challenges faced by the firm. This, however, may complicate the task of incentivising incumbent managers. For instance, if managers anticipate that their tenure at the firm will be short, they will be reluctant to accept any form of deferred compensation, a standard feature of incentive contracts. Thus the firm may face a dilemma: by changing management to adapt to evolving business conditions, it may increase the costs of incentive provision.

To analyze this tension, this paper introduces the idea of *growth-induced* turnover into a dynamic moral hazard framework. Growth-induced turnover refers to the replacement of top management that is motivated by the need to have managers who possess the appropriate skill set and experience to lead the firm in its *current* circumstances. This may involve for instance adopting new production techniques, making acquisitions, launching a new product or expanding into new markets. If the incumbent lacks the vision or skills necessary to implement such transformations, the appointment of new management is the only way for the firm to successfully pursue its course.[1] At the same time, proper dynamic incentive provision requires a combination of deferred compensation and a threat of dismissal following poor performance, both of which constitute agency costs. By introducing the possibility of managerial turnover for the sake of growth as well as for discipline, we show how these costs are affected. The main insight of the paper is that the prospect of growth-induced dismissal effectively increases managers' impatience, thus increasing agency costs and creating a general tendency to front-load compensation. In fact, the firm may actually be better off *ex ante* by committing to pass up otherwise attractive growth opportunities in some circumstances. More generally, our analysis delivers empirical predictions on the effects of a firm's growth prospects on managerial turnover and compensation which we show are broadly supported in the data.

Although our analysis is set up in a continuous-time stationary environment, we first develop the theory in the context of a two-period model. The simplicity of the framework enables us to distill most of the economics of the paper in the most transparent way. In particular, the tradeoff faced by the firm between the benefit of having a manager able to seize new opportunities and the cost of incentive provision appears very starkly in this setting. Moreover, the key empirical implications of the theory are derived analytically.

In the continuous-time model, a long-lived firm is run by a sequence of risk-neutral managers protected by limited liability. A moral hazard problem arises because while they are in charge, managers can divert cashflows for their own private benefit. The firm can fire

[1] In some circumstances, a change of management may be required to avoid decay, rather than to actually grow—e.g., when by sticking with the *status quo*, the firm would fail to face up to a disruptive competitive threat. For instance, in his narrative of the battle waged in Canada around 1820 between the long-established Hudson Bay Company (HB) and its upstart rival North West Company (NW), Roberts (2004) recounts: "HB did respond to the threat, essentially by copying NW's new approach. It did so, however, *only after the leaders of the firm had been replaced by new ones who understood the nature of the threat and were not tied to the old ways that had worked so well for so long.*" (our emphasis)

the incumbent manager at any time and replace him at a cost. Fleeting growth opportunities arrive stochastically over time, and a change of management is needed to seize them. If the firm decides to take up an opportunity, it pays the costs associated with replacing the manager, and its size (or profitability) increases. A long-term incentive contract is signed between the firm and its successive managers at the time they are hired.

As in previous dynamic contracting studies, we show that optimal compensation and turnover policies in this environment can be described in terms of a state variable that coincides with the agent's expected discounted compensation, referred to as his contractual 'promise'. The manager receives cash compensation only when his promise rises to reach an endogenous 'bonus threshold'. When the manager's promise lies below this threshold, cash compensation is deferred, and the promise is increased at a contractually specified rate plus a positive or negative adjustment based on the firm's current performance. If the firm suffers a sustained period of poor performance, the manager's promise can be lowered sufficiently to reach zero, the 'firing threshold', at which point the incumbent is replaced by a new manager who receives an initial promise which is no less than his exogenous reservation value.

In contrast with other studies, the manager's contract in our framework is also contingent on the presence or not of a growth opportunity. If no growth opportunity becomes available, the manager continues his tenure so long as his promise stays above the firing threshold, and he is compensated with bonuses and performance-related changes in his promise as just described. If a growth opportunity arises and the firm takes it, the manager is replaced. However, not all growth opportunities are seized by all firms—even though they would be under first best. Specifically, we show that, depending on the characteristics of the firm and its environment, the optimal growth policy can be one of two types. For some firms, it is optimal to take all growth opportunities as they come. For other firms, it is optimal to forego opportunities that arise after periods of good performance, i.e., when the incumbent manager's promise is above a certain 'growth threshold'. We refer to these two different types of firms as *high-growth* and *low-growth* firms, respectively. In effect, optimal incentive provision in low-growth firms calls for some degree of job protection against the risk of growth-induced termination. Intuitively, the reason why job protection is granted after a spell of good cashflows is that losses due to agency problems are diminished after good performance, thus increasing the value of continuing with the incumbent manager net of the foregone benefit of growth. In high-growth firms, the benefit of growth always dominates.

Under the optimal contract, managerial compensation is affected by the possibility of growth-induced turnover through the drift of the manager's promise during his tenure. In the absence of growth opportunities, this drift would simply be equal to the manager's discount rate. The key novelty in our setup is that, whenever the firm stands ready to take an opportunity that might become available, the drift rate needs to be augmented to compensate the manager for the risk of growth-induced termination, with the drift modification depending on the arrival intensity of growth opportunities. This upwards adjustment of the drift when the firm stands ready to take a growth opportunity explains why firms with better growth prospects tend to have more front-loaded compensation. It also sheds light on why low-growth firms grant job protection when past performance has been good but not if it has been bad. A higher drift is indeed less costly to the firm after poor performance, i.e., when the manager's promise is close to the firing threshold, as it reduces the likelihood of a subsequent inefficient, disciplinary turnover.

Our analysis explicitly allows for the possibility of lump-sum payments, and we show that severance pay is suboptimal in our setting even in the case of growth-induced turnover. Indeed, it is always better for the firm to increase the incumbent's future promise conditional on him being retained, thereby making inefficient termination less likely in the future, than to give cash to a departing manager. However, we establish that an incoming manager may be given a 'signing bonus' when his reservation value is sufficiently high.

To derive these results on the second-best incentive contract, our approach roughly follows the same logic as in previous continuous-time analyses of dynamic moral hazard. First, we establish a state-space representation of long-term incentive contracts, where the state process coincides with the manager's promise as described above. Similar to other studies, no stealing is incentive compatible under a dynamic contract if the sensitivity of the manager's promise to reported cashflows is large enough. We then formulate the firm's contracting problem recursively in order to characterize the optimal incentive-compatible dynamic contract in the presence of stochastic growth opportunities. We show that the firm's size-adjusted value function can be characterized as the solution to a Hamilton-Jacobi-Bellman (HJB) equation that incorporates the possibility of growth-induced turnover in an intuitive way. This crucial step in the analysis is established through a verification theorem from which follow the main properties of optimal compensation and turnover policies. Based on the HJB, we also provide a characterization of the determinants of a firm's growth type. In particular, we show that low-growth firms tend to be those plagued with more severe agency problems. This finding suggests that better governance can work as an effective tool to promote economic growth.

Having characterized the optimal contract, we take full advantage of the dynamic nature of our model and provide a suggestive analysis of its quantitative implications for the distribution of tenure length and for the timing of managerial compensation over tenure. In the model, these are are partly determined by the firm's type and the compensation and growth thresholds, all of which are endogenous. We discuss the impact of a firm's growth prospects on turnover and compensation under the optimal contract through a numerical example. The simulation outcome illustrates the fact that firms with better growth prospects, in particular those with more attractive opportunities (i.e., holding their arrival intensity fixed), tend to have shorter tenure length and more front-loaded compensation.

Finally, we examine the data in light of the theory. Merging data from CRSP, Compustat and ExecuComp for U.S. public companies over the period 1992-2014, we investigate empirically the links between firms' growth prospects, CEO turnover, and CEO compensation. Following an extensive literature in empirical corporate finance, we use average Q to capture firms' growth prospects. Namely, we proxy the *ex ante* growth prospects of a firm at the time a new CEO is appointed by the value of the firm's Q in the year before the CEO's appointment. We first sort CEO episodes along this proxy and compare the distributions of tenure length and compensation duration across the highest and lowest quantiles of growth prospects. In line with the model predictions, we find that the CEOs of firms with better prospects tend to have shorter tenure and more front-loaded compensation. We confirm these findings by regression analysis. In a probit model, our proxy for firms' growth prospects is positively related to the likelihood of turnover, controlling for past performance.

An increase in initial Q by one standard deviation leads to an increase in the probability of turnover by 85 basis points. Since the unconditional frequency of CEO turnover in our sample is 8.4%, the effect is economically significant. We also find that the arrival of an opportunity, proxied by an increase in the firm's average Q since the beginning of a CEO's tenure, increases the probability of a turnover event, consistent with the notion of growth-induced turnover. Furthermore, the likelihood of turnover is less sensitive to the arrival of an opportunity when *ex ante* growth prospects were poor, in line with the prediction that firms with more modest growth prospects are more likely to insulate their managers from the risk of growth-induced turnover. Finally, we find that managerial pay tends to be lower in firms with worse growth prospects, and that the slope of the compensation profile over tenure years tends to be higher in such firms, which can be viewed as a manifestation of their greater reliance on compensation back-loading.

The idea that the pursuit of valuable growth opportunities by a firm may rely on a change of management is found in early contributions to the management literature, going back to Penrose (1959). More recently, Roberts (2004) studies a number of business cases where managerial limitations to firm growth play a prominent role and where a change of management is instrumental in unlocking the growth potential of a firm.[2] Bertrand and Schoar (2003) provide compelling evidence that managers indeed matter for firm performance and that they differ in their management styles. Bennedsen et al. (2012) further report that CEO effects are particularly important in rapidly growing environments. Building on the idea that firm productivity is determined by the quality of the match between the skill set of the manager and the *current* circumstances of the firm, Eisfeldt and Kuhnen (2013) analyze a competitive assignment model of CEO turnover where the skills demanded by the firm are subject to random shocks. In a similar vein, Jenter and Lewellen (2014) extend the standard Bayesian learning model of CEO turnover (e.g., Harris and Holmström (1982)) by allowing the quality of the firm-CEO match to vary over time. In contrast with our work, these papers abstract from agency issues and incentive considerations which occupy centre stage in our analysis.

Our paper relates to a large body of work that applies the tools of dynamic contracting to the study of the firm in the presence of agency conflicts.[3] In particular, Quadrini (2004), Clementi and Hopenhayn (2006), DeMarzo and Fishman (2007a), He (2008), Biais et al. (2010, 2013), Philippon and Sannikov (2011), and DeMarzo et al. (2012) investigate the link between moral hazard and firm growth when the firm can grow with the incumbent. Our main theoretical contribution is to focus instead on growth-induced turnover and its

[2]See in particular his discussion of the British Petroleum and General Motors cases. Both firms achieved major increases in value by undertaking discrete changes in organizational structure implemented by new CEOs with a different vision (John Browne at BP and "Jack" Smith at GM) and only after a sustained period of poor performance. This can be viewed as evidence of the type of behaviour that characterizes low-growth firms in our analysis. Cheng and Hambrick (2012) document the fact that, in turnaround situations, companies substantially improve performance when they replace incumbent CEOs who are poorly suited to the conditions at hand with new ones who are well matched to those conditions.

[3]For seminal contributions to the literature on dynamic moral hazard, see Rogerson (1985) and Spear and Srivastava (1987) in discrete time, as well as Holmström and Milgrom (1987) and Sannikov (2008) in continuous time. Recent applications to the study of CEO turnover and compensation include among others, Spear and Wang (2005), Hoffman and Pfeil (2010), He (2012), Edmans et al. (2012), and Garret and Pavan (2012, 2015).

interactions with incentive provision. To the extent that the optimal contract in our setting is contingent on the realization of observable shocks, our work also bears some similarity with Piskorski and Tchistyi (2010) and Li (2015). More specifically, our continuous-time framework builds on the cash diversion model of DeMarzo and Sannikov (2006),[4] which we extend to incorporate the stochastic arrival of growth opportunities. From a technical point of view, our contributions are as follows. First, we introduce an additional source of uncertainty beyond the Brownian cashflow shocks. Hence, in order to derive the state-space representation of the contract and develop the proof of the verification theorem, our analysis borrows techniques from the credit risk literature. Second, we consider a stationary environment where the firm's continuation value at the time of firing a manager is fully endogenous. Third, we endogenize the initial promise that the firm offers to each manager. In particular, we derive a necessary and sufficient condition for the managers' participation constraint to be binding in high-growth firms. Fourth, we explicitly allow for jumps in the cumulative compensation process, which enables us to assess the optimality of severance pay. Finally, our extensive analysis of the HJB equation and associated free-boundary problems allows us to derive explicit existence and uniqueness results, as well as comparative statics that are new to our setting.

The implications of our model and the evidence we provide are connected to a vast empirical literature on the determinants of turnover and compensation for top management.[5] The literature on CEO turnover has mostly focused on the link between turnover and performance, as recently exemplified by Jenter and Lewellen (2014) and Jenter and Kanaan (2015).[6] We find that, controlling for performance, firms' growth prospects also contribute to explain the likelihood of CEO turnover. In terms of managerial compensation, the model predictions are in line with Murphy (1999) who points out that pay packages often include a bonus system based on the firm's reported earnings in excess of a performance target. They also echoe Kaplan and Minton (2012) who discuss the coincidence of shorter CEO tenures and higher CEO pay in the time series. The degree of reliance on deferred compensation has received relatively little attention in the literature so far. An exception is the analysis by Clementi and Cooley (2010) who exploit information on CEOs' holdings of stocks and stock options to construct a measure of deferred compensation. Gopalan et al. (2014) focus on the duration of a CEO's total compensation award in a given year based on information about the vesting periods of separate components in the package. Instead, we measure the duration of compensation received over the entire tenure of a CEO and we document that this measure varies negatively with the firm's growth prospects at the time the CEO is hired. More broadly, we add to existing empirical studies on CEO compensation by investigating how the profile of CEO pay over tenure relates to firms' growth prospects.

The rest of the paper proceeds as follows. Section 1 develops the theory in a simple two-period framework and analytically derives its empirical implications for managerial turnover and compensation. Section 2 describes the continuous-time modelling setup and derives

[4]See DeMarzo and Fishman (2007b) for a discrete-time version, and Biais et al. (2007) for an analysis of convergence from discrete to continuous time.

[5]For surveys of the literature on CEO compensation and on managerial incentive packages more generally, see for instance Murphy (1999, 2013).

[6]Early studies include, among others, Coughlan and Schmidt (1985), Warner et al. (1988), Weisbach (1988), Kim (1996), and Denis et al. (1997).

the state-space representation of long-term incentive contracts. Section 3 characterizes the optimal dynamic contract for high-growth and low-growth firms, as well as the determinants of firm type, and illustrates the model's implications through simulations. Section 4 presents the empirical evidence. Section 5 concludes.

1 A Two-Period Model

We consider a firm that hires a manager at time $t = 0$ to run its operations for at most two periods. The firm and its manager(s) are risk-neutral and have discount rates r and ϱ, respectively, with $\varrho > r$.[7] The random cashflows generated by the firm's operations at $t = 1$ and $t = 2$ are independently distributed. The first-period cashflow Y_1 is equal to either $y > 0$ with probability $p \in (0, 1)$, or zero with probability $1 - p$. At the end of the first period, a growth opportunity may arrive with probability $q \in (0, 1)$, independently of Y_1. The arrival of a growth opportunity is publicly observable. Crucially, we assume that, in order to take an available opportunity, the firm must dismiss the incumbent manager and appoint a new one. If a growth opportunity arises and the firm hires a new manager to take it, the second-period cashflow Y_2 is either $(1 + \gamma)y$ with probability p, where $\gamma > 0$, or zero with probability $1 - p$. If no growth opportunity arises or if the firm foregoes an available opportunity, the firm may either continue with the incumbent manager or dismiss him and hire a new one, in which cases the distribution of Y_2 is the same as that of Y_1.

The assumption that the firm cannot grow without a change of management occupies center stage in our analysis. This assumption captures circumstances where value creation requires specific managerial skills to carry out radical transformations of the firm and the incumbent does not have the ability to realize the firm's growth potential. We let $\kappa > 0$ denote the exogenous cost of managerial replacement at $t = 1$, which may include search fees as well as indirect costs such as disruption of on-going business, and we assume that the net present value of taking a growth opportunity is positive, i.e.,

$$\frac{p\gamma y}{1 + r} - \kappa > 0. \tag{1}$$

Our analysis focuses on a second-best environment in which cashflows are not observable by the firm. In case of high cashflow realization, the manager can under-report, steal the entire cashflow and get private benefit λ per unit of stealing, where $0 < \lambda \leq 1$ captures the severity of the moral hazard problem. We let $\hat{Y}_t \leq Y_t$ denote the level of reported cashflow in period t, where $\hat{Y}_t$ has the same support as the actual cashflow Y_t. Managers are protected by limited liability and have zero reservation value.[8] Thus, if the firm hires a new manager at $t = 1$, it offers him a one-period incentive contract with compensation $\lambda\hat{Y}_2$ at $t = 2$.[9]

[7] The assumption that the agent is more impatient than the principal is standard in the dynamic contracting literature and allows us to derive sharper predictions on managerial compensation. All the results on managerial turnover derived in the two-period framework go through in the case where $\varrho = r$.

[8] With zero reservation value, limited liability ensures that the manager's participation constraint is satisfied. We allow for a positive reservation value in the continuous-time model that we study in Sections 2 and 3.

[9] This is the optimal one-period contract that supports no stealing. We provide a proof of this standard result in Appendix A (see Lemma A-1).

At $t = 0$, the firm offers a *two-period* incentive contract to the initial manager. Such a contract specifies the firm's dismissal policy along with a compensation policy, and both parties fully commit to the terms of the contract. Specifically, we let $G(\hat{Y}_1) \in [0,1]$ denote the probability of taking a growth opportunity if one arises, thereby replacing management, conditional on first-period reported cashflow. We let $F(\hat{Y}_1) \in [0,1]$ denote the probability of the manager being dismissed at the end of the first period if no growth opportunity arises, conditional on first-period reported cashflow. Hence, G and F determine the occurrence of *growth-induced* and *disciplinary* dismissal, respectively. Furthermore, we let $C_1(\hat{Y}_1)$ and $C_2(\hat{Y}_1, \hat{Y}_2)$ denote the compensation received by the manager in the first and second period, respectively, contingent on reported cashflows, and we denote by $C_g(\hat{Y}_1)$ the amount of severance pay upon growth-induced turnover.[10] Limited liability requires

$$C_1(\hat{Y}_1) \geq 0, \quad C_g(\hat{Y}_1) \geq 0, \quad \text{and} \quad C_2(\hat{Y}_1, \hat{Y}_2) \geq 0. \tag{2}$$

After the adoption of a two-period contract at $t = 0$, the timing is as follows:

- At $t = 1$, the first-period cashflow realizes. The manager reports $\hat{Y}_1$ and receives $C_1(\hat{Y}_1)$. The uncertainty about the availability of a growth opportunity is resolved. The manager is dismissed or retained, as determined by $G(\hat{Y}_1)$ or $F(\hat{Y}_1)$ depending on whether a growth opportunity is available or not.[11] In case of growth-induced dismissal, the departing manager receives $C_g(\hat{Y}_1)$ upon leaving office.
- At $t = 2$, the second-period cashflow realizes and the initial or newly hired manager reports $\hat{Y}_2$. If the initial manager is still in office, he receives $C_2(\hat{Y}_1, \hat{Y}_2)$. Otherwise, the newly hired manager receives $\lambda \hat{Y}_2$.

1.1 The Optimal Two-Period Contract

We look for a two-period contract that maximizes the firm's expected discounted profit while inducing truthful reporting by the manager.[12] Under such a contract, reported cashflows $\hat{Y}_t$ coincide with actual cashflows Y_t, and we therefore dispense with the notational distinction.

In the second period, when the realized cashflow is high ($Y_2 = y$), the manager has the choice between either truthfully reporting good performance or reporting poor performance and steal the cashflow. The incentive compatibility (IC) condition requires that the manager should prefer to report truthfully. This will be the case provided that the difference in compensation upon good and bad reported performance, $C_2(Y_1, y) - C_2(Y_1, 0)$, is sufficiently large, namely,

$$C_2(Y_1, y) \geq \lambda y + C_2(Y_1, 0). \tag{3}$$

Likewise, in the first period, the manager needs to be incentivized to report truthfully when the realized cashflow is high ($Y_1 = y$). At this early stage, incentives are determined by the *total* expected discounted payoff that the manager receives upon reports of either good

[10] It is immediate to show that granting severance pay upon *disciplinary* dismissal is suboptimal.

[11] The contracting environment that we consider allows for randomization of the dismissal outcome. However, our analysis shows that the optimal contract does not involve randomization.

[12] See Appendix A.1 for the expression of the firm's expected discounted profit.

or bad performance. For a given report Y_1, his intertemporal payoff includes first-period compensation, $C_1(Y_1) + qG(Y_1)C_{\mathrm{g}}(Y_1)$, as well as expected second-period pay.[13] The latter depends on the expected pay received at $t = 2$ conditional on being retained, $pC_2(Y_1, y) + (1-p)C_2(Y_1, 0)$, and on the probability of being retained, $1 - [qG(Y_1) + (1-q)F(Y_1)]$. The first-period IC constraint thus requires that

$$C_1(y) + qG(y)C_{\mathrm{g}}(y) + \Big(1 - [qG(y) + (1-q)F(y)]\Big)\frac{pC_2(y,y) + (1-p)C_2(y,0)}{1+\varrho}$$
$$\geq \lambda y + C_1(0) + qG(0)C_{\mathrm{g}}(0) + \Big(1 - [qG(0) + (1-q)F(0)]\Big)\frac{pC_2(0,y) + (1-p)C_2(0,0)}{1+\varrho}, \quad (4)$$

i.e., the difference in the manager's intertemporal payoffs upon good and bad performance needs to be sufficiently large. Importantly, (4) captures the fact that first-period incentives are shaped both by the compensation scheme and by the firm's dismissal policy. In particular, proper incentive provision requires that good reported performance at $t = 1$ be associated with either higher contemporaneous levels of pay, or higher future levels of pay, or a lower likelihood of dismissal. Our analysis elucidates the tradeoffs among these three levers.

Our first lemma characterizes the optimal compensation scheme. The proof of this result, along with the proofs of all results derived in this section, can be found in Appendix A.

Lemma 1. *The compensation policy that maximizes the firm's expected discounted profit while respecting the limited liability constraint (2) and the IC constraints (3) and (4) is such that*

$$C_1(0) = C_{\mathrm{g}}(0) = C_2(0,0) = C_2(y,0) = 0, \quad (5)$$
$$C_2(0,y) = C_2(y,y) = \lambda y, \quad (6)$$
$$C_1(y) + qG(y)C_{\mathrm{g}}(y) = \lambda y - \Big([qG(0) + (1-q)F(0)] - [qG(y) + (1-q)F(y)]\Big)\frac{p\lambda y}{1+\varrho}. \quad (7)$$

Equation (5) shows that, under the optimal contract, the manager receives zero compensation in any given period upon report of poor performance in that period. Limited liability precludes a tougher penalty, i.e., negative compensation. Equation (6) establishes that second-period compensation conditional on good reported cashflow at $t = 2$ is equal to the agency rent λy, independently of first-period performance.[14] Indeed, it is optimal to set $C_2(0, y)$ to the minimum level that satisfies the second-period IC constraint (3) after poor performance, so as to relax the first-period IC constraint (4). On the other hand, the second-period IC constraint after good performance is also binding because, when $\varrho > r$, deferring compensation is costly for the firm. Hence, the optimal two-period contract involves the minimum amount of deferred compensation that is compatible with proper incentive provision.

Equation (7), which directly follows from the binding first-period IC constraint (4), determines the level of first-period compensation upon good performance, $C_1(y)+qG(y)C_{\mathrm{g}}(y)$,

[13] For simplicity, our analysis assumes that the continuation value of a dismissed manager is zero.

[14] Note that (5) and (6) imply that $C_2(Y_1, Y_2) = \lambda Y_2$, i.e., the compensation scheme in the second period does not depend on whether the firm is run by the initial manager or by a new one.

and establishes a crucial link between the compensation and dismissal policies. While the first term on the right-hand side of the equation, λy, is the rent that the manager would get to report performance truthfully at $t = 1$ under a one-period contract, the second term is a distinct feature of the two-period contract. To interpret this term, recall that, if continued into the second period, the manager receives an agency rent λy at $t = 2$ conditional on good second-period performance, independently of his report in the first period. That is, a manager who is retained at $t = 1$ contemplates an expected discounted payoff equal to $p\lambda y/(1 + \varrho)$, whether reported performance in the first period was good or not. Hence, a key determinant of first-period incentives is the *wedge* between dismissal probabilities after poor performance, $qG(0) + (1 - q)F(0)$, and after good performance, $qG(y) + (1 - q)F(y)$. As revealed by (7), a larger wedge serves to incentivize the manager not to steal and reduces the need to use first-period compensation upon good performance to do so—especially when the expected discounted value of second-period compensation is large.

It is worthwhile to note that, in this simple setup, the amount of severance pay upon growth-induced turnover following good performance, $C_{\mathrm{g}}(y)$, is not pinned down under the optimal contract. Indeed, whenever $G(y) > 0$, the firm is indifferent between providing incentives through regular compensation $C_1(y)$ or severance pay upon growth-induced dismissal $C_{\mathrm{g}}(y)$ properly adjusted for the effective probability $qG(y)$ of growth-induced turnover conditional on good first-period performance.[15]

Our next result characterizes the optimal turnover and growth policies. These are captured by $F(Y_1)$ and $G(Y_1)$, which determine the conditional probabilities of disciplinary and growth-induced dismissal, respectively.

Lemma 2. *The optimal contract is such that $F(y) = 0$. Furthermore, the following statements hold true:*

i. Setting $F(0) = 1$ is optimal if and only if

$$\kappa \leq \frac{p}{1-p}\frac{p\lambda y}{1+\varrho} =: \hat{\kappa}_{F(0)}. \tag{8}$$

ii. Setting $G(0) = 1$ is optimal if and only if

$$\kappa \leq \frac{p(1-\lambda)\gamma y}{1+r} + \frac{p}{1-p}\frac{p\lambda y}{1+\varrho} =: \hat{\kappa}_{G(0)}. \tag{9}$$

iii. Setting $G(y) = 1$ is optimal if and only if

$$\kappa \leq \frac{p(1-\lambda)\gamma y}{1+r} - \frac{p\lambda y}{1+\varrho} =: \hat{\kappa}_{G(y)}. \tag{10}$$

If any of the inequalities in (8), (9) or (10) is violated, it is optimal to set the corresponding dismissal probability equal to zero.

[15] In Section 3, we show that positive severance pay is strictly suboptimal in the continuous-time version of our model. The reason why such a 'no-severance' result fails to hold in the simpler setup of this section is that the agency relationship extends for at most two periods, so that the firm's continuation value at the end of the first period is linear in the manager's continuation payoff (see also the discussion following Property 3).

The first part of Lemma 2 characterizes the firm's optimal dismissal policy in the absence of a growth opportunity, $F(Y_1)$. When no growth opportunity is available, resorting to dismissal after good performance (i.e., $F(y) > 0$) is clearly suboptimal. Indeed, this would bring no benefit and would be costly for two reasons: because of the replacement cost $\kappa > 0$ and through a deterioration of incentives in the first period, which would have to be compensated by an increase in first-period compensation upon good performance—as per (7). By contrast, the optimal choice of $F(0)$ involves a tradeoff between incurring the replacement cost κ and improving first-period incentives by increasing the wedge in dismissal probabilities. Equation (8) shows that disciplinary dismissal after poor performance is optimal when κ is low enough or when the impact of the wedge on the firm's net profit is sufficiently large (i.e., high λ, high p or low ϱ).[16]

Statements (ii) and (iii) in Lemma 2 characterize the firm's optimal growth policy, $G(Y_1)$. The direct gain from taking an available growth opportunity is to improve the distribution of the second-period cashflow. The benefit for the firm, captured by the term $p(1-\lambda)\gamma y/(1+r)$ in (9) and (10), is increasing in p and γ, and decreasing in λ and r. Furthermore, the firm's growth policy also affects profit via its impact on the dismissal wedge and first-period incentives. On the one hand, systematically taking growth opportunities after *poor* performance (i.e., $G(0) = 1$) facilitates incentive provision. Thus, as revealed by (9), only very high values of the replacement cost κ would make it optimal for the firm to forego an available growth opportunity after poor performance.[17] On the other hand, replacing management to take a growth opportunity after good performance is detrimental to incentive provision and is therefore less attractive, namely, $\hat{\kappa}_{G(y)} < \hat{\kappa}_{G(0)}$.[18]

In the remainder of Section 1, we assume that

$$\frac{p}{1-p} \geq \frac{1+\varrho}{1+r}\gamma. \tag{11}$$

Combined with (1), this restriction implies that $\kappa < \hat{\kappa}_{G(0)}$ and therefore ensures that the firm systematically takes growth opportunities after poor performance.[19] However, it may or may not be optimal for the firm to also take an available growth opportunity after *good* performance, as our next result shows. In what follows, we distinguish between two types of configurations. Namely, we refer to the case where $G(y) = 1$ as the *high-growth* regime, and we refer to the case where $G(y) = 0$ as the *low-growth* regime. In the former configuration, the firm undertakes any growth opportunity that arises, while in the latter, it undertakes an available growth opportunity only upon poor performance.

Lemma 3. *The high-growth and low-growth regimes can both arise.*

[16] Furthermore, a higher probability of first-period success makes it less likely that the replacement cost κ will have to be paid while increasing the expected gain that the firm obtains from a reduction in $C_1(y) + qG(y)C_{\mathrm{g}}(y)$, which explains why the ratio $p/(1-p)$ appears in the expression for the cutoff value $\hat{\kappa}_{F(0)}$.

[17] Note that $\hat{\kappa}_{G(0)} > \hat{\kappa}_{F(0)}$, so that if the firm stands ready to fire the manager after poor performance absent a growth opportunity, then *a fortiori* it will fire him after poor performance for the sake of growth.

[18] If growing the firm involved a specific cost $\chi > 0$, then $\hat{\kappa}_{G(y)}$ and $\hat{\kappa}_{G(0)}$ would both be translated to the left by χ, making it less likely that growth opportunities are undertaken, but none of our results would be affected.

[19] This assumption, which effectively rules out the possibility that the firm never grows (see Remark A-4 in Appendix A), only serves to simplify the exposition and shorten some of the proofs. In particular, all the empirical implications stated in Propositions 1 and 2 remain true if Condition (11) is relaxed.

The main insight delivered by Lemma 3 is that some firms—namely, low-growth firms—may find it preferable to forego a growth opportunity following good performance even though they would undertake any such opportunity under first best, as implied by (1). Indeed, under second best, it can be optimal for a firm to commit *ex ante* to forego growth opportunities after good performance in order to save on the cost of incentive provision.

The following lemma characterizes the determinants of a firm's growth regime, thus shedding light on the circumstances in which it is optimal for the firm to grant partial job protection to the initial manager.

Lemma 4. *An increase in γ, y, ϱ or p, or a drop in λ, r or κ can induce a switch from a low-growth to a high-growth regime. A change in q has no impact on the optimal growth policy.*

In particular, Lemma 4 establishes that firms with better opportunities (i.e., high γ) tend to fully expose their managers to the risk of growth-induced turnover. Firms with smaller discount rates (i.e., low r) tend to do the same as they give more weight to the future benefit from growth relative to the compensating increase in first-period compensation. By contrast, firms facing larger turnover costs (i.e., high κ) have a natural tendency to grant partial job protection to their managers, thus foregoing growth opportunities after good performance. Low-growth firms also tend to be plagued by severe agency issues (i.e., high λ). Indeed, as reflected in the expression for the threshold $\hat{\kappa}_{G(y)}$ in (10), a high dismissal probability $G(y)$ is less appealing when moral hazard is more severe, both because the fraction of enhanced second-period cashflows accruing to the firm $(1-\lambda)$ is small and because the second-period agency rent is large—implying that any increase in the risk of growth-induced dismissal needs to be matched by a larger increase in $C_1(y) + qG(y)C_{\mathrm{g}}(y)$ to keep the agent incentivized.

1.2 Empirical Implications

In this section, we derive some of the empirical implications that arise in this simple two-period framework combining growth-induced turnover and moral hazard. The following two propositions summarize the theoretical predictions for managerial turnover and managerial compensation, respectively.

Proposition 1. *The following statements hold true:*
i. The likelihood of turnover is decreasing in performance, namely,

$$qG(y) + (1-q)F(y) \leq qG(0) + (1-q)F(0).$$

ii. The likelihood of turnover, $qG(Y_1) + (1-q)F(Y_1)$, is increasing in the quality of growth opportunities, γ, and in their arrival probability, q.

iii. The probability of turnover is higher when a growth opportunity arises, namely,

$$G(Y_1) \geq F(Y_1).$$

Moreover, the impact of the arrival of a growth opportunity on the probability of turnover is stronger in firms with better opportunities, namely, $G(Y_1) - F(Y_1)$ is increasing in γ.

The results derived in Proposition 1 follow almost immediately from Lemmas 2 and 4. Statement (i) establishes a negative relationship between firm performance and turnover. This is a standard prediction of dynamic moral hazard models.[20] In particular, the result would equally hold true under second best in the absence of growth-induced turnover (i.e., in the limit as q goes to zero), when dismissal upon poor performance is used purely as an incentive device. The prediction carries over to our setup because growth-induced dismissal is also less likely to occur after good performance, i.e., $G(y) \leq G(0)$. Statements (ii) and (iii) characterize the impact of *ex ante* growth prospects and the effect of a growth opportunity realization on the likelihood of turnover, respectively. These predictions are primarily driven by the possibility of growth-induced dismissal introduced in our setup.[21]

The second set of empirical implications emphasizes some important features of the compensation scheme under the optimal two-period contract. In particular, when thinking about taking the model predictions to the data, it is useful to consider the average compensation profile $(\bar{C}_1, \bar{C}_2)$, where $\bar{C}_t$ denotes the expected level of compensation that the initial manager receives at time t conditional on running operations in period t, namely,

$$\bar{C}_1 = p[C_1(y) + qG(y)C_{\mathrm{g}}(y)] \quad \text{and} \quad \bar{C}_2 = p\lambda y.$$

Proposition 2. *The following statements hold true:*

i. Compensation is increasing in performance, namely,

$$C_1(y) + qG(y)C_{\mathrm{g}}(y) > C_1(0) + qG(0)C_{\mathrm{g}}(0) \quad \textit{and} \quad C_2(Y_1, y) > C_2(Y_1, 0).$$

ii. The average compensation profile is increasing over tenure, i.e., $\bar{C}_1 \leq \bar{C}_2$.

iii. The average first-period compensation $\bar{C}_1$ *is increasing in the quality of growth opportunities,* γ. *Hence, the slope of the compensation profile,* $\bar{C}_2 - \bar{C}_1$, *is decreasing in* γ.

The first statement in Proposition 2 establishes a positive relationship between firm performance and managerial compensation, which immediately follows from Lemma 1.[22] The other two results in the proposition characterize the shape of the average compensation profile $(\bar{C}_1, \bar{C}_2)$. While statement (ii) shows that the compensation profile is back-loaded, statement (iii) emphasizes the fact that the extent of compensation back-loading depends on the firm's *ex ante* growth prospects.[23] Specifically, the model predicts that firms with better growth prospects tend to have more front-loaded pay. Indeed, as shown in Lemma 4, an improvement in the quality of growth opportunities makes it more likely that the firm finds it optimal to fully expose the initial manager to the risk of growth-induced turnover, setting $G(y) = 1$. In turn, the associated drop in the dismissal wedge needs to be compensated by an increase in $C_1(y) + qG(y)C_{\mathrm{g}}(y)$ to satisfy the first-period IC constraint (see

[20] Under first best, turnover is independent of performance, namely, $F(Y_1) = 0$ and $G(Y_1) = 1$.

[21] It is worthwhile to note, however, that under the assumption that growth is efficient (i.e., Condition (1)), the comparative statics with respect to γ in statements (ii) and (iii) would not hold true under first best.

[22] Note that compensation increases with *recent* performance, not with the entire history of performance. Indeed, $C_2(Y_1, Y_2)$ is independent of Y_1, i.e., second-period pay does not depend on first-period performance.

[23] The inequality in statement (ii), like the one in Proposition 1.(i), is strict unless $F(0) = 0$ and $G(y) = 1$. Necessary and sufficient conditions for this case to arise are provided in the proof of Proposition 2.

Equation (7)), which translates into an increase in the average first-period compensation level $\bar{C}_1$. It is worthwhile to note that, whereas statements (i) and (ii) are driven by moral hazard and would also hold true under second best in the absence of growth-induced dismissal, the last prediction is specific to our setup and is driven by the interaction between moral hazard and the possibility of growth-induced turnover.

2 The Continuous-Time Model

Having previewed the basic economics of growth-induced managerial turnover and its interaction with moral hazard, we now turn to the main focus of our analysis and consider the continuous-time stationary version of the environment introduced in Section 1. In particular, this modelling setup is better adapted to fully capturing the dynamic nature of the agency relationship between a firm and any of its successive managers.

We consider a firm run by a sequence of managers protected by limited liability. The firm and its managers are risk-neutral, with discount rates r and ϱ, respectively. The firm's operations generate a stream of instantaneous cashflows $\Phi_t \, dY_t$, where Φ_t denotes the size of the firm at time t, and the cumulative size-adjusted cashflow process $Y = \{Y_t\}$ follows

$$dY_t = \mu \, dt + \sigma \, dZ_t, \qquad \mu, \sigma > 0,$$

where $Z = \{Z_t\}$ denotes a standard one-dimensional Brownian motion. The firm starts with unit size ($\Phi_0 = 1$) and can later expand. At any point in time, two conditions must be met for the firm to expand: (i) it must have a growth opportunity, and (ii) it must hire a new manager to take up the opportunity. Growth opportunities arrive sequentially, independently of cashflow shocks, and the waiting time for the arrival of the next opportunity is exponentially distributed with parameter q. If not taken immediately, an opportunity is lost and no further growth is possible until a new one arrives.

As in the context of the two-period framework studied in the previous section, the assumption that value creation entails a change of management is central to our analysis. For convenience, we model value enhancement as a discrete change in firm size that scales up the distribution of cashflows. Namely, we assume that when it expands, the size of the firm increases by a factor $1+\gamma > 1$. Firm growth, when it occurs, is the result of bringing in a new manager able to take advantage of newly available opportunities—thereby achieving a permanent increase in expected cashflows.[24]

The second main feature of the model is a standard agency problem arising from the fact that, while running the firm's operations, managers can divert cashflows. The residual cashflow received by the firm is $\Phi_t \, (dY_t - dA_t)$, where $A = \{A_t\}$ denotes the cumulative size-adjusted amount of 'stealing'.[25] Managers enjoy a private benefit $\lambda \in (0, 1]$ for each unit of diverted cashflow, so that λ measures the severity of moral hazard.

The firm has deep pockets and can cover negative cashflows, as well as the costs associated with managerial compensation and turnover. Thus the firm's decisions are not

[24] This may or may not involve an increase in the fixed assets of the firm. If it does, future scaled cashflows should be thought of as net of the financing cost of capital investments.

[25] The stealing strategy A chosen by the manager is adapted to the Brownian filtration and has continuous sample paths. Given that Y is continuous, any jump in A would be immediately detected by the firm.

driven by financing constraints. A manager hired to run the firm at size Φ_t has reservation value $\bar{w}\Phi_t$, and the cost of replacing him is $\kappa\Phi_t$, where $\bar{w}, \kappa > 0$ are given constants. While it is natural to assume that the manager's reservation value (which can be interpreted as a non-pecuniary cost of running the firm) and the cost of managerial replacement (which may include disruption costs) are increasing in firm size, the stronger assumption of proportionality is made to ensure size homogeneity and preserve tractability.[26] The continuation value of a departing manager is equal to zero.[27]

We further assume that

$$\varrho > r, \tag{12}$$

$$r > q\gamma, \tag{13}$$

$$\frac{\gamma\mu}{r} > \kappa + (1+\gamma)\bar{w}, \tag{14}$$

and we refer to parameter values that satisfy these conditions, along with the ones previously imposed in this section, as *permissible*. Condition (12) requires that managers are more impatient than the firm.[28] Condition (13) imposes that the average growth rate when the firm takes all growth opportunities is smaller than the firm's discount rate, which ensures finite valuation. Finally, together with (13), Condition (14) implies that in the absence of moral hazard, it would be optimal for the firm to take all growth opportunities—as we next establish.

2.1 First-Best Policy

The first-best policy can be characterized as follows. First, the optimal compensation policy involves giving to a manager a size-adjusted transfer $\bar{w}$ *at the outset* of his tenure. Indeed, since managers are more impatient than the firm, deferring compensation would affect firm value negatively. Second, in order to save on replacement and hiring costs, managerial turnover never occurs if not for the sake of taking a growth opportunity. Third, the optimal growth policy involves either taking all growth opportunities or never taking any. If the firm takes all opportunities, its expected discounted profit V^* satisfies

$$V^* = -\bar{w} + \mathbb{E}\left[\int_0^\tau e^{-rt} dY_t + e^{-r\tau}\left[(1+\gamma)V^* - \kappa\right]\right],$$

[26]Empirically, executive pay is positively correlated with firm size both over time and across firms, as documented by Kostiuk (1990), Murphy (1999), and Gabaix and Landier (2008). On the other hand, estimates of the various costs associated with CEO transitions for mid-cap companies are roughly twice as large as those borne by small-cap companies, and less than half the costs borne by large-cap companies (Chief Executive Magazine, Nov/Dec 2008). Biais et al. (2010) and DeMarzo et al. (2012) make proportionality assumptions similar to ours.

[27]Allowing for a non-zero continuation value would alter the details of our analysis to the extent that this would affect the dynamics of the manager's promise (20) as well as the HJB equation (28).

[28]This assumption is standard in the dynamic contracting literature (e.g., DeMarzo and Sannikov (2006), Biais et al. (2007, 2010), and DeMarzo et al. (2012)). The wedge in discount rates rules out indefinitely postponing payments to managers.

where τ is the random arrival time of the first growth opportunity. Solving for V^* under the assumption that τ is exponentially distributed with parameter q yields

$$V^* = \frac{\mu - q\kappa}{r - q\gamma} - \frac{r + q}{r - q\gamma}\bar{w}.$$

If instead the firm foregoes all opportunities, its expected discounted profit is given by

$$-\bar{w} + \mathbb{E}\left[\int_0^\infty e^{-rt} dY_t\right] = -\bar{w} + \frac{\mu}{r}.$$

It is straightforward to see that Conditions (13) and (14) are sufficient for the inequality

$$V^* > \max\left\{-\bar{w} + \frac{\mu}{r},\ 0\right\}$$

to hold true. Therefore, as in the two-period framework under Condition (1), our assumptions ensure that it would be optimal for the firm to take all growth opportunities under first best.

2.2 Long-Term Incentive Contract

We now turn to the case where managers can divert cashflows and stealing is not observable by the firm. The firm enters into a long-term contract with each manager at the time of his hiring, and both parties fully commit to the terms of the contract. A contract specifies circumstances upon which the manager will be dismissed, including those when the firm will take a growth opportunity, as well as the manager's pay over the course of his tenure based on the information that will become available to the firm over time. In particular, the arrival of a growth opportunity is assumed to be perfectly observable and contractible. To fix ideas and simplify the exposition, we initially restrict our attention to the contract with the first manager. Readers interested in the technical aspects of the sequential contracting environment are referred to Appendix B.

First, we discuss how dismissal and compensation are determined for a given stealing strategy A chosen by the manager. The information accruing to the firm over time comes from observing the cumulative reported cashflows $\hat{Y} = Y - A$, as well as the arrival of growth opportunities. We denote by $\mathcal{F}_t$ the information gathered by the firm up to time t, which includes information about the occurrence of growth opportunities. We denote by $\hat{\mathcal{F}}_t \subseteq \mathcal{F}_t$ the information coming only from the history of reported cashflows up to time t.

Dismissal of the manager can occur for two distinct reasons in our setting. First, the manager can be sacked after a history of poor reported cashflows. Indeed, committing *ex ante* to fire the incumbent after poor reported performance can be used by the firm as a device to incentivize him not to steal. Second, the manager can be replaced in order to take a growth opportunity that becomes available. Hence, turnover is partly governed by the firm's *growth policy*, which determines the firm's response to the potential arrival of a growth opportunity. This policy is modeled by an $(\hat{\mathcal{F}}_t)$-progressively measurable process $G = \{G_t\}$ taking values in $\{0, 1\}$, with $G_t = 1$ indicating that the firm stands ready to take a growth opportunity at time t, and $G_t = 0$ indicating that it does not.[29] By controlling G, the

[29]Note that G_t is set by the firm without knowledge of whether an opportunity arises or not at time t. In Appendix D, we show that randomization of the growth decision, i.e., $G_t \in (0, 1)$, is suboptimal.

firm effectively determines the instantaneous intensity of growth-induced dismissal, which is equal to qG_t at time t. In view of these observations, the random time τ at which the manager is fired can thus be represented as[30]

$$\tau = \tau_{\mathrm{d}} \wedge \tau_{\mathrm{g}},$$

where τ_{d} denotes an $(\hat{\mathcal{F}}_t)$-stopping time and the random time τ_{g} satisfies[31]

$$\mathbb{P}\Big(\tau_{\mathrm{g}} > t \,\big|\, \hat{\mathcal{F}}_t\Big) = \exp\left(-\int_0^t qG_s\,ds\right). \tag{15}$$

In the event that $\tau = \tau_{\mathrm{d}}$, the manager is replaced for the sake of incentive provision, which we refer to as *disciplinary turnover*. When instead $\tau = \tau_{\mathrm{g}}$, the manager is dismissed for the sake of growth, which we refer to as *growth-induced turnover*.

Compensation to the manager over the course of his tenure is captured by an $(\hat{\mathcal{F}}_t)$-adapted cumulative compensation process $C = \{C_t\}$. Limited liability implies that C is increasing. A positive jump ΔC_t represents a lump-sum payment at time t.[32] In particular, ΔC_0 and $\Delta C_{\tau_{\mathrm{d}}}$ denote a signing bonus and severance pay upon disciplinary dismissal, respectively. To capture severance pay upon growth-induced turnover, we introduce a separate $(\hat{\mathcal{F}}_t)$-progressively measurable process $S = \{S_t\}$. The amount of severance received by a manager dismissed for the sake of growth is given by $S_{\tau_{\mathrm{g}}}$.

Now considering the set $\mathcal{A}$ of all possible stealing strategies, a contract can thus be viewed as a function mapping each stealing strategy $A \in \mathcal{A}$ to a collection

$$C = C(A), \quad S = S(A), \quad G = G(A), \quad \text{and} \quad \tau_{\mathrm{d}} = \tau_{\mathrm{d}}(A),$$

as just described. Such mapping should be consistent across stealing strategies in the sense that any given history of reported cashflows should result in the same compensation and termination outcomes, independently of the underlying combination of true cashflows and stealing that gave rise to that observed history. The contract space $\mathcal{G}$ identifies with the set of all functionals $\Gamma : A \mapsto \big(C(A), S(A), G(A), \tau_{\mathrm{d}}(A)\big)$ on $\mathcal{A}$ that satisfy this requirement.

2.3 The Firm's Problem

Given a contract Γ and a stealing strategy A, the manager's expected discounted payoff at the time of his hiring is given by

$$M(\Gamma, A) = \mathbb{E}\left[\int_{[0,\tau[} e^{-\varrho t}\left(dC_t + \lambda\,dA_t\right) + e^{-\varrho\tau}\Big(\Delta C_{\tau_{\mathrm{d}}}\mathbf{1}_{\{\tau=\tau_{\mathrm{d}}\}} + S_{\tau_{\mathrm{g}}}\mathbf{1}_{\{\tau=\tau_{\mathrm{g}}\}}\Big)\right].$$

[30]We use the notation $x \wedge y$ (resp., $x \vee y$) to denote the minimum (resp., maximum) of x and y.

[31]The left-hand side of (15) denotes the probability that the manager has not been dismissed for the sake of growth by time t, conditional on the history of reported cashflows up to time t. The right-hand side captures the fact that the instantaneous intensity of growth-induced dismissal at time $s \le t$ is qG_s. When the firm stands ready to take all growth opportunities, setting $G \equiv 1$, the probability that the manager survives the threat of growth-induced termination up to time t is given by $\exp(-qt)$, reflecting the fact that the arrival of opportunities is exponentially distributed with parameter q.

[32]We assume that C is right continuous with left limits and $C_{0-} = 0$, therefore $\Delta C_t = C_t - C_{t-}$ and $\Delta C_0 = C_0$. Further technical details on the modelling of long-term incentive contracts are given in Appendix B.1.

For a given contract Γ, a stealing strategy A is said to be *incentive compatible* if it maximizes the manager's payoff. We refer to a contract as *admissible* if it is such that (i) no stealing is incentive compatible, and (ii) the manager's expected discounted payoff under no stealing is greater than or equal to his reservation value $\bar{w}$. Formally, the subset $\mathcal{G}_\mathrm{a}$ of admissible contracts includes all contracts $\Gamma \in \mathcal{G}$ such that

$$M(\Gamma, 0) = \sup_{A \in \mathcal{A}} M(\Gamma, A) \qquad \text{and} \qquad M(\Gamma, 0) \geq \bar{w}.$$

Given an admissible contract Γ, the firm's expected discounted profit at $t = 0$ is

$$F(\Gamma) = \mathbb{E}\Bigg[\int_{[0,\tau[} e^{-rt}\,(\mu\,dt - dC_t) \\ + e^{-r\tau}\Big([V_\mathrm{d} - \Delta C_{\tau_\mathrm{d}} - \kappa]\mathbf{1}_{\{\tau=\tau_\mathrm{d}\}} + [V_\mathrm{g} - S_{\tau_\mathrm{g}} - \kappa]\mathbf{1}_{\{\tau=\tau_\mathrm{g}\}}\Big)\Bigg], \tag{16}$$

where V_d and V_g denote the firm's continuation values after dismissal of the first manager (for disciplinary reasons or upon growth, respectively), which we endogenize later in Section 3.[33] The firm's problem is to find an admissible contract that maximises its expected discounted profit. Formally, the firm's objective is to find Γ^* such that

$$F(\Gamma^*) = \sup_{\Gamma \in \mathcal{G}_\mathrm{a}} F(\Gamma).$$

2.4 Admissible Dynamic Contracts

As observed in previous work on dynamic moral hazard, the challenge of analyzing this type of environment comes from the complexity of the contract space and from the difficulty of evaluating the agents' incentives in a tractable way. In this section, we build on the approach of DeMarzo and Sannikov (2006), Sannikov (2008), and Biais et al. (2007, 2010), and consider a state-space representation of incentive contracts. Under no stealing, the state variable in this representation should coincide with the manager's expected payoff. As a preliminary step, we therefore characterize the process followed by

$$M_t = \mathbb{E}\Bigg[\int_{]t,\tau[} e^{-\varrho(s-t)} dC_s + e^{-\varrho(\tau-t)}\Big(\Delta C_{\tau_\mathrm{d}}\mathbf{1}_{\{\tau=\tau_\mathrm{d}\}} + S_{\tau_\mathrm{g}}\mathbf{1}_{\{\tau=\tau_\mathrm{g}\}}\Big)\,\Big|\,\mathcal{F}_t\Bigg], \quad t < \tau,$$

which corresponds to the manager's expected future payoff at time $t < \tau$ when he refrains from stealing.

Lemma 5. *For any given any contract $\Gamma \in \mathcal{G}$, there exists a process $\beta = \{\beta_t\}$ such that*

$$dM_t = \big[\varrho M_t + qG_t(M_t - S_t)\big]dt - dC_t + \sigma\beta_t\,dZ_t, \qquad \textit{for } t < \tau. \tag{17}$$

[33]In Appendix B.2, we provide an expression for the firm's value at $t = 0$ for a given sequence of admissible contracts. In particular, when the same admissible contract Γ is offered to all managers, we show that the firm's size-adjusted expected discounted profit $F(\Gamma)$ satisfies (16) with $V_\mathrm{d} = F(\Gamma)$ and $V_\mathrm{g} = (1+\gamma)F(\Gamma)$. We restrict our attention to contracts that implement no stealing, which is standard in the literature when moral hazard is modelled as a cash diversion problem (e.g., see DeMarzo and Sannikov (2006)).

Proof. See Appendix C.1.[34]

The presence of the diffusion term in the dynamics of the agent's expected payoff is very natural. Since compensation and dismissal policies are contingent on the history of reported cashflows, the evolution of the manager's expected future payoff under a long-term incentive contract is sensitive to currently reported cashflows. The process β can precisely be interpreted as the sensitivity induced by a long-term contract. Since reported cashflows coincide with true cashflows when the manager refrains from stealing, the stochastic evolution of the manager's expected payoff under no stealing M_t is directly driven by the true cashflow shocks dZ_t.

In light of Lemma 5, we consider *dynamic contracts* whose implementation is driven by a state process $W = \{W_t\}$ that evolves as

$$dW_t = \big[\varrho W_t + qG_t(W_t - S_t)\big]\, dt - dC_t + \beta_t\, (d\hat{Y}_t - \mu\, dt). \tag{18}$$

Along with compensation and growth policies, a dynamic contract specifies the sensitivity β of the state variable W to the reported cashflows. Importantly, since the dynamics of the state variable are driven by processes that are either observed or controlled by the firm, its evolution over time can be tracked by the firm. While growth-induced turnover is jointly determined by the growth policy and the random arrival of opportunities, disciplinary dismissal occurs when the state process W hits zero, namely,

$$\tau_{\rm d} = \inf\{t \geq 0 : \; W_t = 0\}. \tag{19}$$

Noting that $d\hat{Y}_t - \mu\, dt = -dA_t + \sigma\, dZ_t$, it is straightforward to see that, when the manager refrains from stealing, the dynamics of the state process become

$$dW_t = \big[\varrho W_t + qG_t(W_t - S_t)\big]\, dt - dC_t + \sigma\beta_t\, dZ_t \tag{20}$$

and therefore mirror (17). Indeed, when the manager refrains from stealing, the value taken by the state variable at any time during his tenure does coincide with his expected future compensation under the contract, as stated in the following lemma.

Lemma 6. *Consider a dynamic contract with termination occurring at time $\tau = \tau_{\rm g} \wedge \tau_{\rm d}$, where $\tau_{\rm g}$ satisfies (15) and $\tau_{\rm d}$ is defined by (19) where W follows (18) for some initial condition $W_{0-} = w_{\rm init} > 0$. Then the manager's expected future payoff at time $t < \tau$ if he refrains from stealing is equal to W_t, namely,*

$$W_t = \mathbb{E}\left[\int_{]t,\tau[} e^{-\varrho(s-t)} dC_s + e^{-\varrho(\tau-t)}\Big(\Delta C_{\tau_{\rm d}} \mathbf{1}_{\{\tau=\tau_{\rm d}\}} + S_{\tau_{\rm g}} \mathbf{1}_{\{\tau=\tau_{\rm g}\}}\Big) \,\Big|\, \mathcal{F}_t\right] \tag{21}$$

on the event $\{t < \tau\}$. Moreover, if $\beta \geq \lambda$, it is optimal for the manager not to steal.

[34] It is worthwhile to note that, in our model, uncertainty is not only driven by the Brownian cashflow shock but also by the stochastic arrival of growth opportunities. As a result, the derivation of (17) does not simply rely on the martingale representation theorem, as in the standard martingale approach developed by Sannikov (2008), but also on a "change of filtration" formula and other techniques borrowed from the credit risk literature.

Proof. See Appendix C.3.

Equation (21) confirms that the state process W under a dynamic contract can be interpreted as the manager's expected future payoff if he refrains from stealing, which we shall refer to as the manager's *promise*. Lemma 6 also establishes an incentive compatibility condition.[35] The condition is intuitive: since he enjoys a private benefit λ per unit of diverted cash, incentivizing the manager not to steal requires that his promise increases by at least λ for each extra unit of reported cashflow, namely, $\beta \geq \lambda$. A dynamic contract is admissible if it satisfies this condition as well as the initial promise condition $W_{0-} \geq \bar{w}$. Since $\beta \geq \lambda > 0$ under an admissible dynamic contract, (18) and (19) imply that inefficient disciplinary turnover occurs as the result of *poor* reported cashlows.

Before proceeding further, it is important to observe that, relative to the environment considered in DeMarzo and Sannikov (2006), the introduction of growth-induced turnover affects the dynamics of the agent's promise in a substantial way. The key difference lies in the drift of the promise, which in our setup is equal to $\varrho W_t + qG_t(W_t - S_t)$ instead of simply ϱW_t. The reason for this difference is that, whenever the manager is put at risk of being fired for the sake of growth (i.e., whenever $G_t = 1$), he needs to be 'compensated' for the loss that he would incur in case a growth opportunity arises. The potential loss corresponds to the difference $(W_t - S_t)$ between the manager's current promise and the amount of severance pay that he would receive if replaced for the sake of growth,[36] while the chances of incurring such loss are determined by the instantaneous intensity of growth-induced dismissal qG_t. Compensation for the risk of growth-induced termination comes in the form of an augmented drift, which translates into a faster increase of the manager's promise in states of the world where no growth opportunity materializes. In other words, the law of motion for the agent's promise is modified in our setup in such a way that the *promise keeping* condition remains satisfied.

3 Optimal Dynamic Contract

Having characterized the set of admissible dynamic contracts, we reformulate the firm's optimization problem as a stochastic control problem. We denote by $V(\phi, w)$ the firm's value function, which gives the firm's expected discounted profit at a given current size ϕ and for a given size-adjusted promise w to the incumbent manager. The firm's value function satisfies the recursive dynamic programming equation

$$V(\phi, w) = \sup_{C,S,G,\beta} \mathbb{E}\Big[\phi \int_{[0,\tau[} e^{-rt}(\mu dt - dC_t) - \phi e^{-r\tau}\Big(\Delta C_{\tau_{\rm d}} \mathbf{1}_{\{\tau=\tau_{\rm d}\}} + S_{\tau_{\rm g}} \mathbf{1}_{\{\tau=\tau_{\rm g}\}}\Big) + e^{-r\tau}\Big(-\kappa\phi + V_{\rm d}\mathbf{1}_{\{\tau=\tau_{\rm d}\}} + V_{\rm g}\mathbf{1}_{\{\tau=\tau_{\rm g}\}}\Big)\Big], \quad (22)$$

[35] This result extends the incentive compatibility condition derived in DeMarzo and Sannikov (2006) to our environment with growth-induced turnover.

[36] We later show that, under the optimal dynamic contract, a manager receives no severance when dismissed for the sake of growth. The potential loss upon growth-induced termination is therefore equal to W_t under the optimal contract.

in which expression

$$\tau = \tau_{\mathrm{d}} \wedge \tau_{\mathrm{g}}, \quad V_{\mathrm{d}} = V(\phi, w_{\mathrm{h}}^{\phi}) \quad \text{and} \quad V_{\mathrm{g}} = V\Big((1+\gamma)\phi, w_{\mathrm{h}}^{(1+\gamma)\phi}\Big), \tag{23}$$

subject to the incentive compatibility constraint $\beta \geq \lambda$, and subject to (15), (19) and (20) with initial condition $W_{0-} = w$. In this formulation, C and S denote the manager's size-adjusted cumulative compensation and size-adjusted severance upon growth, respectively, while

$$w_{\mathrm{h}}^{\phi} = \bar{w} \vee \arg\max_{w>0} V(\phi, w) \quad \text{and} \quad w_{\mathrm{h}}^{(1+\gamma)\phi} = \bar{w} \vee \arg\max_{w>0} V\big((1+\gamma)\phi, w\big). \tag{24}$$

In particular, (24) captures the possibility that a new manager's participation constraint may not be binding, as it may be optimal for the firm to give him a rent in excess of his reservation value. Since cashflows, turnover costs and reservation values are all proportional to firm size, it follows that firm value itself is homogenous in size, namely,

$$V(\phi, w) = \phi V(1, w) =: \phi v(w). \tag{25}$$

In particular, stationarity and size homogeneity imply that the firm offers the same dynamic contract to all successive managers. Using (22)–(25), the size-adjusted value function $v(w)$ is determined along with the optimal contract by

$$\begin{aligned} v(w) = \sup_{C,S,G,\beta} \mathbb{E}\bigg[\int_{[0,\tau[} e^{-rt}(\mu dt - dC_t) - e^{-r\tau}\Big(\Delta C_{\tau_{\mathrm{d}}} \mathbf{1}_{\{\tau=\tau_{\mathrm{d}}\}} + S_{\tau_{\mathrm{g}}} \mathbf{1}_{\{\tau=\tau_{\mathrm{g}}\}}\Big) \\ + e^{-r\tau}\Big(-\kappa + v(w_{\mathrm{h}})\mathbf{1}_{\{\tau=\tau_{\mathrm{d}}\}} + (1+\gamma)v(w_{\mathrm{h}})\mathbf{1}_{\{\tau=\tau_{\mathrm{g}}\}}\Big)\bigg] \end{aligned} \tag{26}$$

subject to the same constraints as above, where the size-adjusted *hiring promise* w_{h} satisfies

$$w_{\mathrm{h}} = \bar{w} \vee \arg\max_{w>0} v(w). \tag{27}$$

The following proposition is central to our characterization of the optimal dynamic contract.

Proposition 3. *Let $u : \mathbb{R}_+ \to \mathbb{R}$ be a concave C^2 function that satisfies the Hamilton-Jacobi-Bellman (HJB) equation*

$$\begin{aligned} \max\bigg\{ \frac{\sigma^2\lambda^2}{2} u''(w) + \varrho w u'(w) - r u(w) + \mu \\ + q\left[(1+\gamma)u(w_{\mathrm{h}}) - \kappa + w u'(w) - u(w)\right]^+, \ -u'(w) - 1 \bigg\} = 0, \end{aligned} \tag{28}$$

with boundary condition

$$u(0) = u(w_{\mathrm{h}}) - \kappa, \tag{29}$$

where $w_{\mathrm{h}} = \bar{w} \vee \arg\max_{w>0} u(w)$. Also, suppose that $\lim_{w\downarrow 0} |u'(w)| < \infty$ and $u'(w) = -1$ for some $w < \infty$. Then the function u identifies with the value function v defined by (26), namely, $v(w) = u(w)$ for all $w \geq 0$. Moreover, the optimal dynamic contract satisfies Properties 1-5 listed below.

Proof. See Appendix D.

We rely on Proposition 3 to construct the firm's value function and solve for the optimal dynamic contract. As observed in previous work on dynamic moral hazard, the concavity of the value function is related to the fact that a change in w affects firm value not only directly by increasing the amount of compensation owed to the manager, but also via its impact on the likelihood of disciplinary turnover. Indeed, by reducing the prospect of a costly disciplinary turnover, an increase in the agent's promise by one dollar effectively costs less than one dollar to the firm. Moreover, since the probability of disciplinary turnover is higher after poor performance, the reduction in agency costs induced by a marginal increase in the agent's promise is larger for low values of w. This is what gives rise to concavity.

3.1 Optimality Properties

We now turn to the properties satisfied by the optimal dynamic contract, as implied by Proposition 3. These impose restrictions on the cashflow sensitivity, on the compensation policy, and on the growth policy. The first two properties are standard. In particular, they also hold in the absence of growth opportunities, and are derived in that context by DeMarzo and Sannikov (2006), and Biais et al. (2007).[37]

Property 1. *The optimal contract has sensitivity to reported cashflows $\beta = \lambda$.*

The fact that the incentive compatibility constraint should hold as an equality ($\beta = \lambda$) is related to the concavity of the value function. Intuitively, reducing the volatility of the manager's promise as much as possible while satisfying incentive compatibility is optimal for the firm because it lowers the probability that the promise hits zero, which would result in *ex post* inefficient disciplinary turnover.

Property 2. *The optimal compensation policy is such that the manager receives transfers only if his current promise w is at least w_c. The compensation threshold w_c satisfies*

$$v'(w_c) = -1.$$

This property can be explained heuristically by observing that, at any instant, the firm has the option to make an immediate transfer to the manager and continue optimally. Hence, the inequality $v(w) \geq -\varepsilon + v(w - \varepsilon)$ holds for any transfer ε, which implies $v'(w) \geq -1$. When the manager's current promise w is such that $v'(w) > -1$, deferring compensation is optimal. By concavity of the value function, this happens when w is below the point w_c that satisfies $v'(w_c) = -1$. In this case, the manager receives no compensation until his promise reaches the compensation threshold. If $\bar{w} > w_c$, the manager receives a signing bonus $\Delta C_0 = \bar{w} - w_c$ when appointed, and his promise later remains in the interval $[0, w_c]$.[38]

[37] From a broader perspective, the 'smooth pasting' condition in Property 2 is a standard feature of the solution to singular stochastic control problems such as the one given by (26); see Beneš et al. (1980) and Karatzas (1983) for early references.

[38] Technically, the agent's promise W is *reflected* at w_c by the cumulative compensation process. A rigorous construction of this process is provided in Appendix D (see Theorem D-2). One way in which the introduction of stochastic growth opportunities and growth-induced turnover modifies the firm's pay policy is by affecting the value of the optimal threshold w_c, as shown in Section 3.3 for the high-growth case.

Property 3. *The optimal compensation policy involves no severance payment, namely,* $\Delta C_{\tau_{\mathrm{d}}} = 0$ *and* $S = 0$*.*

Property 3 establishes that severance pay is strictly suboptimal in the continuous-time setting, even in the case of growth-induced termination. The reason is that, rather than give cash to a departing manager, the firm is always better off increasing the promise of the incumbent conditional on him being retained, which has the benefit of reducing the likelihood of inefficient turnover later on.[39] This result is in contrast with the indifference result that holds in the two-period model where disciplinary dismissal can only occur at the end of the first period (see the discussion following Lemma 1 and Footnote 15). It is also worthwhile to note that the no-severance result upon growth-induced dismissal relies crucially on the assumption that the arrival of a growth opportunity is contractible.[40]

Property 4. *It is optimal for the firm to stand ready to take a growth opportunity if and only if the manager's current promise* w *is such that*

$$(1+\gamma)v(w_{\mathrm{h}}) - \kappa + wv'(w) \geq v(w). \tag{30}$$

Condition (30), which we shall refer to as the *growth optimality* condition, determines the circumstances under which growth-induced turnover can occur. The inequality reveals that the optimal growth policy does not just rely on a comparison between the *status quo* continuation value $v(w)$ and the continuation value upon growth $(1+\gamma)v(w_{\mathrm{h}}) - \kappa$. The extra term $wv'(w)$ accounts for the fact that putting the manager at risk of being fired if a growth opportunity arrives requires to compensate him in the form of an augmented drift, as discussed in Section 2.4. When the firm's value function is decreasing at the current value of the agent's promise (i.e., $v'(w) < 0$), this higher drift constitutes a cost. If this cost is high relative to the potential gains from growth, so that $(1+\gamma)v(w_{\mathrm{h}}) - \kappa + wv'(w) - v(w) < 0$, it is optimal for the firm to insulate the incumbent manager from the risk of being replaced, and thus forego growth opportunities when they become available. We refer to this possibility as contractual *job protection.*

Property 5. *If partial job protection arises as part of the optimal contract, the firm foregoes growth opportunities if the manager's promise* w *is above* w_{g}*, where the growth threshold* w_{g} *satisfies*

$$w_{\mathrm{g}} = \sup\big\{w \geq 0 : \ (1+\gamma)v(w_{\mathrm{h}}) - \kappa + wv'(w) - v(w) \geq 0\big\} < w_{\mathrm{c}}.$$

This property indicates that, if some degree of job protection arises as part of the optimal incentive contract, managers are shielded from the risk of growth-induced turnover

[39]By the same logic, severance pay would be suboptimal in a simpler setting with exogenous random exit of the manager. We are grateful to an anonymous referee for making this observation.

[40]In an earlier version of this paper, we analyzed the case in which the availability of a growth opportunity is privately observed by the incumbent manager and showed that, in that case, severance upon growth-induced dismissal arises as part of the optimal contract (See Anderson et al. (2012), Section 6). This result is reminiscent of Eisfeldt and Rampini (2008) and Inderst and Mueller (2010), although in our model severance is used to incentivize the incubent to reveal *good* news. Malenko (2013) also considers an environment with privately observed investment opportunities. Severance pay upon *disciplinary* dismissal arises in the setup analyzed by He (2012) with risk-averse agent and private savings.

after good performance. Intuitively, the benefit of retaining the incumbent, net of the foregone gains from growth, is increasing in w because losses due to moral hazard under the incumbent are dimished after good performance.[41]

3.2 Two Types of Firms

In light of our discussion of Properties 4 and 5, two configurations can arise—echoing the two 'growth regimes' encountered in the analysis of the two-period model. In the first one, the growth optimality condition (30) holds for all values of the manager's promise $w \in [0, w_c]$. We refer to firms falling into this configuration as *high-growth firms.* In such firms, managers are fully exposed to the risk of being fired for the sake of growth, and the instantaneous rate of growth-induced turnover is always equal to q. Over the course of the manager's tenure, the firm keeps track of the evolution of

$$dW_t = (\varrho + q)W_t\,dt - dC_t + \lambda\,(d\hat{Y}_t - \mu\,dt), \quad W_{0-} = w_h,$$

where transfers dC reflect the manager's promise W at the endogenous compensation threshold w_c. Transfers to the manager can be interpreted as bonuses indexed on reported performance.[42] The manager is dismissed when a growth opportunity arises or when W hits zero, whichever comes first.

By contrast, in the second possible configuration, the growth optimality condition does not hold everywhere on the interval $[0, w_c]$, and some degree of job protection is part of the optimal contract. We refer to firms falling into the latter configuration as *low-growth firms.* The contract offered by a low-growth firm specifies, along with a compensation threshold w_c, a growth threshold $w_g < w_c$. Over the course of the manager's tenure, the firm keeps track of

$$dW_t = \big[\varrho + q\mathbf{1}_{[0,w_g]}(W_t)\big]W_t\,dt - dC_t + \lambda\,(d\hat{Y}_t - \mu\,dt), \quad W_{0-} = w_h,$$

where transfers dC reflect W at w_c. The manager is dismissed if a growth opportunity arises at a time when $W_t \leq w_g$, or when W hits zero, whichever comes first. Consistent with our discussion of Property 5, the optimal contract in low-growth firms commands that, whenever the manager's promise is above the growth threshold w_g, the firm foregoes any growth opportunity that becomes available.[43]

[41] In other words, the net benefit of exposing a manager to the risk of growth-induced termination is decreasing in the manager's promise, which can be seen from the fact that $(1+\gamma)v(w_h) - \kappa + wv'(w) - v(w)$ is decreasing in w, by concavity of v. This is in line with the observation that $\hat{\kappa}_{G(y)} < \hat{\kappa}_{G(0)}$ in the two-period model (see our discussion of the optimal growth policy following Lemma 2).

[42] In both types of firms, transfers to the manager are increasing in reported cashflows net of the expected level of cashflows. This feature of the contract is qualitatively in line with the use of *bonus systems* based on reported earnings in excess of a performance target, as documented in Murphy (1999).

[43] The manager being partially shielded from the risk of growth-induced turnover might be described as an endogenous form of 'entrenchment'. We do not use this word because it more commonly connotes actions taken by a manager to make his replacement costly. A number of recent papers explore frameworks very different from ours where they establish conditions under which managers are protected from termination. See, e.g., Atkeson and Cole (2008), Casamatta and Guembel (2010), and Garrett and Pavan (2012).

The insights gathered through our analysis of the two-period model in Section 1 are useful to understand why the low-growth configuration may sometimes be optimal even though Condition (14) guarantees that foregoing growth opportunities is inefficient under first best. We saw in that context that exposing managers to early termination risk can make it more expensive to incentivize them. In the continuous-time model, this is manifested by the fact that putting a manager at risk of being replaced for the sake of growth effectively makes him more impatient, as revealed by the augmentation of the drift of the contractual promise (which reflects the manager's 'effective' discount rate) from ϱ to $\varrho + q$. In the presence of moral hazard, a firm thus faces this *ex ante* tradeoff: a policy of always standing ready to pursue growth by appointing a new, more suitable manager has the advantage of producing higher expected cashflows, but it entails increased early termination risk for incumbent managers and a higher cost of incentive provision during their tenure. In low-growth firms, the resolution of the tradeoff between efficient turnover and the cost of incentive provision gives rise to an 'interior' solution, whereby the optimal contract allows for job protection after good performance.[44]

3.3 High-Growth Firms

In this section, we further characterize the optimal contract offered by a high-growth firm. To this end, we consider the free-boundary problem that consists in finding a free-boundary point w_{c} and a function u that satisfies the ODE

$$\frac{\sigma^2\lambda^2}{2}u''(w) + (\varrho + q)wu'(w) - (r+q)u(w) + \mu + q\big[(1+\gamma)u(w_{\mathrm{h}}) - \kappa\big] = 0 \qquad (31)$$

in the interval $(0, w_{\mathrm{c}})$, is given by

$$u(w) = u(w_{\mathrm{c}}) - (w - w_{\mathrm{c}}), \quad \text{if } w > w_{\mathrm{c}}, \qquad (32)$$

and satisfies the boundary conditions

$$u(0) = u(w_{\mathrm{h}}) - \kappa, \quad u'(w_{\mathrm{c}}) = -1 \quad \text{and} \quad u''(w_{\mathrm{c}}) = 0, \qquad (33)$$

where

$$w_{\mathrm{h}} = \bar{w} \vee \arg\max_{w>0} u(w). \qquad (34)$$

Proposition 4. *Given any permissible values of* $(r, \varrho, \mu, \sigma, q, \gamma, \lambda, \kappa, \bar{w})$ *in* $\mathbb{R}^9$*, there exists a unique solution* (u, w_{c}) *to the free-boundary problem defined by (31)–(34). The function* u *is* C^2 *and concave, and satisfies the HJB equation*

$$\max\left\{\frac{\sigma^2\lambda^2}{2}u''(w) + (\varrho + q)wu'(w) - (r+q)u(w) + \mu + q\big[(1+\gamma)u(w_{\mathrm{h}}) - \kappa\big],\ -u'(w) - 1\right\} = 0.$$

[44] A third possible configuration involves fully isolating the managers from the risk of growth-induced termination, which corresponds to (30) being violated for all values of $w \in [0, w_{\mathrm{c}}]$. However, we show in Appendix F.5 that this 'no-growth' policy can only be optimal if $v(w_{\mathrm{h}}) < 0$, so that the firm would rather not operate. We do not expand further on this case in the remainder of our analysis.

Furthermore, the following statements hold true:

i. The set of permissible parameter values for which u satisfies (30) for all $w \in [0, w_c]$, and therefore the HJB equation (28), has non-empty interior in $\mathbb{R}^9$.

ii. There exists a unique point $\bar{w}_\dagger = \bar{w}_\dagger\left(\frac{\varrho+q}{\sigma^2\lambda^2}, \frac{r+q}{\sigma^2\lambda^2}, \kappa\right)$ such that

$$w_\circ := \arg\max_{w>0} u(w) \geq \bar{w} \;\Leftrightarrow\; \bar{w} \leq \bar{w}_\dagger \qquad \text{and} \qquad w_\circ = \bar{w} \;\Leftrightarrow\; \bar{w} = \bar{w}_\dagger.$$

iii. There exists a unique point $\bar{w}_\ddagger = \bar{w}_\ddagger\left(\frac{\varrho+q}{\sigma^2\lambda^2}, \frac{r+q}{\sigma^2\lambda^2}, \kappa\right) > \bar{w}_\dagger$ such that

$$w_c \geq w_h \;\Leftrightarrow\; \bar{w} \leq \bar{w}_\ddagger \qquad \text{and} \qquad w_c = w_h = \bar{w} \;\Leftrightarrow\; \bar{w} = \bar{w}_\ddagger.$$

iv. If $\bar{w} = \bar{w}_\ddagger$, u satisfies (30) for all $w \in [0, w_c]$ and the HJB equation (28) if and only if

$$\gamma\mu \geq r\kappa + (r + \gamma\varrho)\bar{w}. \tag{35}$$

Proof. See Appendix F.2.

In view of Proposition 3 and the general properties of the solution to the free-boundary problem established in Proposition 4, statement (i) implies that, for a large set of permissible parameter values, the firm is of the high-growth type. For such parameter values, the firm's size-adjusted value function and the optimal compensation threshold are given, along with the hiring promise, by the solution to the free-boundary problem (31)–(34). Figure 1 illustrates the firm's value function and the optimal compensation threshold in the high-growth configuration for particular parameter values.

[FIGURE 1 HERE]

Statement (ii) determines conditions under which it is optimal for a high-growth firm to grant to a new manager a compensation rent in excess of his reservation value. The hiring promise w_h is optimally set above the manager's reservation promise $\bar{w}$ when the latter is sufficiently low, namely, $\bar{w} \leq \bar{w}_\dagger$. In this case, the initial promise w_h is equal to the level $w_\circ \geq \bar{w}$ that maximizes the firm's value. Otherwise, the manager's participation constraint is binding so that the hiring promise coincides with his reservation value (i.e., $w_h = \bar{w} > w_\circ$).

Statement (iii) sheds light on how the optimal compensation policy plays out at the start of a manager's tenure. Depending on the value of the reservation promise $\bar{w}$, three scenarios can arise. When the reservation value is relatively low, in the sense that $\bar{w} < \bar{w}_\ddagger$, the compensation threshold is optimally set above the hiring promise ($w_c > w_h$). In this scenario, a newly hired manager does not receive any pay for some time until the effect of the positive drift $\varrho + q$, possibly combined with good cashflow realizations, finally takes his promise up to the compensation threshold w_c. In contrast, if the reservation promise is high enough ($\bar{w} > \bar{w}_\ddagger$), a manager receives a signing bonus $\Delta C_0 = \bar{w} - w_c > 0$ when appointed. In the particular case where $\bar{w} = \bar{w}_\ddagger$, the hiring promise and compensation threshold are such that $w_c = w_h = \bar{w}$, and the manager starts receiving compensation immediately after taking office.

Statement (iv) provides an explicit condition on exogenous parameter values for the firm to be a high-growth type. Condition (35) suggests that high-growth firms tend to be the ones that are more productive (high μ) or have better opportunities (high γ). Our next proposition gives further insight on the characteristics of high-growth firms.

Proposition 5. *Consider any permissible values of* $(r, \varrho, \mu, \sigma, q, \gamma, \lambda, \kappa, \bar{w})$ *in* $\mathbb{R}^9$ *such that* $\bar{w} = \bar{w}_{\ddagger}$ *and condition (35) holds with equality. A marginal increase in* λ, σ *or* κ, *or a marginal decrease in* γ, q *or* μ, *leads condition (35) to fail.*

Proof. See Appendix F.3.

In view of statement (iv), the proposition suggests that high-growth firms also tend to be characterized by not-too-severe moral hazard and low turnover costs (low λ and κ). These results confirm the insights derived in the two-period framework (see Lemma 4).[45] Proposition 5 further suggests that frequent growth opportunities (high q) and not-too-volatile cashflows (low σ) are other attributes of high-growth firms. In particular, because of the cumulative nature of growth in our stationary environment, having more frequent growth opportunities makes it more valuable to undertake any such opportunity.

Finally, we characterize the determinants of the compensation threshold in high-growth firms with the following proposition. At this point, it is worthwile to note that, holding the dynamics of the manager's promise constant, a lower (resp., higher) compensation threshold results in more front-loaded (resp., back-loaded) compensation.

Proposition 6. *Consider* $(r, \varrho, \mu, \sigma, q, \gamma, \lambda, \kappa, \bar{w})$ *in the interior of the set of permissible parameter values for which the firm is a high-growth type.*

i. The optimal compensation threshold w_c *is increasing in* κ, *and independent of* μ *and* γ.

ii. If the parameter values are initially such that $\bar{w} = \bar{w}_{\ddagger}$, *then a marginal increase in* λ *or* σ *leads to an increase in* w_c, *whereas a marginal increase in* q *leads to a reduction in* w_c.

Proof. See Appendix F.4.

As the severity of moral hazard, the volatility of cashflows or the cost of managerial replacement increases, the compensation threshold is raised to reduce the likelihood of inefficient turnover. On the other hand, an increase in the arrival rate of growth opportunities, by increasing the manager's effective discount rate, results in a lower compensation threshold. Furthermore, the quality of growth opportunities, γ, has no impact on the optimal compensation scheme conditional on the firm being of the high-growth type, although it may affect the shape of the compensation profile to the extent that it alters the firm's growth regime (as in the two-period model).[46] Section 3.5 further illustrates the implications of our model for the timing of compensation.

3.4 Low-Growth Firms

We now turn to the low-growth configuration. In light of Proposition 3 and Property 5, we consider the free-boundary problem that consists in finding two free-boundary points w_c and $w_g < w_c$ and a function u that satisfies the ODE

$$\frac{\sigma^2\lambda^2}{2}u''(w) + (\varrho + q)wu'(w) - (r + q)u(w) + \mu + q\big[(1 + \gamma)u(w_h) - \kappa\big] = 0 \tag{36}$$

[45] The assumption $\bar{w} = \bar{w}_{\ddagger}$, under which Propositions 5 and 6.(ii) are derived, implies the identities $w_c = w_h = \bar{w}$, which facilitate the proofs of our results. Establishing such results globally is beyond the scope of this paper.

[46] The result that the threshold w_c is unaffected by the mean size-adjusted cashflow μ differs from the one derived in DeMarzo and Sannikov (2006), where w_c is increasing in μ. This is because the firm's continuation value upon termination is exogenously given in their setup, whereas it is endogenously determined in ours.

in the interval $(0, w_g)$, satisfies the ODE

$$\frac{\sigma^2\lambda^2}{2}u''(w) + \varrho w u'(w) - ru(w) + \mu = 0 \tag{37}$$

in the interval (w_g, w_c), is given by

$$u(w) = u(w_c) - (w - w_c), \quad \text{if } w > w_c, \tag{38}$$

satisfies the boundary conditions given by (33), and satisfies the requirement that

$$u(w_g) - w_g u'(w_g) = (1+\gamma)u(w_h) - \kappa, \tag{39}$$

where w_h is defined as in (34). The analysis of this problem allows us to establish that, despite the assumption that foregoing growth opportunities is suboptimal under first best, the low-growth configuration can arise, as stated in the following proposition.

Proposition 7. *The set of permissible parameter values for which the solution u to the free-boundary problem defined by (33)–(34) and (36)–(39) is C^2 and concave, and satisfies the HJB equation (28), has non-empty interior in $\mathbb{R}^9$. For such parameter values, the firm is a low-growth type.*

Proof. See Appendix G.3.

In low-growth firms, the optimal contract is as described in Section 3.2, with growth and compensation thresholds given by the free-boundary points w_g, w_c of the free-boundary problem defined above and with the hiring promise w_h endogenously determined as part of the problem.[47] Figure 2 depicts the firm value function along with the optimal thresholds in the low-growth configuration for particular parameter values. The figure also represents what the value of the firm would be if it were constrained to systematically take all growth opportunities as they come. The distance between the two curves on the figure illustrates the benefit that a low-growth firm derives from offering partial job protection to its managers, which can ultimately be traced back to a reduction in agency costs.

[FIGURE 2 HERE]

It is worthwhile to note that our finding that, in some firms, growth may only occur after poor performance is in contrast with the result obtained in setups where the firm can grow through investment with the incumbent (see, e.g., DeMarzo and Fishman (2007a)). In such settings, growth is positively related to past performance because the return on investment is higher after good cashflows, due to a reduction in agency costs. The opposite prediction arises in our setup because the net benefit of exposing a manager to the risk of growth-induced termination is lower after good performance, also due to a reduction in agency costs. In practice, the relevance of each of these two mechanisms should depend on the extent to which growth is of a 'transformative' nature or not, i.e., on whether taking a growth opportunity requires a change of management or not.[48]

[47] In Appendix G.1, we derive five possible systems of highly non-linear equations that should be solved to determine the points w_g, w_c and w_h; see in particular Problem G-0. Given the complexity of this problem, providing a complete characterization of its solution with a view to deriving a suitable solution to the HJB equation (28) with boundary condition (29) is beyond the scope of this paper.

[48] The investment-cashflow sensitivity literature points to a positive relationship between investment and past

3.5 A Numerical Example

We now use numerical simulations to illustrate how a firm's growth prospects may affect managerial turnover and pay in our setup. The numerical example also provides a sense of the quantitative properties of the continuous-time model.

Frequency of Managerial Turnover. In our setting, the probability of an incumbent manager being dismissed depends on the past performance of the firm under his tenure, but also on the availability of a growth opportunity and on the *ex ante* characteristics of the firm that affect the turnover policy. First, the likelihood of dismissal increases with poor performance—both because a string of bad cashflows can result in disciplinary turnover and because, in some firms, growth-induced turnover only occurs after poor performance. Second, holding performance and firm characteristics constant, the probability of dismissal also increases (at least weakly) upon arrival of a growth opportunity. Finally, the probability of turnover depends on firm characteristics, to the extent that these affect the contract specification and the degree of protection granted to the manager. In particular, firms with better growth prospects should show a higher turnover rate.

To see this last point, we consider two firms that are identical in every dimension except for the size (γ) of the growth opportunities they might receive. For the sake of illustration, we take as common parameter values across the two firms $r = 7\%$, $\varrho = 16\%$, $\mu = 1$, $\sigma = 1$, $q = 0.2$, $\lambda = 0.4$, $\kappa = 0.3$, $\bar{w} = 1$. In the firm with better growth prospects, we set $\gamma = 0.25$, while we set $\gamma = 0.10$ in the other firm.[49] The difference in the quality of growth opportunities faced by the two firms makes the former a high-growth type (i.e., a manager in this firm is never immune to the risk of growth-induced termination), and the latter a low-growth type (i.e., managers are protected from growth-induced turnover after good performance). The average annualized turnover rate in these two firms are 21.4% and 5.5%, respectively. Changes in growth prospects driven by the arrival rate (q) of growth opportunities have similar effects. To see this, we consider variations in q around the high-growth and low-growth baselines. For the high-growth firm, an increase in the frequency of growth opportunities from $q = 0.20$ to $q = 0.22$ causes the average turnover rate to rise to 23.2%, while setting $q = 0.18$ causes the turnover rate to drop to 19.4%. For the low-growth firm, the same variations in q cause the average turnover rate to rise to 5.7% or drop to 5.3%, respectively.

Figure 3 depicts the cumulative distribution of tenure length in the two baseline examples. The probability distribution of tenure length for the low-growth firm first-order

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performance. However, these studies do not really shed light on the empirical validity of the mechanism at play in our model because they do not account for the fact that firms may grow both by 'marginal' changes or by radical transformations requiring a change in top management, nor for the possibility that an increase in the firm's value may be obtained without capital investment.

[49] These parameter values are permissible (see Conditions (12)–(14)). In particular, the firms' growth prospects are sufficiently attractive as to make taking all growth opportunities optimal in the absence of moral hazard. Discount rates r and ϱ, and the intensity rate q, are expressed on an annual basis. Given the normalization $\mu = 1$, parameters σ, κ, and $\bar{w}$ are effectively expressed in terms of annual mean cashflow. For given parameter values, we first determine the firm's type and the optimal contractual threshold(s) by solving numerically the free-boundary problems associated with the HJB equation (28)–(29), as described in Sections 3.3 and 3.4. The average turnover rate is then obtained from simulating the dynamics of the promise W under the optimal contract until dismissal for a very large number of managers.

stochastically dominates the one for the high-growth firm, i.e., the probability of a manager reaching any given number of tenure years is higher in the low-growth firm. The median tenure length of a manager is 3.3 years in the firm with better growth prospects ($\gamma = 0.25$), whereas it is 12.6 years in the firm with poorer growth prospects ($\gamma = 0.10$). Changes in the quality of growth prospects driven by the arrival rate of growth opportunities q affect tenure length in a similar way: increasing the frequency of growth opportunities from $q = 0.20$ to $q = 0.22$ causes the median time in office to drop to 3.0 years and 12.1 years, respectively; whereas switching to $q = 0.18$ causes that time to rise to 3.6 years and 13.1 years, respectively.

[FIGURE 3 HERE]

Extent of Compensation Back-Loading. Deferred compensation constitutes an essential feature of the optimal dynamic contract under moral hazard.[50] It is well-understood that the degree of compensation back-loading should depend on the severity of moral hazard (λ), the cashflow volatility (σ), and the wedge between the manager's and the firm's discount rates ($\varrho - r$). In our setup, the extent of back-loading also depends on the prospect of growth-induced turnover. Indeed, as revealed by our analysis, firms' growth prospects may affect the timing of pay both through the drift of the manager's promise and the level of the compensation threshold.

By way of illustrating this aspect of our model, we simulate managerial pay under the optimal contract and characterize the degree of compensation back-loading using the notion of *compensation duration*. Namely, for a given sequence of bonuses received by a manager over his entire tenure, we compute the weighted average of the points in time when compensation is received—with weights equal to the fraction of the total discounted pay (using the agent's discount rate ϱ) received at each point in time.[51] Figure 4 depicts the cumulative distribution of realized compensation duration in the two baseline configurations introduced in the previous subsection. The relative position of the two distributions reflects the fact that, holding the level of expected discounted pay $\bar{w}$ constant across firms, compensation is more front-loaded in firms with better growth prospects. On average, compensation duration is 2.2 years in the high-growth firm, versus 4.8 years in the low-growth firm.[52]

[FIGURE 4 HERE]

4 Empirical Evidence

In this section, we present evidence that is consistent with the notion of growth-induced turnover and the empirical implications derived in the context of the two-period framework

[50] Compensation under the optimal contract comes in the form of bonuses that the manager receives whenever his promise reaches the endogenous compensation threshold w_{c} (see Property 2 for a precise characterization).

[51] Compensation duration is formally defined as $\left(\int_0^\tau e^{-\varrho t}\, dC_t\right)^{-1} \int_0^\tau t e^{-\varrho t}\, dC_t$, in line with the notion of bond duration used in interest rate risk management. In simulations, we use the discretized version of this expression.

[52] The high-growth and low-growth baselines only differ in terms of the size of growth opportunities (γ). An increase in the arrival rate of growth opporunities (q) also results in lower compensation duration.

in Section 1.2 and illustrated for the continuous-time model in Section 3.5. We first investigate the empirical determinants of CEO turnover in the light of the theory. We then explore the relation between the timing of CEO compensation and firms' growth prospects.

4.1 Data

Our empirical analysis relies on information on CEO tenure episodes in US public firms as reported in the Standard & Poor's ExecuComp database for the period 1992-2014.[53] Once we merge the ExecuComp sample with accounting information from Compustat and stock return data from CRSP, our sample comprises 4,514 CEO episodes. Out of these, 2,510 episodes cover the full tenure of the CEO from the year the CEO is appointed until the year of leaving post. The total number of CEO-year observations in our sample is 27,992.[54] The minimum number of firms covered in a given year is 760 in 1992, and the maximum is 1,416 in 2005.

Using information from ExecuComp, we identify the beginning and end years of each completed CEO episode. The variable *TotTenure* is defined as the total number of years in which the CEO is running the firm. Within an episode, the variable *Turnover* is a dummy variable which equals 1 in the last year of the CEO's tenure and zero otherwise. We also use Execucomp to construct the variable *TotPay* defined as the total compensation awarded to a CEO in a given year.[55] Table 1 reports the summary statistics of our sample. In particular, the average and median CEO tenure lengths are 6.5 years and 5 years, respectively, while the average annual turnover rate is 8.4%.

Our analysis centers on CEO turnover and the timing of CEO compensation in relation to the quality of a firm's growth prospects. Our empirical proxy for the growth prospects of a firm during a given CEO episode is based on the 'average Q' of each firm. The use of average Q as a proxy for a firm's growth opportunities is standard in the empirical corporate finance literature.[56] We construct the average Q for each firm-year in our sample and we denote it simply by Q. As a proxy for the quality of growth prospects during a given episode, we use the value of Q in the year before the CEO is appointed, which we denote by *QInit*. We interpret a higher value of *QInit* as capturing better *ex ante* growth prospects at the time a new CEO is hired.

Managerial turnover in our setup is also affected by the availability of growth opportunities. Capturing the arrival of a growth opportunity in any given year during a CEO

[53]The ExecuComp database covers all firms included in the S&P 500, MidCap, and SmallCap indexes. It would be interesting to extend the analysis to smaller public firms and private firms. Kaplan, Sensoy, and Stromberg (2008) document high turnover in the management teams of VC-backed private companies before going public.

[54]All CEO-year observations are from 1992 onwards. While the ExecuComp dataset starts in 1992, it covers episodes in which the CEO was appointed earlier. Our sample includes observations pertaining to such episodes when the required information (e.g., the firm's average Q at the time the CEO was appointed) is available.

[55]In common with many studies of CEO compensation (see, e.g., Murphy (1999, 2013)), we use total compensation in the year awarded. Further details on variable definitions and on the construction of our dataset are provided in Appendix H.

[56]The handbook by Eckbo et al (2001) surveys multiple studies in which average Q is used as a proxy for growth opportunities. This practice stems from Hayashi (1982), who derives sufficient conditions such that a firm's average Q coincides with its marginal product of capital. See Caballero (1997) and Bond and Van Reenen (2007) for surveys of the empirical literature assessing the links between average Q, marginal q, and investment.

episode is challenging empirically. As a proxy, we construct the variable *RatioQ* defined as the ratio of the lagged value of a firm's Q in any given year to *QInit*. A higher value of *RatioQ* is more likely to be observed when the firm has new growth opportunities available.

To control for past performance in any given year within a CEO episode, we use the cumulative abnormal return of the firm measured over the previous two years, denoted by CAR.[57] We also consider the lagged return on assets (ROA) as an additional control for performance. Finally, we use the logarithm of the lagged value of total assets ($LnAssets$) to control for firm size.

[TABLE 1 HERE]

4.2 Determinants of CEO Turnover

We first examine the relation between turnover and the quality of growth prospects. As a first pass, Figure 5 depicts the cumulative distribution of CEO tenure length conditional on *ex ante* growth prospects proxied by *QInit*. The solid line plots the kernel estimate of the distribution for the upper twenty percent of CEO episodes ranked by *QInit*, whereas the dashed line corresponds to the bottom twenty percent. The cumulative distribution for the upper *QInit* sub-sample lies significantly above the one for the bottom *QInit* sub-sample.[58] That is, the likelihood that a CEO will not 'survive' beyond any number of years is higher for CEOs entering firms with good growth prospects than for CEOs entering firms with poor growth prospects, consistent with the notion of growth-induced turnover. Figure 5 constitutes the empirical counterpart to the simulation results depicted in Figure 3.

[FIGURE 5 HERE]

We further assess the implications of our model for managerial turnover by running a probit regression where the dependent variable is the *Turnover* indicator variable. The probit specification is as follows:

$$Prob(Turnover_{j,t} = 1) = \Phi[\psi_0 + \psi_1 QInit_j + \psi_2 RatioQ_{j,t-1} + \psi_3 CAR_{j,t-1} + \boldsymbol{\alpha}' \boldsymbol{X}_{j,t-1}],$$

where Φ is the standard normal cumulative distribution function, j denotes a CEO episode, t is calendar year, and $\mathbf{X}$ denotes a vector of control variables. We control for the return on assets and the size of the firm, both lagged by one year. Calendar year fixed effects are also included. In view of the comparative static results derived in Proposition 1, we hypothesize that the coefficients on *QInit* and *RatioQ* should be positive, while the coefficient on CAR and ROA should be negative.

Table 2 summarizes the results of the probit regression. Column A reports the estimated coefficients of the probit model and their standard errors.[59] All explanatory variables

[57] Our qualitative results are unaffected when one-year or three-year cumulative abnormal returns are used as a measure of past performance, or when the initial years of a CEO's tenure are removed from the sample. Our results are also robust to the use of industry-level measures of Q in the construction of *QInit* and *RatioQ*.

[58] The two-sample Wilcoxon-Mann-Whitney test shows that the two distributions are significantly different from each other, with a p-value of zero. Similarly, the Kolmogorov-Smirnov test rejects the hypothesis that the two samples are drawn from the same distribution.

[59] Robust standard errors are clustered at the firm level in all the regressions.

have the expected signs and are highly statistically significant. The coefficient on *QInit* is positive, in line with our model's prediction that turnover is more frequent in firms with better *ex ante* growth prospects. The coefficient on *RatioQ* is also positive, in accord with the idea that turnover is sometimes triggered by the arrival of growth opportunities. Finally, the coefficients on *CAR* and on *ROA* are negative, in line with the theoretical prediction that turnover is more likely after poor performance.

Column B reports the implied marginal effects, which give the impact on the probability of turnover of a unit increase in an explanatory variable, when all variables are evaluated at the sample means. In Column C, the marginal effects are multiplied by the sample standard deviation of the corresponding explanatory variables. A one standard deviation increase in *QInit* is associated with an increase in the probability of turnover by 85 basis points. Similarly, a one standard deviation increase in *RatioQ* leads to a 59 basis point increase in the probability of turnover. Since the unconditional frequency of CEO turnover in our sample is 8.4%, these results support the view that the growth-related drivers of managerial turnover emphasized in this paper are economically significant. Of course, this is in addition to the disciplinary role of turnover, which we also find to be important. In our sample, a one standard error increase in past abnormal returns is associated with a drop in the probability of turnover by 2.2 percentage points.

[TABLE 2 HERE]

An additional implication of our model is a tendency for firms with relatively poor *ex ante* growth prospects to grant partial job protection to their CEOs. That is, our theory predicts that in such firms, a CEO is less likely to be dismissed for the sake of growth when an opportunity arises calling for his replacement. We explore the empirical validity of this prediction by evaluating the marginal effect of *RatioQ* on the probability of CEO turnover at different levels of *QInit*. According to Proposition 1.(iii), the impact of *RatioQ* on turnover should be greater for firms with relatively better *ex ante* growth prospects, i.e., for higher values of *QInit*. Table 3 reports these differential marginal effects. The marginal effect of *RatioQ* is strictly positive at all levels of *QInit* and is indeed increasing in *QInit*.

[TABLE 3 HERE]

4.3 Growth Prospects and CEO Compensation

A key insight from our theory is that the managers of firms with better and more frequent growth opportunities should have more front-loaded compensation. In this subsection, we provide evidence on the empirical relation between CEO compensation and firms' growth prospects. We explore the data in two ways.

As a first pass, we compute a measure of realized compensation duration for each CEO episode and investigate how it varies across episodes that differ in terms of the firm's growth prospects at the time the CEO was appointed. For a given CEO episode j lasting for N_j years, our measure of compensation duration, labelled *PayDuration*, is obtained as:

$$PayDuration_j = \sum_{n=1}^{N_j} \frac{DiscPay_{j,n}}{\sum_{k=1}^{N_j} DiscPay_{j,k}} \times n, \tag{40}$$

where $DiscPay_{j,n} = TotPay_{j,n}/(1+\varrho)^n$ corresponds to the present value of the compensation received by the CEO in his n-th tenure year.[60] Setting the discount rate ϱ at 10%, we find that, in the sub-sample of episodes over which this measure is computed, average CEO compensation duration is 3.7 years, while the median is 3.2 years.[61]

Figure 6 provides an empirical counterpart to Figure 4, depicting kernel estimates of the cumulative distribution of *PayDuration* conditional on *ex ante* growth prospects proxied by *QInit*. The solid line pertains to the upper twenty percent of *QInit* in our sample, while the dashed line pertains to the bottom twenty percent. The cumulative distribution for the upper *QInit* sub-sample lies everywhere above the one for the bottom *QInit* sub-sample, which is consistent with our model's insight that firms with better growth prospects should have more front-loaded compensation.[62]

[FIGURE 6 HERE]

To further investigate how profiles of CEO pay over tenure vary with firms' growth prospects, we consider the regression equation

$$Ln(TotPay_{j,t}) = \psi_0 + \psi_1 \, TenureYear_{j,t} + \psi_2 \, QInit_j + \\ + \psi_3 \, QInit_j \times TenureYear_{j,t} + \boldsymbol{\alpha}' \mathbf{X}_{j,t-1} + \epsilon_{j,t},$$

where j denotes a CEO episode, t is calendar year, $TenureYear_{j,t}$ (resp., $TotPay_{j,t}$) denotes the number of years in tenure of CEO j in year t (resp., the total compensation received by the CEO in that year), and $\mathbf{X}$ is a vector of control variables. We control for past performance and firm size, as well as for calendar year fixed effects and industry or firm fixed effects. Our theory predicts (see Proposition 2.(iii), in particular) that firms with better *ex ante* growth prospects are characterized by a higher initial level of pay per period (i.e., ψ_2 positive), and slower growth in compensation over tenure years (i.e., ψ_3 negative). Table 4 summarizes our empirical findings for two alternative specifications, controlling for industry and firm fixed effects, respectively. The coefficients of interest are significant with the expected signs, and the results are very similar across both specifications. We also note that the coefficient on past abnormal returns is positive and significant, in line with the theoretical prediction that CEO pay is positively related to past performance.

[TABLE 4 HERE]

[60] This empirical measure of pay duration is analogous to the one introduced in Section 3.5 to illustrate the implications of our model for managerial compensation.

[61] The measure of compensation duration is not very sensitive to the value of the discount rate. Our conclusions are robust to alternative values of ϱ.

[62] The results from the two-sample Wilcoxon-Mann-Whitney test and from the Kolmogorov-Smirnov test both confirm that the difference between the two empirical distributions is statistically significant. Furthermore, controlling for year fixed effects, *PayDuration* and *QInit* are significantly negatively correlated across CEO episodes. When controlling for firm fixed effects, the correlation remains negative but becomes insignificant due to the small number of observations.

5 Conclusion

This paper introduces growth-induced turnover in a dynamic moral hazard framework and analyzes the interaction between this type of turnover and managerial incentive provision. In our model, growth opportunities arrive stochastically over time and the firm must appoint a new management to be able to seize them. Our analysis highlights the tradeoff that a firm faces between the benefit of always having at the helm a manager who is the right man for the job at hand and the cost of incentive provision. The key new insight is that exposing incumbent managers to the risk of growth-induced dismissal effectively increases their discount rate, thus increasing the cost of incentive provision. As a result, some firms find it optimal to provide some degree of job protection to their managers, at the cost of foregoing growth opportunities. Across firms, a higher likelihood of growth-induced turnover translates into a greater tendency to front-load compensation. Our empirical findings are consistent with these predictions of the model.

An essential feature of our model is that non-disciplinary managerial turnover can be triggered by the firm contingent on the arrival of exogenous contractible shocks. In our setup, shocks correspond to the arrival of growth opportunities, and it is first-best efficient for the firm to replace the incumbent manager upon arrival of an opportunity. Our analysis could be applied to alternative forms of exogenous contractible shocks. First, transformative managerial change may also be important for firms in decline. For instance, a change of management may be required for a firm to respond to increased product market competition or to the threat of a disruptive new technology. Second, the firm may face opportunities to transform—through a change of management—that would bring gains that are too modest to outweigh the cost of implementing them, so that they would not be taken up under first best. Yet, in a second-best world, it may be optimal to take these inefficient opportunities when the agency costs associated with the current manager are high. We believe that a number of theoretical insights of the paper would carry through in these alternative settings, although the empirical implications would be quite different.

The existing empirical literature on managerial turnover and compensation has been mostly informed by two paradigms from the contracting literature—the moral hazard model in which pay and dismissal are used to incentivize the agent, and the learning model in which the principal learns over time about the unknown quality of the agent. In our view, transformative change can be another powerful driver of managerial turnover and compensation. We document the fact that industries with better growth prospects experience higher CEO turnover and rely on more front-loaded compensation schemes. These findings are consistent with the assumption of growth-induced turnover and the predictions of our model. Nonetheless, other theories may be consistent with these findings. Identifying the specific channel through which firms' growth prospects relate to CEO turnover and compensation deserves further empirical work.

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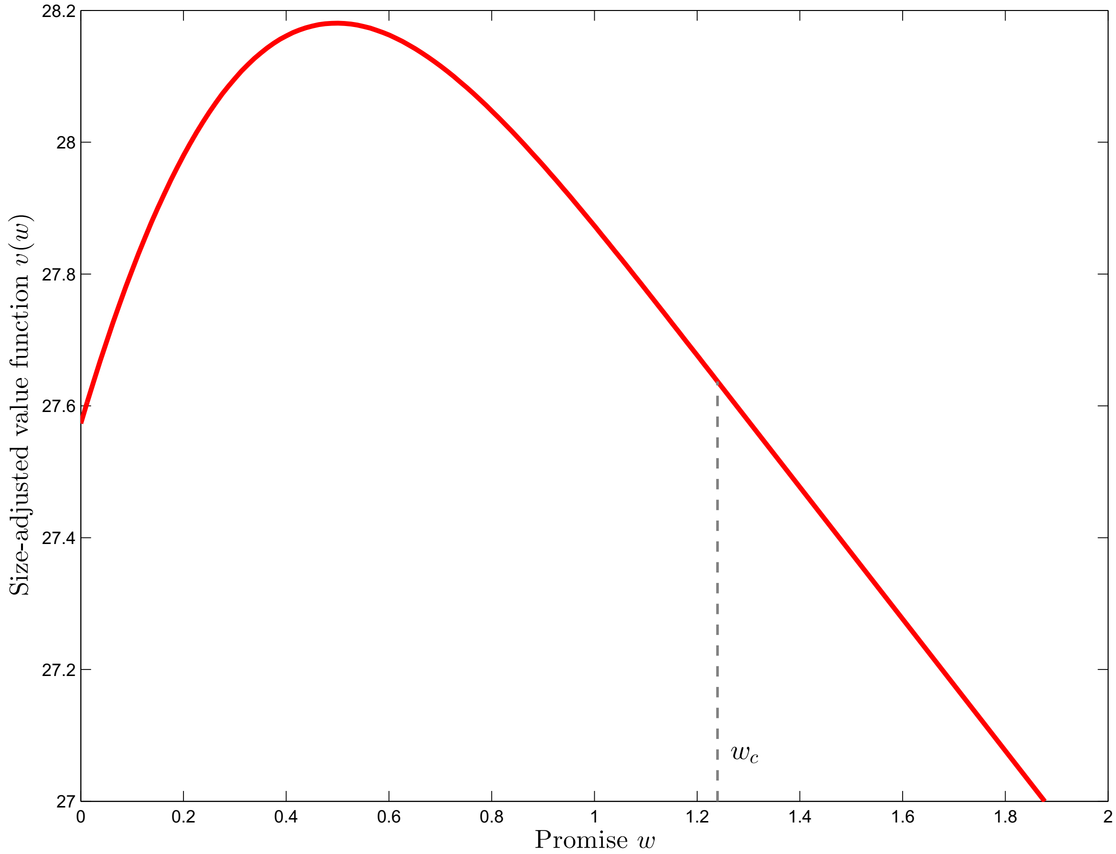

Figure 1: Value Function, High-Growth Firm

Notes: The figure depicts the firm's value function and the optimal compensation threshold w_c for parameter values $r = 7\%$, $\varrho = 16\%$, $\mu = 1$, $\sigma = 1$, $q = 0.2$, $\gamma = 0.25$, $\lambda = 0.4$, $\kappa = 0.3$, $\bar{w} = 1$. The firm's value function and the compensation threshold are determined by solving the free-boundary problem defined in Section 3.3. The growth-optimality condition (30) holds for all values of the manager's promise.

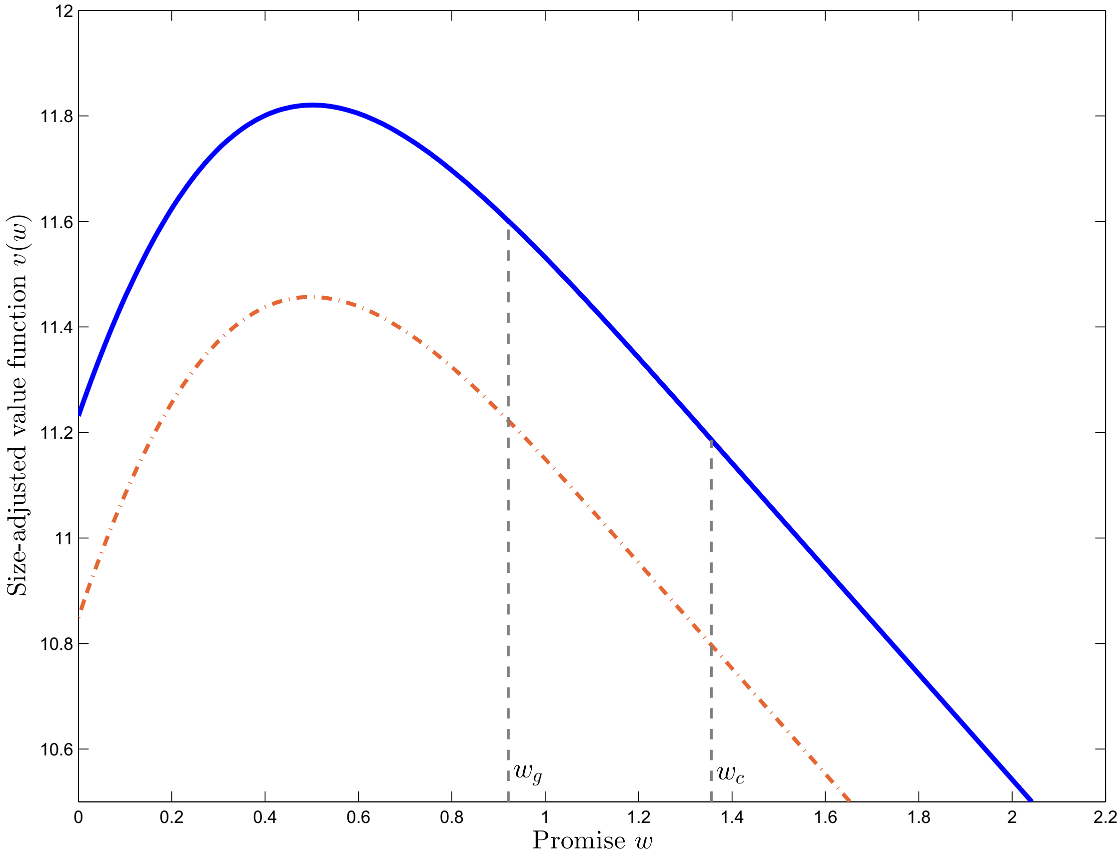

Figure 2: Value Function, Low-Growth Firm

Notes: The figure depicts the firm's value function (solid blue line), along with the optimal growth and compensation thresholds $(w_{\mathrm{g}}, w_{\mathrm{c}})$, for parameter values $r = 7\%$, $\varrho = 16\%$, $\mu = 1$, $\sigma = 1$, $q = 0.2$, $\gamma = 0.1$, $\lambda = 0.4$, $\kappa = 0.3$, $\bar{w} = 1$. The firm's value function and the thresholds are determined by solving the free-boundary problem defined in Section 3.4. The growth-optimality condition (30) holds on $[0, w_g]$ but is violated for $w > w_{\mathrm{g}}$. The figure also represents (dashed orange line) what firm value would be if the firm were constrained to take all growth opportunities, with the compensation threshold optimally determined by the solution to the free-boundary problem defined in Section 3.3.

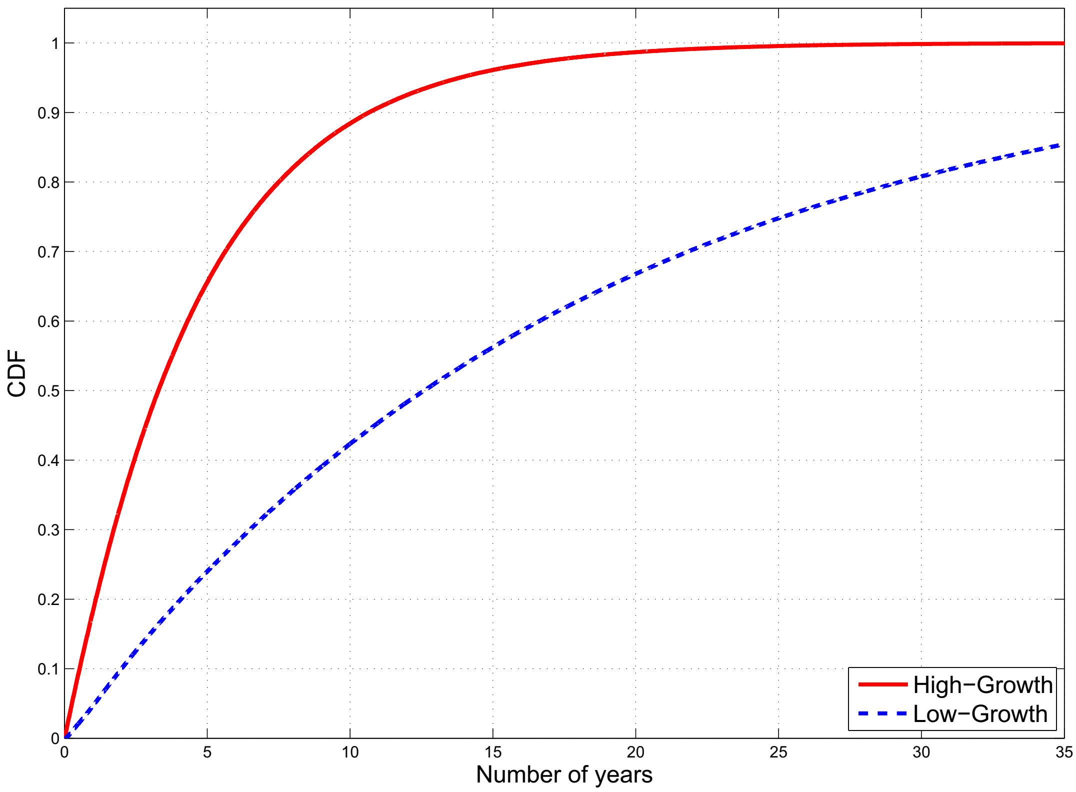

Figure 3: Distribution of Tenure Length: High-Growth vs. Low-Growth

Notes: The figure depicts model-implied cumulative distribution functions obtained from simulations for parameter values $r = 7\%$, $\varrho = 16\%$, $\mu = 1$, $\sigma = 1$, $q = 0.2$, $\gamma = 0.25$ (high-growth) or $\gamma = 0.10$ (low-growth), $\lambda = 0.4$, $\kappa = 0.3$, $\bar{w} = 1$. Optimal contractual thresholds are $w_{\rm c} = 1.24$ in the high-growth case, and $w_{\rm g} = 0.92$ and $w_{\rm c} = 1.36$ in the low-growth case. In both cases, the hiring promise $w_{\rm h}$ coincides with the reservation promise $\bar{w}$.

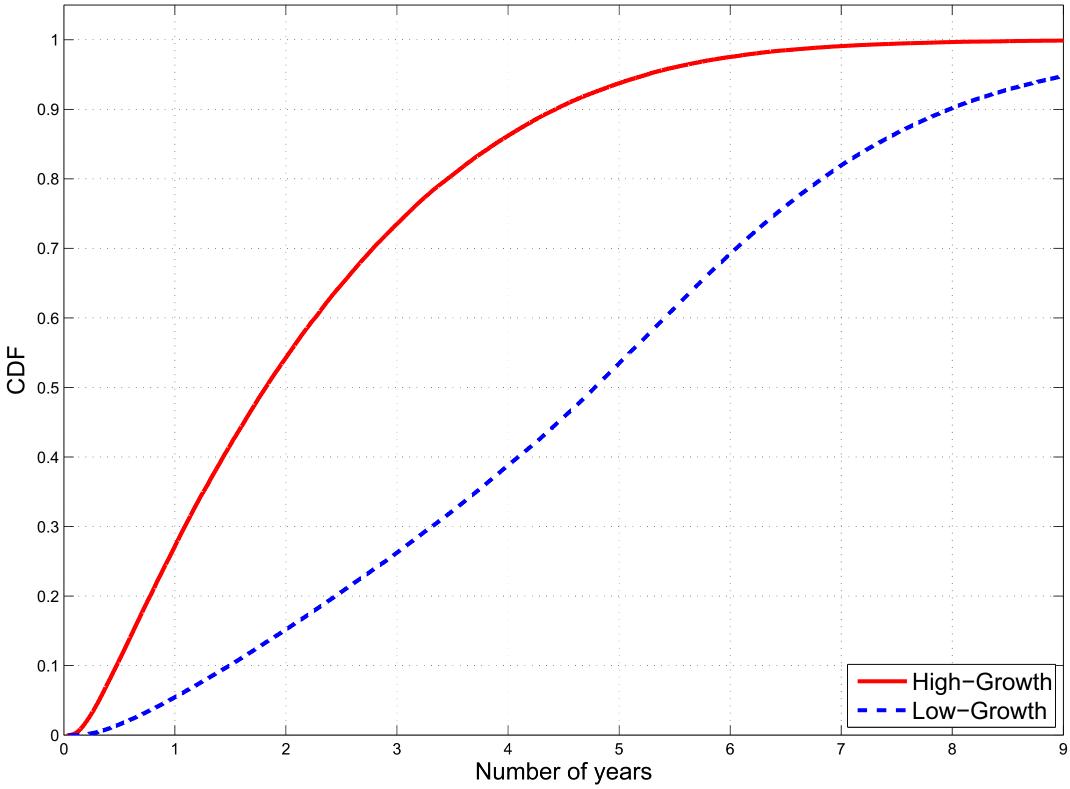

Figure 4: Distribution of Realized Compensation Duration: High-Growth vs. Low-Growth

Notes: The figure depicts model-implied cumulative distribution functions obtained from simulations for parameter values $r = 7\%$, $\varrho = 16\%$, $\mu = 1$, $\sigma = 1$, $q = 0.2$, $\gamma = 0.25$ (high-growth) or $\gamma = 0.10$ (low-growth), $\lambda = 0.4$, $\kappa = 0.3$, $\bar{w} = 1$. Optimal contractual thresholds are $w_{\mathrm{c}} = 1.24$ in the high-growth case, and $w_{\mathrm{g}} = 0.92$ and $w_{\mathrm{c}} = 1.36$ in the low-growth case. In both cases, the hiring promise w_{h} coincides with the reservation promise $\bar{w}$.

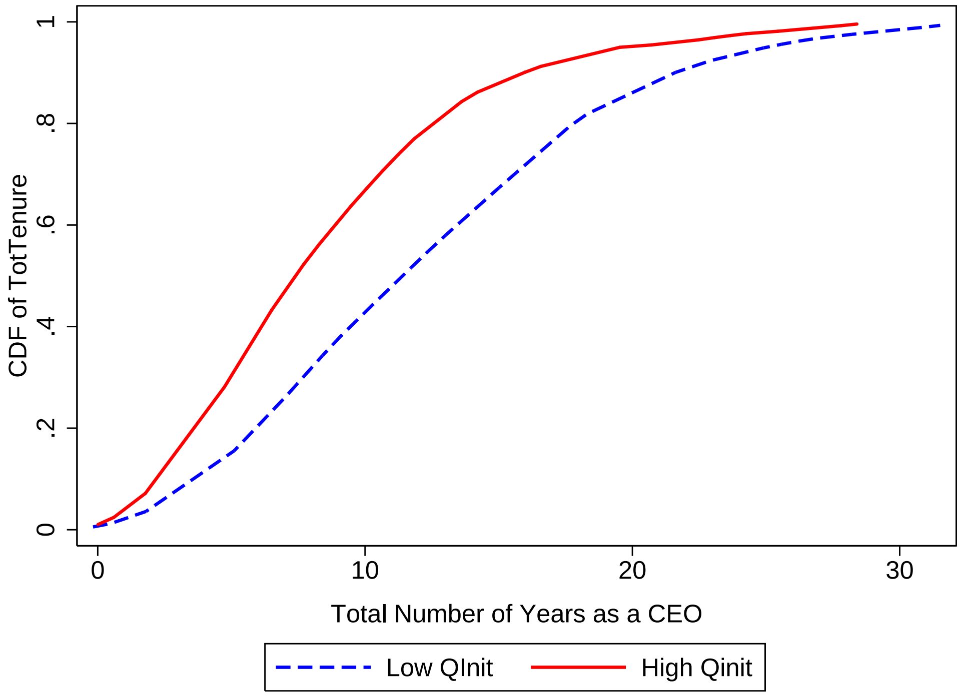

Figure 5: Distribution of CEO Tenure Length Conditional on Initial Growth Prospects
Notes: The figure depicts kernel estimates of the empirical cumulative distribution of CEO tenure length (*TotTenure*) for two sub-samples. The first sub-sample ('Low QInit') consists of the bottom quintile of CEO episodes sorted by initial Q; the corresponding distribution is plotted as a dashed line. The second sub-sample ('High QInit') consists of the top quintile of episodes sorted by initial Q; the corresponding distribution is plotted as a solid line. Details on variable definitions are provided in the main text and in Appendix H.

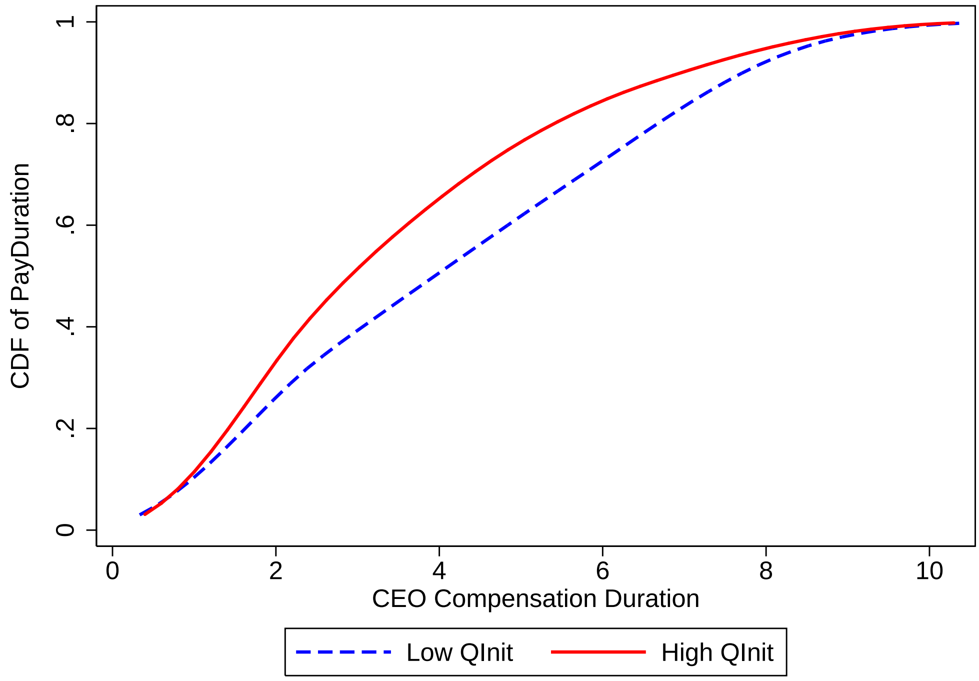

Figure 6: Distribution of CEO Pay Duration Conditional on Initial Growth Prospects
Notes: The figure depicts kernel estimates of the empirical cumulative distribution of realized compensation duration (*PayDuration*) for the bottom and top quintiles of CEO episodes sorted by initial Q (*QInit*), denoted by 'Low QInit' and 'High QInit', respectively. Details on variable definitions are provided in the main text and in Appendix H.

Table 1: Summary Statistics

Variable	Mean	Sd	p25	p50	p75	N
TotTenure	6.492	5.001	3.000	5.000	9.000	2,510
Turnover	0.084	0.278	0.000	0.000	0.000	27,992
LnTotPay	7.915	1.053	7.172	7.927	8.663	27,958
QInit	1.788	1.212	1.099	1.371	1.981	27,992
RatioQ	1.069	0.420	0.876	1.000	1.164	27,992
CAR	-0.001	0.244	-0.147	-0.005	0.129	27,992
ROA	0.038	0.078	0.013	0.040	0.075	27,992
LnAssets	7.700	1.699	6.452	7.602	8.887	27,992

Notes: The table reports summary sample statistics for the merged ExecuComp/Compustat/CRSP data set, which covers CEO episodes reported in ExecuComp over the period 1992-2014. *TotTenure* is the total number of tenure years for CEO episodes that are completed within the sample period. *Turnover* is a dummy variable which equals 1 in the last year of a CEO's tenure and zero otherwise. *LnTotPay* is the logarithm of total CEO compensation awarded in a given calendar year. *QInit* is the average Q of the firm in the year before the CEO was appointed; the same value is repeated throughout each CEO episode. *RatioQ* is the ratio of the lagged average Q of a firm in a given year divided by *QInit*. *CAR* is the two-year cumulative abnormal return of the firm (annualized). *ROA* is return on assets. *LnAssets* is the logarithm of the book value of total assets. Further details on variable definitions are provided in the main text and in Appendix H.

Table 2: Determinants of CEO Turnover

	(A) Coefficients b/se	(B) Marginal Effects b/se	(C) Coefficients of Variation (in percentage points)
QInit	0.047***	0.007***	0.848
	(0.009)	(0.001)	
RatioQ	0.089***	0.014***	0.588
	(0.030)	(0.005)	
CAR	-0.595***	-0.090***	-2.196
	(0.053)	(0.008)	
ROA	-0.806***	-0.122***	-0.952
	(0.131)	(0.020)	
LnAssets	0.040***	0.006***	1.019
	(0.006)	(0.001)	
N	27,992	27,992	27,992
Year fixed effects	Yes	Yes	Yes

Notes: The table summarizes the evidence on the probability of CEO turnover from the probit regression estimated over the merged ExecuComp/Compustat/CRSP data set from 1992 to 2014. The dependent variable is the *Turnover* indicator variable. *QInit* is the average Q of the firm in the year before the CEO was appointed; the same value is repeated throughout each CEO episode. *RatioQ* is the ratio of the lagged average Q of the firm in a given year divided by *QInit*. *CAR* is the two-year cumulative abnormal return of the firm (annualized) over the previous two years. *ROA* is return on assets, lagged by one year. *LnAssets* is the logarithm of the lagged book value of total assets. Calendar year fixed effects are included in the regression. Robust standard errors are clustered at the firm level.

Table 3: Initial Growth Prospects and Growth-Induced Turnover

	Marginal Effect of *RatioQ* b/se
Low *QInit*	0.0124***
	(0.0040)
Median *QInit*	0.0126***
	(0.0041)
High *QInit*	0.0133***
	(0.0044)
N	27,992

Notes: The table reports the marginal effect of *RatioQ* on the likelihood of turnover for different levels of initial average Q, as implied by the probit model estimated over the merged ExecuComp/Compustat/CRSP data set from 1992 to 2014. *QInit* is the average Q of the firm in the year before the CEO was appointed. *RatioQ* is the ratio of lagged Q in a given year divided by *QInit*. The marginal effect of *RatioQ* is evaluated at different quantiles of the distribution of *QInit*. 'Low', 'Median', and 'High' quantiles correspond to the 20-th, 50-th, and 80-th percentiles of the distribution of *QInit*, respectively.

Table 4: Determinants of CEO Compensation

	(A) *LnTotPay*	(B) *LnTotPay*
TenureYear	0.017***	0.008*
	(0.004)	(0.003)
QInit	0.073***	0.054***
	(0.011)	(0.014)
QInit × *TenureYear*	-0.008***	-0.005**
	(0.002)	(0.002)
CAR	0.556***	0.435***
	(0.026)	(0.026)
LnAssets	0.430***	0.199***
	(0.007)	(0.019)
Firm Fixed Effects	No	Yes
Industry Fixed Effects	Yes	No
Year Fixed Effects	Yes	Yes
R-squared	0.532	0.663
N	27,615	27,615

Notes: This table summarizes the evidence on the profile of CEO compensation over tenure. *LnTotPay* is the logarithm of total CEO pay awarded in a given year as reported in ExecuComp. *TenureYear* is the number of years in tenure of the CEO in a given calendar year. *QInit* is the average Q of the firm in the year before the CEO was appointed. *CAR* is the two-year cumulative abnormal return of the firm (annualized) over the previous two years. *ROA* is return on assets, lagged by one year. *LnAssets* is the logarithm of the lagged book value of total assets. The regression is estimated over all episode-year observations in our sample, some of which pertain to CEO episodes that have not finished by the end of the sample period. Robust standard errors are clustered at the firm level.

A The Two-Period Model

This appendix proves the results pertaining to the two-period model that we consider in Section 1 of the paper. In this context, the set of permissible parameter values is

$$\mathcal{P} = \Big\{(r, \varrho, y, p, q, \gamma, \lambda, \kappa) \in \mathbb{R}^8 \mid \varrho > r \geq 0,\ \gamma, y > 0,\ p \in \,]0,1[,\ q \in \,]0,1[,$$
$$\lambda \in [0,1] \text{ and } 0 < \kappa < \frac{p\gamma y}{1+r}\Big\}. \tag{A.1}$$

The inequality $\kappa < p\gamma y/(1+r)$, referred to as Condition (1) in the main text, ensures that growth is efficient under first best.

As a preliminary step, we consider the one-period contracting problem faced by the firm at $t = 1$ if it hires a new manager to run its operations for the second period. The following standard result holds true in our setting.

Lemma A-1. *If managers are protected by limited liability and have zero reservation value, then the optimal one-period contract that induces truthful reporting offers compensation $\lambda\hat{Y}_2$, where $\hat{Y}_2$ denotes the reported cashflow at $t = 2$.*

Proof. It suffices to consider the case where a new manager is hired at $t = 1$ after disciplinary dismissal, so that the actual cashflow at $t = 2$ is either y with probability p or zero with probability $1-p$. In this case, the contracting problem boils down to determining a compensation policy $C : \{0, y\} \to \mathbb{R}^+$ that specifies the agent's payoff at $t = 2$ as a function of reported output. The incentive compatibility constraint is

$$C(y) \geq \lambda y + C(0),$$

while the firm's expected profit under no stealing is $py - \big(pC(y) + (1-p)C(0)\big)$. Furthermore, the limited liability constraint $C(\hat{Y}_2) \geq 0$ ensures that the manager's participation constraint is satisfied. It is immediate to see that the solution to the firm's constrained maximization problem yields $C(0) = 0$ and $C(y) = \lambda y$, namely, $C(\hat{Y}_2) = \lambda\hat{Y}_2$. ∎

A.1 The Firm's Problem

We now turn to the firm's contracting problem at $t = 0$ when it hires the initial manager.

Definition A-1. A *two-period contract* is a quintuple (C_1, C_g, C_2, F, G), where

$$C_1 : \{0, y\} \to \mathbb{R}^+, \quad C_{\mathrm{g}} : \{0, y\} \to \mathbb{R}^+, \quad C_2 : \{0, y\}^2 \to \mathbb{R}^+,$$
$$F : \{0, y\} \to [0, 1] \quad \text{and} \quad G : \{0, y\} \to [0, 1]$$

have the interpretation given at the beginning of Section 1. ∎

The firm's problem is to choose a two-period contract (C_1, C_g, C_2, F, G) that maximizes its expected discounted profit

$$\begin{aligned} V = p\Bigg(& y - C_1(y) + qG(y)[V_{\rm g} - C_{\rm g}(y)] + (1-q)F(y)V_{\rm d} \\ & + \Big(1 - [qG(y) + (1-q)F(y)]\Big)\frac{p[y - C_2(y,y)] - (1-p)C_2(y,0)}{1+r}\Bigg) \\ +(1-p)\Bigg(& -C_1(0) + qG(0)[V_{\rm g} - C_{\rm g}(0)] + (1-q)F(0)V_{\rm d} \\ & + \Big(1 - [qG(0) + (1-q)F(0)]\Big)\frac{p[y - C_2(0,y)] - (1-p)C_2(0,0)}{1+r}\Bigg), \end{aligned} \tag{A.2}$$

in which expression

$$V_{\rm d} = \frac{p(1-\lambda)y}{1+r} - \kappa \quad \text{and} \quad V_{\rm g} = \frac{p(1-\lambda)(1+\gamma)y}{1+r} - \kappa, \tag{A.3}$$

subject to the first-period incentive compatibility (IC) constraint

$$\begin{aligned} & C_1(y) + qG(y)C_{\rm g}(y) + \Big(1 - [qG(y) + (1-q)F(y)]\Big)\frac{pC_2(y,y) + (1-p)C_2(y,0)}{1+\varrho} \\ \geq\ & \lambda y + C_1(0) + qG(0)C_{\rm g}(0) + \Big(1 - [qG(0) + (1-q)F(0)]\Big)\frac{pC_2(0,y) + (1-p)C_2(0,0)}{1+\varrho} \end{aligned} \tag{A.4}$$

and subject to the second-period IC constraints

$$C_2(y,y) \geq \lambda y + C_2(y,0), \tag{A.5}$$

$$\text{and} \quad C_2(0,y) \geq \lambda y + C_2(0,0). \tag{A.6}$$

Note that the expressions in (A.3) for the firm's continuation values upon disciplinary and growth-induced dismissal follow immediately from the form of the optimal one-period contract derived in Lemma A-1. Also, note that, in order to simplify the expression for the firm's objective, the firm's net cashflows in (A.2) are discounted as of $t = 1$.

A.2 Proofs

Proof of Lemma 1. In view of the the firm's optimization problem formulated in Section A.1, we can see that lowering $C_1(0)$, $C_{\rm g}(0)$ and $C_2(0,0)$ relaxes the IC constraints (A.4) and (A.6) and improves the firm's profit (A.2), while leaving (A.5) unaffected. Hence, it is optimal to set

$$C_1(0) = C_{\rm g}(0) = C_2(0,0) = 0.$$

Lowering $C_2(y,0)$ while raising $C_2(y,y)$ to leave $pC_2(y,y) + (1-p)C_2(y,0)$ constant relaxes (A.5) while leaving (A.4) unaffected. Hence, it is optimal to set

$$C_2(y,0) = 0.$$

Since an increase in $C_2(0,y)$ tightens the first-period IC constraint (A.4), the second-period IC constraint after poor performance (A.6) is binding. Taking into account the fact that $C_2(0,0)=0$, we can see that

$$C_2(0,y) = \lambda y$$

is optimal. The firm's problem thus simplifies to choosing $C_1(y)$, $C_{\mathrm{g}}(y)$, $C_2(y,y)$, as well as $F(0)$, $F(y)$ and $G(0)$, $G(y)$ to maximize

$$\begin{aligned} V = p\Bigg(& y - C_1(y) + qG(y)[V_{\mathrm{g}} - C_{\mathrm{g}}(y)] + (1-q)F(y)V_{\mathrm{d}} \\ & + \Big(1 - [qG(y) + (1-q)F(y)]\Big)p\frac{y - C_2(y,y)}{1+r}\Bigg) \\ + (1-p)\Bigg(& qG(0)V_{\mathrm{g}} + \Big(1 - [qG(0) + (1-q)F(0)]\Big)p\frac{(1-\lambda)y}{1+r} + (1-q)F(0)V_{\mathrm{d}}\Bigg), \end{aligned} \tag{A.7}$$

subject to first-period IC constraint

$$\begin{aligned} C_1(y) + qG(y)C_{\mathrm{g}}(y) + \Big(1 - [qG(y) + (1-q)F(y)]\Big)\frac{pC_2(y,y)}{1+\varrho} \\ \geq \lambda y + \Big(1 - [qG(0) + (1-q)F(0)]\Big)\frac{p\lambda y}{1+\varrho}, \end{aligned} \tag{A.8}$$

and second-period IC constraint after good performance

$$C_2(y,y) \geq \lambda y. \tag{A.9}$$

Since lowering $C_1(y)$, $C_{\mathrm{g}}(y)$ or $C_2(y,y)$ increases the firm's profit, the constraint (A.8) should be binding. Substituting the expression for $C_1(y) + qG(y)C_{\mathrm{g}}(y)$ implied by the resulting equality into (A.7) and maximizing with respect to $C_2(y,y)$ subject to (A.9), we can see that

$$C_2(y,y) = \lambda y$$

is optimal, and the expression for first-period compensation conditional on good performance given by (7) follows immediately. ∎

Remark A-1. For future reference, we note that (7) implies that

- If the optimal contract sets $F(0)=1$ and $G(y)=1$, then

$$C_1(y) + qC_{\mathrm{g}}(y) = \lambda y - (1-q)\frac{p\lambda y}{1+\varrho}. \tag{A.10}$$

- If the optimal contract sets $F(0)=1$ and $G(y)=0$, then

$$C_1(y) = \lambda y - \frac{p\lambda y}{1+\varrho}. \tag{A.11}$$

- If the optimal contract sets $F(0) = 0$ and $G(y) = 1$, then

$$C_1(y) + qC_{\rm g}(y) = \lambda y. \tag{A.12}$$

- If the optimal contract sets $F(0) = 0$ and $G(y) = 0$, then

$$C_1(y) = \lambda y - q\frac{p\lambda y}{1+\varrho}. \tag{A.13}$$

∎

Proof of Lemma 2. Substituting $C_2(y, y) = \lambda y$ as well as the expression for $C_1(y) + qG(y)C_{\rm g}(y)$ given by (7) into (A.7), we obtain

$$\begin{aligned} V = p\Big((1-\lambda)y + \Big([qG(0) + (1-q)F(0)] - [qG(y) + (1-q)F(y)]\Big)p\frac{\lambda y}{1+\varrho} \\ + qG(y)V_{\rm g} + (1-q)F(y)V_{\rm d} + \Big(1 - [qG(y) + (1-q)F(y)]\Big)p\frac{(1-\lambda)y}{1+r}\Big) \\ + (1-p)\Big(qG(0)V_{\rm g} + (1-q)F(0)V_{\rm d} + \Big(1 - [qG(0) + (1-q)F(0)]\Big)p\frac{(1-\lambda)y}{1+r}\Big), \end{aligned} \tag{A.14}$$

where $V_{\rm d}$ and $V_{\rm g}$ are given by (A.3). All the statements in the lemma follow immediately from maximizing this expression with respect to each of $F(y)$, $F(0)$, $G(0)$, and $G(y)$, subject to the constraints $F(Y_1), G(Y_1) \in [0, 1]$. ∎

Remark A-2. It is worth noting that, when maximizing (A.14) with respect to $F(y)$, the linearity of the firm's objective gives rise to the unique corner solution $F(y) = 1$ for any permissible parameter values. On the other hand, when maximizing with respect to $F(0)$, $G(0)$ or $G(y)$, the linearity of the firm's objective gives rise to either a unique corner solution or to a continuum of solutions, depending on the parameter values. For instance, if $\kappa = \hat{\kappa}_{F(0)}$, where $\hat{\kappa}_{F(0)} \equiv \hat{\kappa}_{F(0)}(\varrho, y, p, \lambda)$ is defined as in (8), then any value of $F(0)$ in $[0, 1]$ is optimal. We adopt the convention that when the optimal dismissal probability $F(0)$, $G(0)$ or $G(y)$ is not uniquely determined, which happens when κ coincides with $\hat{\kappa}_{F(0)}$, $\hat{\kappa}_{G(0)}$ or $\hat{\kappa}_{G(y)}$, the firm sets the dismissal probability equal to one. ∎

Remark A-3. For future reference, we observe that, for given values of r, ϱ, p and γ, the relative location of the two thresholds $\hat{\kappa}_{F(0)}$ and $\hat{\kappa}_{G(y)}$ defined in Lemma 2 depends on the value of λ. Indeed,

$$\hat{\kappa}_{G(y)} < \hat{\kappa}_{F(0)} \;\Leftrightarrow\; \lambda > \lambda_\dagger \quad \text{and} \quad \hat{\kappa}_{G(y)} = \hat{\kappa}_{F(0)} \;\Leftrightarrow\; \lambda = \lambda_\dagger, \tag{A.15}$$

where the cutoff value $\lambda_\dagger \equiv \lambda_\dagger(r, \varrho, p, \gamma) < 1$ is such that

$$\frac{\lambda_\dagger}{1-\lambda_\dagger} = \frac{1+\varrho}{1+r}(1-p)\gamma. \tag{A.16}$$

∎

In the rest of this appendix, we restrict attention to permissible parameter values such that Condition (11) also holds true, namely, we consider parameter values in the set

$$\tilde{\mathcal{P}} = \Big\{(r, \varrho, y, p, q, \gamma, \lambda, \kappa) \in \mathcal{P} \mid p \geq \hat{p}(r, \varrho, \gamma)\Big\},$$

where $\hat{p} \equiv \hat{p}(r, \varrho, \gamma) < 1$ satisfies

$$\frac{\hat{p}}{1-\hat{p}} = \frac{1+\varrho}{1+r}\gamma. \tag{A.17}$$

It is immediate to see that the threshold $\hat{\kappa}_{G(0)}$ defined in (9) satisfies $\hat{\kappa}_{G(0)} \geq p\gamma y/(1+r)$ if and only if Condition (11) holds true. In view of Lemma 2.(ii), it follows that Conditions (1) and (11) together imply that it is strictly optimal to set $G(0) = 1$, as claimed in the main text.

Remark A-4. The thresholds $\hat{\kappa}_{F(0)}$, $\hat{\kappa}_{G(0)}$ and $\hat{\kappa}_{G(y)}$ defined in Lemma 2 are such that

$$\max\{\hat{\kappa}_{F(0)}, \hat{\kappa}_{G(y)}\} < \hat{\kappa}_{G(0)}.$$

Therefore, if Condition (11) were violated and the parameter values were such that $\kappa > \hat{\kappa}_{G(0)}$, then the optimal contract would set $G(y) = G(0) = 0$ and $F(0) = F(y) = 0$, i.e., the firm would never grow and the manager would never be dismissed. ∎

Proof of Lemma 3. In view of the definition of the high-growth (resp., low-growth) regime given in the text above the statement of the lemma, we need to show that the set of parameter values in $\tilde{\mathcal{P}}$ such that the inequality $\kappa \leq \hat{\kappa}_{G(y)} \equiv \hat{\kappa}_{G(y)}(r, \varrho, y, p, \gamma, \lambda)$ is satisfied (resp., violated) has non-empty interior in $\mathbb{R}^8$ (see also Lemma 2.(iii) and Remark A-2). Recall that permissibility requires in particular that $0 < \kappa < p\gamma y/(1+r)$.

The high-growth regime arises whenever $\kappa \leq \hat{\kappa}_{G(y)}$. This can happen if and only if $\hat{\kappa}_{G(y)} > 0$, which is equivalent to $\lambda < \lambda_\ddagger$ where $\lambda_\ddagger \equiv \lambda_\ddagger(r, \varrho, \gamma) \in]\lambda_\dagger, 1[$ is such that

$$\frac{\lambda_\ddagger}{1-\lambda_\ddagger} = \frac{1+\varrho}{1+r}\gamma. \tag{A.18}$$

The inequality $\lambda_\ddagger > \lambda_\dagger$ follows from a straightforward comparison of (A.16) and (A.18). The low-growth regime arises whenever $\kappa > \hat{\kappa}_{G(y)}$. This can happen for any values of the parameters r, ϱ, y, p, q, γ, λ in the relevant projection of $\tilde{\mathcal{P}}$ because $\hat{\kappa}_{G(y)} < p\gamma y/(1+r)$. ∎

Proof of Lemma 4. The threshold $\hat{\kappa}_{G(y)}$ defined in Lemma 2.(iii) is increasing in γ and ϱ, and decreasing in λ and r. Furthermore, as long as $\hat{\kappa}_{G(y)} > 0$, which is equivalent to $\lambda < \lambda_\ddagger$, where $\lambda_\ddagger$ is defined by (A.18), this threshold is also increasing in p and y. Combining these observations with statement (iii) in Lemma 2, we can see that a change in parameter values (within the permissible set $\tilde{\mathcal{P}}$) induces a switch from the low-growth to the high-growth regime arise in the following situation. We start with any values of r, ϱ, y, p, q, γ, λ in the relevant projection of $\tilde{\mathcal{P}}$, with $\lambda < \lambda_\ddagger$. For such parameter values, $0 < \hat{\kappa}_{G(y)} < p\gamma y/(1+r)$. If κ is initially in the right neighbourhood of $\hat{\kappa}_{G(y)}$, any change in r, ϱ, y, p, γ or λ that causes an increase in $\hat{\kappa}_{G(y)}$ induces a switch from the low-growth to the high-growth regime.

A drop in κ has the same effect. Furthermore, a change in q has no impact on $G(y)$ because $\hat{\kappa}_{G(y)}$ is independent of q. ∎

Proof of Proposition 1. In view of Lemma 2, statement (i) holds true because

$$F(0) \geq F(y) = 0 \quad \text{and} \quad G(0) = 1 \geq G(y),$$

where the first inequality is strict when $\kappa \leq \hat{\kappa}_{F(0)}$, in which case $F(0) = 1$, while the second inequality is strict when $\kappa \geq \hat{\kappa}_{G(y)}$, in which case $G(y) = 0$.

The first part of statement (ii) follows from the fact that an increase in γ that induces a change of regime from low-growth to high-growth, as described in the proof of Lemma 4, causes an increase in $G(y)$ from 0 to 1, while $F(y)$, $F(0)$ and $G(0)$ are left unaffected.

The second part of (ii) and the first part of (iii) follow from the fact that

$$G(y) \geq F(y) = 0 \quad \text{and} \quad G(0) = 1 \geq F(0),$$

where the first inequality is strict if $\kappa \leq \hat{\kappa}_{G(y)}$, in which case $G(y) = 1$, while the second inequality is strict when $\kappa \geq \hat{\kappa}_{F(0)}$, in which case $F(0) = 0$. In particular, the likelihood of dismissal, $qG(Y_1) + (1-q)F(Y_1) = F(Y_1) + q[G(Y_1) - F(Y_1)]$, is increasing in q.

To prove the second part of the last statement, we first note that $G(0) - F(0) = 1 - F(0)$ is not affected by γ. We also note that $G(y) - F(y) = G(y)$ is equal to one if $\kappa \leq \hat{\kappa}_{G(y)}$ and zero otherwise. Therefore, an increase in γ that induces a change from the low-growth to the high-growth regime, as described in the proof of Lemma 4, increases the impact that the arrival of a growth opportunity has on the likelihood of dismissal. ∎

Proof of Proposition 2. Statement (i) follows from Lemma 1, which implies that

$$\begin{aligned} &C_1(y) + qC_{\text{g}}(y) > 0 = C_1(0) = C_{\text{g}}(0), \\ \text{and} \quad &\lambda y = C_2(Y_1, y) > C_2(Y_1, 0) = 0. \end{aligned}$$

To prove statement (ii), we note that (A.10)–(A.13) imply that $C_1(y) + qG(y)C_{\text{g}}(y) \leq \lambda y$, and therefore $\bar{C}_1 \leq \bar{C}_2$, where $\bar{C}_1$, $\bar{C}_2$ are defined above the statement of the proposition. The inequality is strict unless $F(0) = 0$, $G(y) = 1$, which is optimal if and only if $\lambda < \lambda_\dagger$ and $\kappa \in \,]\hat{\kappa}_{F(0)}, \hat{\kappa}_{G(y)}]$ (see also Remark A-3).

Finally, consider an increase in γ that causes a switch from the low-growth to the high-growth regime, as described in the proof of Lemma 4. If $\lambda \in \,]\lambda_\dagger, \lambda_\ddagger[$, where $\lambda_\dagger$ and $\lambda_\ddagger$ are defined as per (A.16) and (A.18) given the initial value of γ, respectively, the change of regime is accompanied by a change in first-period compensation conditional on good performance from (A.11) to (A.10) (see also Remark A-3 and Lemma 2.(i)). On the other hand, if $\lambda \leq \lambda_\dagger$, first-period compensation conditional on good performance goes from (A.13) to (A.12). In both cases, the compensation profile $(\bar{C}_1, \bar{C}_2)$ starts higher and has a lower slope as a result, which proves statement (iii). ∎

B The Continuous-Time Setting

In this appendix, we provide a complete description of the environment that we consider in Sections 2 and 3, in which an infinitely-lived firm is run by a sequence of managers who can divert cashflows for their own benefit. We build the model that we study on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ supporting a sequence of independent standard one-dimensional Brownian motions $Z^1, Z^2, \ldots, Z^n, \ldots$ as well as an independent sequence of independent and identically distributed random variables $U^1, U^2, \ldots, U^n, \ldots$, each having the uniform distribution on $[0, 1]$. We denote by $(\mathcal{F}_t^{Z^n})$ the natural filtration of Z^n. We assume that these filtrations as well as any other one we consider in this and the following appendices have been regularised to satisfy the "usual conditions", namely, to be right-continuous and augmented by the $\mathbb{P}$-negligible sets in $\mathcal{F}$.

B.1 The n-th Manager's Contract

In this section, we describe the contract of the n-th manager. To simplify the notation, we let $t = 0$ refer to the time at which the n-th manager takes office. We model the n-th manager's size-adjusted cumulative stealing strategy by an increasing continuous $(\mathcal{F}_t^{Z^n})$-adapted process A^n such that $A_0^n = 0$. We denote by $\mathcal{A}^n$ the family of all such processes. Given a stealing strategy $A^n \in \mathcal{A}^n$, we denote by $(\hat{\mathcal{F}}_t^n) = \big(\hat{\mathcal{F}}_t^n(A)\big)$ the information flow generated by the size-adjusted reported cashflows during the tenure of the n-th manager, which is the natural filtration of the process $\hat{Y}^n$ defined by

$$\hat{Y}_t^n = \mu t - A_t^n + \sigma Z_t^n.$$

It is worth noting that $\hat{\mathcal{F}}_t^n \subseteq \mathcal{F}_t^{Z^n}$ for all $t \geq 0$, with equality holding if $A^n = 0$.

We assume that the firm's growth policy is based on the history of reported cashflows during the tenure of each manager. Accordingly, we model the firm's growth policy during the tenure of the n-th manager by a càdlàg $(\hat{\mathcal{F}}_t^n)$-progressively measurable process process G^n with values in the interval $[0, 1]$. We define the time τ_{g}^n that elapses between the appointment of the n-th manager and his growth-induced dismissal (if he is not fired for disciplinary reasons before) by

$$\tau_{\mathrm{g}}^n = \inf \left\{ t \geq 0 \,\middle|\, \exp\left(-q \int_0^t G_s^n \, ds\right) \leq U^n \right\}, \tag{B.1}$$

with the usual convention that $\inf \emptyset = \infty$. In view of the independence of $(\hat{\mathcal{F}}_t^n)$ and U^n, we can see that

$$\mathbb{P}\left(\tau_{\mathrm{g}}^n > t \mid \hat{\mathcal{F}}_t^n\right) = \mathbb{P}\left(U^n < \exp\left(-q \int_0^t G_s^n \, ds\right) \,\middle|\, \hat{\mathcal{F}}_t^n\right) = \exp\left(-q \int_0^t G_s^n \, ds\right).$$

We assume that disciplinary dismissal is also based on the history of each manager's reported cashflows. Accordingly, the time τ_{d}^n that elapses between the appointment of the n-th manager and his disciplinary firing (if he is not replaced for the sake of taking a growth opportunity before) is an $(\hat{\mathcal{F}}_t^n)$-stopping time. Furthermore, we assume that the manager's compensation is also determined based on the history of reported cashflows. Accordingly, the manager's size-adjusted cumulative compensation is given by an increasing càdlàg $(\hat{\mathcal{F}}_t^n)$-adapted process C^n such that $C_{0-} = 0$, while the manager's size-adjusted severance upon growth-induced dismissal is $S^n_{\tau^n_{\mathrm{g}}}$, where S^n is a positive $(\hat{\mathcal{F}}_t^n)$-progressively measurable process.

Remark B-1. If the firm stands ready to take all growth opportunities, setting $G^n \equiv 1$ for all n, then the times τ^n_{g} are independent and exponentially distributed with parameter q. Indeed, the choice $G^n \equiv 1$ gives rise to the identities

$$\mathbb{P}\left(\tau^n_{\mathrm{g}} > t \mid \hat{\mathcal{F}}_t^n\right) = e^{-qt} = \mathbb{P}(\tau^n_{\mathrm{g}} > t).$$

In other words, the random times between the arrival of two consecutive growth opportunities are independent random variables that are exponentially distributed with parameter q. ∎

For technical reasons, we assume that the family $\mathcal{A}^n$ is restricted to include only processes satisfying the integrability condition

$$\mathbb{E}\left[\int_{[0,\infty[} e^{-\varrho t}\, dA_t^n\right] < \infty. \tag{B.2}$$

Given any such stealing strategy $A^n \in \mathcal{A}^n$, we use the following notation:

$\mathcal{P}^n_C(A^n)$ is the family of all increasing càdlàg $(\hat{\mathcal{F}}_t^n)$-adapted processes C^n such that $C^n_{0-} = 0$;

$\mathcal{P}^n_S(A^n)$ is the family of all positive càdlàg $(\hat{\mathcal{F}}_t^n)$-adapted processes S^n;

$\mathcal{P}^n_G(A^n)$ is the family of all $(\hat{\mathcal{F}}_t^n)$-progressively measurable processes G^n with values in $[0,1]$;

$\mathcal{P}^n_{\tau_{\mathrm{d}}}(A^n)$ is the set of all $(\hat{\mathcal{F}}_t^n)$-stopping times.

These families depend on the choice of $A^n \in \mathcal{A}^n$ through the dependence on A^n of the natural filtration $(\hat{\mathcal{F}}_t^n)$ of the reported cashflows process $\hat{Y}^n$. In particular, it is worth noting that

$$\begin{gathered} \mathcal{P}^n_C(A^n) \subseteq \mathcal{P}^n_C(0), \quad \mathcal{P}^n_S(A^n) \subseteq \mathcal{P}^n_S(0), \\ \mathcal{P}^n_G(A^n) \subseteq \mathcal{P}^n_G(0) \quad \text{and} \quad \mathcal{P}^n_{\tau_{\mathrm{d}}}(A^n) \subseteq \mathcal{P}^n_{\tau_{\mathrm{d}}}(0) \quad \text{for all } A^n \in \mathcal{A}^n \end{gathered} \tag{B.3}$$

because $\hat{\mathcal{F}}_t^n \subseteq \mathcal{F}_t^{Z^n}$ for all $t \geq 0$, with equality if $A^n = 0$. Furthermore, we assume that the families $\mathcal{P}^n_C(A^n)$ and $\mathcal{P}^n_S(A^n)$ are restricted to include only processes satisfying the integrability condition

$$\mathbb{E}\left[\int_{[0,\infty[} e^{-\varrho t}\, dC_t^n + \sup_{t\geq 0}\bigl(e^{-\varrho t} S_t^n\bigr)\right] < \infty. \tag{B.4}$$

We can now introduce the formal definition of the n-th manager's contract.

Definition B-1. A *long-term incentive contract*, or just *incentive contract*, for the n-th manager is a function

$$\Gamma^n = (\Gamma^n_C, \Gamma^n_S, \Gamma^n_G, \Gamma^n_{\tau_\mathrm{d}}) : \mathcal{A}^n \to \mathcal{P}^n_C(0) \times \mathcal{P}^n_S(0) \times \mathcal{P}^n_G(0) \times \mathcal{P}^n_{\tau_\mathrm{d}}(0)$$

such that

$$\Gamma^n_C(A^n) \in \mathcal{P}^n_C(A^n), \quad \Gamma^n_S(A^n) \in \mathcal{P}^n_S(A^n),$$
$$\Gamma^n_G(A^n) \in \mathcal{P}^n_G(A^n) \quad \text{and} \quad \Gamma_{\tau_\mathrm{d}}(A^n) \in \mathcal{P}^n_{\tau_\mathrm{d}}(A^n) \quad \text{for all } A^n \in \mathcal{A}^n.$$

We denote by $\mathcal{G}^n$ the family of all such contracts. ∎

Remark B-2. It would be natural to include additional requirements as part of the definition of a long-term incentive contract. For instance, for any two stealing strategies that coincide up to a certain stopping time, the evaluation of a contract at these two strategies should result in the same compensation and termination outcomes up to that stopping time. We have opted for not spelling out explicitly such constraints in Definition B-1 because they do not affect the remainder of our analysis. ∎

B.2 The Managers' and the Firm's Payoffs

We define

$$\tau^1_\mathrm{h} = 0, \quad \tau^{n+1}_\mathrm{h} = \sum_{j=1}^{n} \tau^j_\mathrm{d} \wedge \tau^j_\mathrm{g}, \quad \Phi^1 = 1 \quad \text{and} \quad \Phi^{n+1} = (1+\gamma)^{\mathbf{1}_{\{\tau^1_\mathrm{g} \le \tau^1_\mathrm{d}\}} + \cdots + \mathbf{1}_{\{\tau^n_\mathrm{g} \le \tau^n_\mathrm{d}\}}}, \qquad \text{(B.5)}$$

for $n \geq 1$, and we note that τ^n_h is the time at which the n-th manager is hired, while Φ^n is the size of the firm during the n-th manager's tenure. Accordingly, $\Phi^n A^n$, $\Phi^n C^n$ and $\Phi^n S^n$ model the n-th manager's actual stealing strategy, cumulative compensation and severance upon dismissal, respectively. We also consider the σ-algebras

$$\mathcal{I}^1 = \{\emptyset, \Omega\} \quad \text{and} \quad \mathcal{I}^{n+1} = \sigma\left(\hat{Y}^j_{t \wedge \tau^j_\mathrm{d} \wedge \tau^j_\mathrm{g}}, \, \tau^j_\mathrm{d}, \, \tau^j_\mathrm{g}, \, j = 1, \ldots, n, \, t \geq 0\right), \qquad \text{(B.6)}$$

for $n \geq 1$, and we note that $\mathcal{I}^n$ is the information that is available to the firm at the hiring time τ^n_h of the n-th manager. In view of the independence of (Z^1, U^1), …, (Z^n, U^n), … and the structure of each manager's contract that we considered in the previous section, we can see that

$$\tau^n_\mathrm{d}, \ \tau^n_\mathrm{g}, \ \Phi^{n+1} \text{ are } \mathcal{I}^{n+1}\text{-measurable}, \qquad \text{(B.7)}$$

while

$$A^j, \ S^j, \ C^j, \ \tau^j_\mathrm{d}, \ \tau^j_\mathrm{g}, \ \frac{\Phi^{j+1}}{\Phi^n}, \text{ for } j \geq n, \text{ are independent of } \mathcal{I}^n. \qquad \text{(B.8)}$$

Given an incentive contract $\Gamma^n \in \mathcal{G}^n$ (see Definition B-1), the n-th manager's total expected discounted payoff as of the time τ_{h}^n of his hiring is given by

$$\tilde{M}^n(\Gamma^n, A^n \mid \mathcal{I}^n) = \mathbb{E}\biggl[\int_{[0,\tau_{\mathrm{d}}^n \wedge \tau_{\mathrm{g}}^n[} e^{-\varrho t}\Phi^n \, dC_t^n + e^{-\varrho \tau_{\mathrm{d}}^n}\Phi^n \Delta C_{\tau_{\mathrm{d}}^n}^n \mathbf{1}_{\{\tau_{\mathrm{d}}^n < \tau_{\mathrm{g}}^n\}} + e^{-\varrho \tau_{\mathrm{g}}^n}\Phi^n S_{\tau_{\mathrm{g}}^n}^n \mathbf{1}_{\{\tau_{\mathrm{g}}^n \le \tau_{\mathrm{d}}^n\} \cap \{\tau_{\mathrm{g}}^n < \infty\}} + \lambda \int_0^{\tau_{\mathrm{d}}^n \wedge \tau_{\mathrm{g}}^n} e^{-\varrho t}\Phi^n \, dA_t^n \biggm| \mathcal{I}^n \biggr].$$

Here,

we write C^n, S^n and τ_{d}^n in place of $\Gamma_C^n(A^n)$, $\Gamma_S^n(A^n)$ and $\Gamma_{\tau_{\mathrm{d}}}^n(A^n)$,

respectively, and we note that

τ_{g}^n is defined as in (B.1) for $G^n = \Gamma_G^n(A^n)$.

In view of (B.7)–(B.8), we can see that this expression is equivalent to

$$\begin{aligned}\tilde{M}^n(\Gamma^n, A^n \mid \mathcal{I}^n) = \Phi^n \, \mathbb{E}\biggl[\int_{[0,\tau_{\mathrm{d}}^n \wedge \tau_{\mathrm{g}}^n[} e^{-\varrho t} \, dC_t^n + e^{-\varrho \tau_{\mathrm{d}}^n} \Delta C_{\tau_{\mathrm{d}}^n}^n \mathbf{1}_{\{\tau_{\mathrm{d}}^n < \tau_{\mathrm{g}}^n\}} \\ + e^{-\varrho \tau_{\mathrm{g}}^n} S_{\tau_{\mathrm{g}}^n}^n \mathbf{1}_{\{\tau_{\mathrm{g}}^n \le \tau_{\mathrm{d}}^n\} \cap \{\tau_{\mathrm{g}}^n < \infty\}} + \lambda \int_0^{\tau_{\mathrm{d}}^n \wedge \tau_{\mathrm{g}}^n} e^{-\varrho t} \, dA_t^n \biggr] \\ =: \Phi^n M(\Gamma^n, A^n), \qquad \text{(B.9)}\end{aligned}$$

where $M(\Gamma^n, A^n)$ is the n-th manager's size-adjusted total expected discounted payoff as of time τ_{h}^n.

From this point onward, we restrict our attention to admissible contracts. These contracts are such that they make no stealing optimal for the manager, i.e., no stealing is "incentive compatible". Additionally, these contracts are such that the manager's size-adjusted expected discounted compensation under no stealing is greater than or equal to the size-adjusted reservation value $\bar{w}$.

Definition B-2. An *admissible long-term incentive contract*, or just *admissible contract*, for the n-th manager is any incentive contract $\Gamma^n \in \mathcal{G}^n$ (see Definition B-1) satisfying the admissibility constraints

$$M(\Gamma^n, 0) = \sup_{A^n \in \mathcal{A}^n} M(\Gamma^n, A^n) \quad \text{and} \quad M(\Gamma^n, 0) \ge \bar{w}. \qquad \text{(B.10)}$$

We denote by $\mathcal{G}_{\mathrm{a}}^n \subseteq \mathcal{G}^n$ the family of all such contracts. ∎

Remark B-3. The inequality $M(\Gamma^n, 0) \ge \bar{w}$ reflects the fact that the firm must offer to each manager a total discounted payoff that is at least equal to the manager's reservation value. Note that the firm may be willing to offer a manager a total discounted payoff that is *strictly* greater than his reservation value as long as this enhances the firm's own expected discounted profit. ∎

We henceforth assume that the firm offers admissible contracts to all managers. Furthermore, we make the usual assumption that, if no stealing is incentive compatible, then managers refrain from stealing, namely, $A^n = 0$ for all $n \geq 1$. The expected discounted profit as of time τ_{h}^n that the firm receives during the tenure of the n-th manager is

$$\begin{aligned}\tilde{\Pi}^n(\Gamma^n \mid \mathcal{I}^n) = \Phi^n \, \mathbb{E}\bigg[\int_0^{\tau_{\mathrm{d}}^n \wedge \tau_{\mathrm{g}}^n} e^{-rt} \mu \, dt - \int_{[0, \tau_{\mathrm{d}}^n \wedge \tau_{\mathrm{g}}^n[} e^{-rt} \, dC_t^n \\ - e^{-r\tau_{\mathrm{d}}^n} \Delta C_{\tau_{\mathrm{d}}^n}^n \mathbf{1}_{\{\tau_{\mathrm{d}}^n < \tau_{\mathrm{g}}^n\}} - e^{-r\tau_{\mathrm{g}}^n} S_{\tau_{\mathrm{g}}^n}^n \mathbf{1}_{\{\tau_{\mathrm{g}}^n \leq \tau_{\mathrm{d}}^n\} \cap \{\tau_{\mathrm{g}}^n < \infty\}} \bigg] \\ =: \Phi^n \Pi(\Gamma^n), \end{aligned} \tag{B.11}$$

where $\Pi(\Gamma^n)$ is the size-adjusted expected discounted profit as of time τ_{h}^n that the firm receives during the tenure of the n-th manager. Here, as well as in what follows,

$$\text{we write } C^n, \; S^n \text{ and } \tau_{\mathrm{d}}^n \text{ in place of } \Gamma_C^n(0), \; \Gamma_S^n(0) \text{ and } \Gamma_{\tau_{\mathrm{d}}}^n(0),$$

respectively, and we note that

$$\tau_{\mathrm{g}}^n \text{ is defined as in (B.1) for } G^n = \Gamma_G^n(0).$$

In view of (B.7)–(B.8), we can see that the expected discounted profit of the firm at the start of the n-th manager's tenure is

$$\begin{aligned}\tilde{F}^n\left((\Gamma^j)_{j \geq n} \mid \mathcal{I}^n\right) &= \mathbb{E}\left[\Phi^n \Pi(\Gamma^n) + \sum_{j=n+1}^{\infty} e^{-r(\tau_{\mathrm{h}}^j - \tau_{\mathrm{h}}^n)} \left(\Phi^j \Pi(\Gamma^j) - \Phi^{j-1} \kappa \right) \,\middle|\, \mathcal{I}^n \right] \\ &= \Phi^n \left(\Pi(\Gamma^n) + \mathbb{E}\left[\sum_{j=n+1}^{\infty} e^{-r(\tau_{\mathrm{h}}^j - \tau_{\mathrm{h}}^n)} \left(\frac{\Phi^j}{\Phi^n} \Pi(\Gamma^j) - \frac{\Phi^{j-1}}{\Phi^n} \kappa \right) \right] \right) \\ &=: \Phi^n F^n\left((\Gamma^j)_{j \geq n}\right), \end{aligned}$$

where $F^n\left((\Gamma^j)_{j \geq n}\right)$ is the size-adjusted expected discounted profit of the firm at the start of the n-th manager's tenure. Note that this expression incorporates the turnover costs associated with the n-th manager and his successors. In view of the identities

$$\begin{aligned}\tau_{\mathrm{h}}^j - \tau_{\mathrm{h}}^n &= \tau_{\mathrm{d}}^n \wedge \tau_{\mathrm{g}}^n + \cdots + \tau_{\mathrm{d}}^{j-1} \wedge \tau_{\mathrm{g}}^{j-1} \\ &= \tau_{\mathrm{d}}^n \wedge \tau_{\mathrm{g}}^n + \tau_{\mathrm{h}}^j - \tau_{\mathrm{h}}^{n+1} \quad \text{for all } j \geq n+1, \end{aligned}$$

which follow from (B.5), we can see that

$$\begin{aligned}
&\mathbb{E}\Bigg[\sum_{j=n+1}^{\infty} e^{-r(\tau_{\mathrm{h}}^{j}-\tau_{\mathrm{h}}^{n})}\left(\frac{\Phi^{j}}{\Phi^{n}}\Pi(\Gamma^{j})-\frac{\Phi^{j-1}}{\Phi^{n}}\kappa\right)\Bigg] \\
&\quad=\mathbb{E}\Bigg[e^{-r(\tau_{\mathrm{d}}^{n}\wedge\tau_{\mathrm{g}}^{n})}\Bigg\{\frac{\Phi^{n+1}}{\Phi^{n}}\Pi(\Gamma^{n+1})+\sum_{j=n+2}^{\infty} e^{-r(\tau_{\mathrm{h}}^{j}-\tau_{\mathrm{h}}^{n+1})}\left(\frac{\Phi^{j}}{\Phi^{n}}\Pi(\Gamma^{j})-\frac{\Phi^{j-1}}{\Phi^{n}}\kappa\right)-\kappa\Bigg\}\Bigg] \\
&\quad=\mathbb{E}\Bigg[e^{-r(\tau_{\mathrm{d}}^{n}\wedge\tau_{\mathrm{g}}^{n})}\frac{1}{\Phi^{n}}\Bigg\{\mathbb{E}\Bigg[\Phi^{n+1}\Pi(\Gamma^{n+1}) \\
&\qquad\quad+\sum_{j=n+2}^{\infty} e^{-r(\tau_{\mathrm{h}}^{j}-\tau_{\mathrm{h}}^{n+1})}\Big(\Phi^{j}\Pi(\Gamma^{j})-\Phi^{j-1}\kappa\Big)\;\Big|\;\mathcal{I}^{n+1}\Bigg]-\Phi^{n}\kappa\Bigg\}\Bigg] \\
&\quad=\mathbb{E}\left[e^{-r(\tau_{\mathrm{d}}^{n}\wedge\tau_{\mathrm{g}}^{n})}\frac{\Phi^{n+1}}{\Phi^{n}}F^{n+1}\left((\Gamma^{j})_{j\geq n+1}\right)-e^{-r(\tau_{\mathrm{d}}^{n}\wedge\tau_{\mathrm{g}}^{n})}\kappa\right].
\end{aligned}$$

We thus obtain the recursive expression

$$F^{n}\left((\Gamma^{j})_{j\geq n}\right)=\Pi(\Gamma^{n})+\mathbb{E}\left[e^{-r(\tau_{\mathrm{d}}^{n}\wedge\tau_{\mathrm{g}}^{n})}\frac{\Phi^{n+1}}{\Phi^{n}}F^{n+1}\left((\Gamma^{j})_{j\geq n+1}\right)-e^{-r(\tau_{\mathrm{d}}^{n}\wedge\tau_{\mathrm{g}}^{n})}\kappa\right].$$

Combining this result with the calculation

$$\begin{aligned}
\mathbb{E}\left[e^{-r(\tau_{\mathrm{d}}^{n}\wedge\tau_{\mathrm{g}}^{n})}\frac{\Phi^{n+1}}{\Phi^{n}}\right]&=\mathbb{E}\left[e^{-r(\tau_{\mathrm{d}}^{n}\wedge\tau_{\mathrm{g}}^{n})}(1+\gamma)^{\mathbf{1}_{\{\tau_{\mathrm{g}}^{n}\leq\tau_{\mathrm{d}}^{n}\}}}\right] \\
&=\mathbb{E}\left[e^{-r\tau_{\mathrm{d}}^{n}}\mathbf{1}_{\{\tau_{\mathrm{d}}^{n}<\tau_{\mathrm{g}}^{n}\}}+(1+\gamma)e^{-r\tau_{\mathrm{g}}^{n}}\mathbf{1}_{\{\tau_{\mathrm{g}}^{n}\leq\tau_{\mathrm{d}}^{n}\}}\right]
\end{aligned}$$

and (B.11), we can see that the size-adjusted expected discounted profit of the firm at the hiring time τ_{h}^{n} of the n-th manager satisfies the recursive equation

$$\begin{aligned}
&F^{n}\left((\Gamma^{j})_{j\geq n}\right) \\
&\quad=\mathbb{E}\Bigg[\int_{0}^{\tau_{\mathrm{d}}^{n}\wedge\tau_{\mathrm{g}}^{n}} e^{-rt}\mu\,dt-\int_{[0,\tau_{\mathrm{d}}^{n}\wedge\tau_{\mathrm{g}}^{n}[} e^{-rt}\,dC_{t}^{n} \\
&\qquad+e^{-r\tau_{\mathrm{d}}^{n}}\left[F^{n+1}\left((\Gamma^{j})_{j\geq n+1}\right)-\Delta C_{\tau_{\mathrm{d}}^{n}}^{n}-\kappa\right]\mathbf{1}_{\{\tau_{\mathrm{d}}^{n}<\tau_{\mathrm{g}}^{n}\}} \\
&\qquad+e^{-r\tau_{\mathrm{g}}^{n}}\left[(1+\gamma)F^{n+1}\left((\Gamma^{j})_{j\geq n+1}\right)-S_{\tau_{\mathrm{g}}^{n}}^{n}-\kappa\right]\mathbf{1}_{\{\tau_{\mathrm{g}}^{n}\leq\tau_{\mathrm{d}}^{n}\}\cap\{\tau_{\mathrm{g}}^{n}<\infty\}}\Bigg]. \qquad \text{(B.12)}
\end{aligned}$$

B.3 The Firm's Problem: Take One

The uncertainty under the n-th manager's tenure is driven by (Z^n, U^n), which is an independent copy of (Z^1, U^1). In particular, $(\mathcal{A}^n)$ is a sequence of independent copies of $\mathcal{A}^1$. As a result, if a contract is admissible (in the sense of Definition B-2) for the first manager, then the same contract is admissible for all managers. In view of this observation and a simple induction argument along the lines that lead to the recursive equation (B.12), we can see that it is optimal for the firm seeking to implement no stealing to give the same admissible contract to all managers.

If all successive managers are offered the same contract $\Gamma \in \mathcal{G}_{\mathrm{a}}$, then the firm's size-adjusted expected discounted profit at the start of any manager's tenure is

$$F^1\big((\Gamma,\ldots,\Gamma,\ldots)\big) = F^n\big((\Gamma,\ldots,\Gamma,\ldots)\big) =: F(\Gamma) \quad \text{for all } n \geq 1.$$

In particular, (B.12) implies that

$$\begin{aligned} F(\Gamma) = \mathbb{E}\bigg[\int_0^{\tau_{\mathrm{d}}\wedge\tau_{\mathrm{g}}} e^{-rt}\mu\,dt - \int_{[0,\tau_{\mathrm{d}}\wedge\tau_{\mathrm{g}}[} e^{-rt}\,dC_t + e^{-r\tau_{\mathrm{d}}}\Big[F(\Gamma) - \Delta C_{\tau_{\mathrm{d}}} - \kappa\Big]\mathbf{1}_{\{\tau_{\mathrm{d}}<\tau_{\mathrm{g}}\}} \\ + e^{-r\tau_{\mathrm{g}}}\Big[(1+\gamma)F(\Gamma) - S_{\tau_{\mathrm{g}}} - \kappa\Big]\mathbf{1}_{\{\tau_{\mathrm{g}}\leq\tau_{\mathrm{d}}\}\cap\{\tau_{\mathrm{g}}<\infty\}}\bigg], \end{aligned} \tag{B.13}$$

where C, S, and τ_{d} stand for $\Gamma_C(0)$, $\Gamma_S(0)$ and $\Gamma_{\tau_{\mathrm{d}}}(0)$, respectively, and τ_{g} is defined as in (B.1) for $G = \Gamma_G(0)$.

We conclude with the statement of the contracting problem that the firm faces.

Problem B-1. Determine an admissible contract $\Gamma^\star \in \mathcal{G}_{\mathrm{a}}$ (see Definition B-2) such that

$$F(\Gamma^\star) = \sup_{\Gamma\in\mathcal{G}_{\mathrm{a}}} F(\Gamma).$$

∎

C Admissible Dynamic Contracts

In light of the analysis in Appendix B, we now focus on the first manager's contract. To simplify the notation, we write Z, $(\mathcal{F}_t^Z)$, $\mathcal{A}$, $(\hat{\mathcal{F}}_t) = \big(\hat{\mathcal{F}}_t(A)\big)$, etc, instead of Z^1, $(\mathcal{F}_t^{Z^1})$, $\mathcal{A}^n$, $(\hat{\mathcal{F}}_t^n) = \big(\hat{\mathcal{F}}_t^n(A)\big)$, etc, in what follows. In particular, given a stealing strategy $A \in \mathcal{A}$,

$$
\begin{aligned}
\mathcal{P}_C(A) &\text{ is the family of all increasing càdlàg } (\hat{\mathcal{F}}_t)\text{-adapted process } C \\
&\quad \text{such that } C_{0-} = 0; && \text{(C.1)} \\
\mathcal{P}_S(A) &\text{ is the family of all positive càdlàg } (\hat{\mathcal{F}}_t)\text{-adapted processes } S; && \text{(C.2)} \\
\mathcal{P}_G(A) &\text{ is the family of all } (\hat{\mathcal{F}}_t)\text{-progressively measurable processes } G \\
&\quad \text{with values in } [0,1]; && \text{(C.3)} \\
\mathcal{P}_{\tau_\mathrm{d}}(A) &\text{ is the set of all } (\hat{\mathcal{F}}_t)\text{-stopping times} && \text{(C.4)}
\end{aligned}
$$

(recall that the families $\mathcal{P}_C(A)$ and $\mathcal{P}_S(A)$ are restricted to include only processes satisfying the integrability condition (B.4)). We will also need the filtration $(\mathcal{F}_t^{Z,\tau_\mathrm{g}})$ that is larger than $(\mathcal{F}_t^Z)$ and incorporates the information on the occurrence of growth-induced termination, which is defined by

$$
\mathcal{F}_t^{Z,\tau_\mathrm{g}} = \mathcal{F}_t^Z \vee \sigma\Big(\{\tau_g \le s\}\,,\, s \le t\Big)
$$

(see also the discussion on filtrations at the very beginning of Appendix B). Notice that if the manager refrains from stealing ($A = 0$), then $\hat{\mathcal{F}}_t = \mathcal{F}_t^Z$ for all $t \ge 0$, in which case the sigma-algebra $\mathcal{F}_t$ introduced in Section 2.2 coincides with $\mathcal{F}_t^{Z,\tau_\mathrm{g}}$.

C.1 Proof of Lemma 5

Lemma 5 is a direct consequence of the following result (see Remark C-1 below).

Lemma C-1. *Consider any processes $C \in \mathcal{P}_C(0)$, $S \in \mathcal{P}_S(0)$, $G \in \mathcal{P}_G(0)$ together with any stopping time $\tau_\mathrm{d} \in \mathcal{P}_{\tau_\mathrm{d}}(0)$, and let τ_g be the random time that is defined as in (B.1). Also, consider the processes M, $\tilde{M}$ defined by*

$$
\begin{aligned}
M_t = \mathbf{1}_{\{t<\tau_\mathrm{d}\wedge\tau_\mathrm{g}\}}\, \mathbb{E}\bigg[\int_{]t,\tau_\mathrm{d}\wedge\tau_\mathrm{g}[} & e^{-\varrho(s-t)}\, dC_s \\
& + e^{-\varrho(\tau_\mathrm{d}-t)} \Delta C_{\tau_\mathrm{d}} \mathbf{1}_{\{\tau_\mathrm{d}<\tau_\mathrm{g}\}} + e^{-\varrho(\tau_\mathrm{g}-t)} S_{\tau_\mathrm{g}} \mathbf{1}_{\{\tau_\mathrm{g}\le\tau_\mathrm{d}\}\cap\{\tau_\mathrm{g}<\infty\}} \;\Big|\; \mathcal{F}_t^{Z,\tau_\mathrm{g}}\bigg], \\
\tilde{M}_t = \mathbf{1}_{\{t<\tau_\mathrm{d}\}} \mathfrak{d}_t^{-1}\, \mathbb{E}\bigg[\int_{]t,\tau_\mathrm{d}]\cap\mathbb{R}_+} & \mathfrak{d}_s\, dC_s + q\int_t^{\tau_\mathrm{d}} \mathfrak{d}_s G_s S_s\, ds \;\Big|\; \mathcal{F}_t^Z\bigg],
\end{aligned}
$$

where

$$
\mathfrak{d}_t = \exp\left(-\varrho t - \int_0^t qG_s\, ds\right), \tag{C.5}
$$

and let $T^0_{\tilde M}$ be the first hitting time of 0 by $\tilde M$, namely,

$$T^0_{\tilde M} = \inf\{t \geq 0 \mid \ \tilde M_t = 0\}.$$

The following statements hold true:
(I) $M_t = \mathbf{1}_{\{t<\tau_{\mathrm{g}}\}} \tilde M_t$ *for all* $t \geq 0$.
(II) *There exists an* $(\mathcal{F}^Z_t)$*-progressively measurable process* β *such that* $\int_0^T \mathfrak{d}_t^2 \beta_t^2 \, dt < \infty$ *for all* $T > 0$, *and*

$$dM_t = \big[\varrho M_t + qG_t(M_t - S_t)\big]\, dt - dC_t + \sigma\beta_t \, dZ_t \tag{C.6}$$

on the event $\{t < \tau_{\mathrm{d}} \wedge \tau_{\mathrm{g}}\}$.
(III) $T^0_{\tilde M} \leq \tau_{\mathrm{d}}$, *and the processes* $(C - C_{T^0_{\tilde M}})\mathbf{1}_{]T^0_{\tilde M}, \tau_{\mathrm{d}}]}$, $GS\mathbf{1}_{]T^0_{\tilde M}, \tau_{\mathrm{d}}]}$ *and* $\beta\mathbf{1}_{]T^0_{\tilde M}, \tau_{\mathrm{d}}]}$ *are indistinguishable from 0.*

Proof. Claim (I) follows from standard credit risk theory (e.g., see Bielecki and Rutkowski (2002), Section 5.1.1). To establish the rest of the claims, we first observe that

$$\begin{aligned}
\mathbb{E}\bigg[\int_{[0,\tau_{\mathrm{d}}]\cap\mathbb{R}_+} \mathfrak{d}_s \, dC_s + q\int_0^{\tau_{\mathrm{d}}} \mathfrak{d}_s G_s S_s \, ds \ \Big| \ \mathcal{F}^Z_t\bigg]& \\
= \int_{[0,t\wedge\tau_{\mathrm{d}}]} \mathfrak{d}_s \, dC_s &+ q\int_0^{t\wedge\tau_{\mathrm{d}}} \mathfrak{d}_s G_s S_s \, ds \\
&+ \mathbf{1}_{\{t<\tau_{\mathrm{d}}\}} \mathbb{E}\left[\int_{]t,\tau_{\mathrm{d}}]\cap\mathbb{R}_+} \mathfrak{d}_s \, dC_s + q\int_t^{\tau_{\mathrm{d}}} \mathfrak{d}_s G_s S_s \, ds \ \Big| \ \mathcal{F}^Z_t\right] \\
= \int_{[0,t\wedge\tau_{\mathrm{d}}]} \mathfrak{d}_s \, dC_s &+ q\int_0^{t\wedge\tau_{\mathrm{d}}} \mathfrak{d}_s G_s S_s \, ds + \mathfrak{d}_t \tilde M_t.
\end{aligned}$$

In view of the martingale representation theorem, there exists an $(\mathcal{F}^Z_t)$-progressively measurable process β such that $\int_0^T \mathfrak{d}_s^2 \beta_s^2 \, ds < \infty$ for all $T > 0$, and

$$\mathbb{E}\left[\int_{[0,\tau_{\mathrm{d}}]\cap\mathbb{R}_+} \mathfrak{d}_s \, dC_s + q\int_0^{\tau_{\mathrm{d}}} \mathfrak{d}_s G_s S_s \, ds \ \Big| \ \mathcal{F}^Z_t\right] = \tilde M_{0-} + \int_0^t \mathfrak{d}_s \beta_s \, dZ_s,$$

where

$$\tilde M_{0-} = \mathbb{E}\left[\int_{[0,\tau_{\mathrm{d}}]\cap\mathbb{R}_+} \mathfrak{d}_s \, dC_s + q\int_0^{\tau_{\mathrm{d}}} \mathfrak{d}_s G_s S_s \, ds\right] = \Delta C_0 + \tilde M_0.$$

Rearranging terms, we obtain

$$\mathfrak{d}_t \tilde M_t = \tilde M_{0-} - \int_{[0,t\wedge\tau_{\mathrm{d}}]} \mathfrak{d}_s \, dC_s - q\int_0^{t\wedge\tau_{\mathrm{d}}} \mathfrak{d}_s G_s S_s \, ds + \int_0^t \mathfrak{d}_s \beta_s \, dZ_s. \tag{C.7}$$

Using the definition of $\mathfrak{d}$ in (C.5), (C.7), and the integration by parts formula, we can see that $\tilde M$ satisfies

$$d\tilde M_t = \big[\varrho \tilde M_t + qG_t(\tilde M_t - S_t)\big]\, dt - dC_t + \sigma\beta_t \, dZ_t$$

on the event $\{t < \tau_{\rm d}\}$, which, combined with part (I) of the lemma, implies the claim in (II). Furthermore, the definitions of $\tilde{M}$ and $T^0_{\tilde{M}}$, along with (C.7), imply all the properties listed in (III). ∎

Remark C-1. It is straightforward to see that Lemma 5 follows immediately from Lemma C-1. Indeed, consider a long-term incentive contract $\Gamma \in \mathcal{G}$ and suppose the manager refrains from stealing, so that $\mathcal{F}_t^{Z,\tau_{\rm g}}$ represents all the information accumulated by the firm up to time t. Also, let $C = \Gamma_C(0)$, $S = \Gamma_S(0)$ and $\tau_{\rm d} = \Gamma_{\tau_{\rm d}}(0)$, and let $\tau_{\rm g}$ be the random time that is defined as in (B.1) for $G = \Gamma_G(0)$. In view of Definition B-1, $C \in \mathcal{P}_C(0)$, $S \in \mathcal{P}_S(0)$, $G \in \mathcal{P}_G(0)$ and $\tau_{\rm d} \in \mathcal{P}_{\tau_{\rm d}}(0)$. Therefore, the results of Lemma C-1 apply, and Lemma 5 follows from Claim (II). ∎

C.2 Dynamic Contracts

Under no stealing, the dynamics given by (C.6) identify with

$$dM_t = \big[\varrho M_t + qG_t(M_t - S_t)\big]\, dt - dC_t + \beta_t\, (d\hat{Y}_t - \mu\, dt)$$

because, in this case, $\sigma\, dZ_t = d\hat{Y}_t - \mu\, dt$. This observation motivates us to restrict our attention to "dynamic" contracts that track the state process W whose stochastic dynamics are modelled by

$$\begin{aligned} dW_t &= \big[\varrho W_t + qG_t(W_t - S_t)\big]\, dt - dC_t + \beta_t\, (d\hat{Y}_t - \mu\, dt) \\ &= \big[\varrho W_t + qG_t(W_t - S_t)\big]\, dt - dC_t - \beta_t\, dA_t + \sigma\beta_t\, dZ_t, \end{aligned} \tag{C.8}$$

the second equality following from the fact that, in general, $d\hat{Y}_t - \mu\, dt = -dA_t + \sigma dZ_t$.

In view of these considerations, we adopt the following definition of dynamic contracts, where, in line with (C.1)–(C.4),

$$\begin{aligned} &\mathcal{P}_\beta(A) \text{ is the family of all positive } (\hat{\mathcal{F}}_t)\text{-progressively measurable processes } \beta \\ &\qquad \text{such that } \mathbb{E}\left[\int_0^\infty e^{-2rt}\beta_t^2\, dt\right] < \infty, \end{aligned} \tag{C.9}$$

where $r < \varrho$ is the firm's discount rate. Note that, for the purposes of our analysis below, we impose technical conditions on the process β that are stronger than the ones appearing in Lemma C-1.(II).

Definition C-1. A *dynamic contract* is a function

$$\overline{\rm D} = (\overline{\rm D}_C, \overline{\rm D}_S, \overline{\rm D}_G, \overline{\rm D}_{\tau_{\rm d}}, \overline{\rm D}_\beta) : \mathcal{A} \to \mathcal{P}_C(0) \times \mathcal{P}_S(0) \times \mathcal{P}_G(0) \times \mathcal{P}_{\tau_{\rm d}}(0) \times \mathcal{P}_\beta(0)$$

together with a constant $w_{\rm init} > 0$ such that

(I) $(\overline{\rm D}_C, \overline{\rm D}_S, \overline{\rm D}_G, \overline{\rm D}_{\tau_{\rm d}})$ is a contract in the sense of Definition B-1;

(II) $\overline{\rm D}_\beta(A) \in \mathcal{P}_\beta(A)$ for all $A \in \mathcal{A}$;

(III) given any $A \in \mathcal{A}$, the solution W to the SDE (C.8) for

$$C = \overline{\rm D}_C(A), \quad S = \overline{\rm D}_S(A), \quad G = \overline{\rm D}_G(A), \quad \beta = \overline{\rm D}_\beta(A) \quad \text{and} \quad W_{0-} = w_{\rm init},$$

is such that

(a) $\Delta C_t \le W_{t-}$ for all $t \le \tau_{\mathrm{d}} := \overline{\mathrm{D}}_{\tau_{\mathrm{d}}}(A)$, and

(b) if $T_W^0 = \inf\{t \ge 0 \mid W_t = 0\}$, then $T_W^0 \le \tau_{\mathrm{d}}$ and the processes $(C - C_{T_W^0})\mathbf{1}_{]T_W^0, \tau_{\mathrm{d}}]}$, $GS\mathbf{1}_{]T_W^0, \tau_{\mathrm{d}}]}$ and $\beta\mathbf{1}_{]T_W^0, \tau_{\mathrm{d}}]}$ are indistinguishable from 0.

We denote by $\overline{\mathcal{D}}^{w_{\mathrm{init}}}$ the family of all dynamic contracts that are associated with the initial condition $W_{0-} = w_{\mathrm{init}}$. ∎

Remark C-2. From a mathematical point of view, the *raison d'être* of the extra component that differentiates dynamic contracts from contracts in the sense of Definition B-1, namely, the process $\beta = \overline{\mathrm{D}}_\beta(A)$, is to give sense to the constraints in (III) that involve the solution W to the SDE (C.8). The constraints in (III).(b) are imposed in the definition of a dynamic contract to mimic the properties stated in Lemma C-1.(III). ∎

Remark C-3. The general definition of a dynamic contract that we have adopted here is slightly different from the one given in Section 2.4, in the sense that Definition C-1 allows for any random disciplinary dismissal time $\tau_{\mathrm{d}} \ge T_W^0$, whereas the exposition in the main text assumes that this random time identifies with the first hitting time of zero by W, namely, $\tau_{\mathrm{d}} = T_W^0$. The advantage of adopting a more general definition, as we do here, is that it allows us to derive the identity $\tau_{\mathrm{d}} = T_W^0$ as an incentive-compatibility requirement (see Lemma C-2). However, note that the notion of admissible dynamic contract that we define in the next section (see Definition C-3) coincides with the one introduced in Section 2.4. ∎

We can now introduce a general notion of admissibility for dynamic contracts, in line with Definition B-2. Before doing so, we note that, given a dynamic contract $\overline{\mathrm{D}}$ and a stealing strategy A, the same credit risk theory results as the ones we used in the proof of Lemma C-1.(I) imply that the manager's total expected discounted payoff at the time of his hiring, which is defined by (B.9), admits the expression

$$M(\overline{\mathrm{D}}, A) = \mathbb{E}\left[\int_{[0,\tau_{\mathrm{d}}]\cap\mathbb{R}_+} \mathfrak{d}_t \left(dC_t + \lambda\, dA_t + qG_tS_t\, dt\right)\right], \tag{C.10}$$

where $(C, S, G, \tau_{\mathrm{d}}) = \big(\overline{\mathrm{D}}_C(A), \overline{\mathrm{D}}_S(A), \overline{\mathrm{D}}_G(A), \overline{\mathrm{D}}_{\tau_{\mathrm{d}}}(A)\big)$ and $\mathfrak{d}$ is defined by (C.5).

Definition C-2. A dynamic contract $\overline{\mathrm{D}} = (\overline{\mathrm{D}}_C, \overline{\mathrm{D}}_S, \overline{\mathrm{D}}_G, \overline{\mathrm{D}}_{\tau_{\mathrm{d}}}, \overline{\mathrm{D}}_\beta) \in \overline{\mathcal{D}}^{w_{\mathrm{init}}}$ is *generally admissible* if

$$M(\overline{\mathrm{D}}, 0) = \max_{A\in\mathcal{A}} M(\overline{\mathrm{D}}, A) \quad \text{and} \quad M(\overline{\mathrm{D}}, 0) \ge \bar{w}, \tag{C.11}$$

where the manager's total expected discounted payoff M is defined by (C.10).

We denote by $\overline{\mathcal{D}}_{\mathrm{ga}}^{w_{\mathrm{init}}} \subseteq \overline{\mathcal{D}}^{w_{\mathrm{init}}}$ the family of all generally admissible dynamic contracts with initial condition $W_{0-} = w_{\mathrm{init}}$. ∎

C.3 Proof of Lemma 6

The following result establishes sufficient conditions for a dynamic contract to be generally admissible.

Lemma C-2. *Consider a dynamic contract $\overline{\mathrm{D}} \in \overline{\mathcal{D}}^{w_{\mathrm{init}}}$ in the sense of Definition C-1. If*

$$\overline{\mathrm{D}}_\beta(A) \geq \lambda \quad \text{and} \quad \overline{\mathrm{D}}_{\tau_{\mathrm{d}}}(A) = T_W^0 \quad \text{for all } A \in \mathcal{A}, \tag{C.12}$$

and the associated solution to (C.8) for $A = 0$ satisfies the transversality condition

$$\lim_{T\to\infty} e^{-rT}\mathbb{E}\left[W_T \mathbf{1}_{\{T\leq\tau_{\mathrm{d}}\}}\right] = 0, \tag{C.13}$$

where r is the firm's discount rate, then

$$w_{\mathrm{init}} = M(\overline{\mathrm{D}}, 0) = \max_{A\in\mathcal{A}} M(\overline{\mathrm{D}}, A).$$

If (C.12)–(C.13) hold true and $w_{\mathrm{init}} \geq \bar{w}$, then $\overline{\mathrm{D}}$ is generally admissible in the sense of Definition C-2.

Proof. Consider any dynamic contract $\overline{\mathrm{D}} = (\overline{\mathrm{D}}_C, \overline{\mathrm{D}}_S, \overline{\mathrm{D}}_G, \overline{\mathrm{D}}_{\tau_{\mathrm{d}}}, \overline{\mathrm{D}}_\beta) \in \overline{\mathcal{D}}^{w_{\mathrm{init}}}$ and let C, S, G, τ_{d}, β be the evaluations of the contract at any given $A \in \mathcal{A}$. Using (C.5), (C.8), and the integration by parts formula, we calculate

$$\mathfrak{d}_{T\wedge\tau_{\mathrm{d}}} W_{T\wedge\tau_{\mathrm{d}}} = w_{\mathrm{init}} - \int_{[0,T\wedge\tau_{\mathrm{d}}]} \mathfrak{d}_t \left(dC_t + \beta_t\, dA_t + qG_tS_t\, dt\right) + \sigma N_{T\wedge\tau_{\mathrm{d}}}, \tag{C.14}$$

where N is the stochastic integral defined by

$$N_T = \int_0^T \mathfrak{d}_t \beta_t\, dZ_t.$$

In view of (C.14) and the positivity of the stopped process $W^{\tau_{\mathrm{d}}}$, which follows from the properties of a dynamic contract, we can see that

$$0 \leq \mathfrak{d}_{T\wedge\tau_{\mathrm{d}}} W_{T\wedge\tau_{\mathrm{d}}} \leq w_{\mathrm{init}} + \sigma N_{T\wedge\tau_{\mathrm{d}}}. \tag{C.15}$$

On the other hand, Doob's L^2-inequality, Itô's isometry, (C.5) and (C.9) imply that

$$\begin{aligned}\mathbb{E}\left[\left(\sup_{T\geq 0}|N_T|\right)^2\right] &\leq 4\sup_{T\geq 0}\mathbb{E}\left[N_T^2\right] \\ &= 4\sup_{T\geq 0}\mathbb{E}\left[\int_0^T \mathfrak{d}_t^2\beta_t^2\, dt\right] \leq 4\,\mathbb{E}\left[\int_0^\infty e^{-2rt}\beta_t^2\, dt\right] < \infty,\end{aligned} \tag{C.16}$$

therefore, N is a martingale in H^2.

Taking expectations in (C.14) and using the monotone convergence theorem, we derive the expression

$$\begin{aligned} w_{\text{init}} &= \mathbb{E}\left[\int_{[0,\tau_{\text{d}}]\cap\mathbb{R}_+} \mathfrak{d}_t\left(dC_t + \beta_t\, dA_t + qG_tS_t\, dt\right)\right] + \lim_{T\to\infty} \mathbb{E}\left[\mathfrak{d}_{T\wedge\tau_{\text{d}}} W_{T\wedge\tau_{\text{d}}}\right] \\ &= M\big(\overline{\mathrm{D}}, A\big) + \mathbb{E}\left[\int_0^{\tau_{\text{d}}} \mathfrak{d}_t\left(\beta_t - \lambda\right) dA_t\right] + \lim_{T\to\infty} \mathbb{E}\left[\mathfrak{d}_{T\wedge\tau_{\text{d}}} W_{T\wedge\tau_{\text{d}}}\right], \end{aligned}$$

where M is the manager's total expected discounted payoff, which is defined by (C.10). In light of this calculation and the positivity of the stopped process $W^{\tau_{\text{d}}}$, we can see that, if $\beta_t \geq \lambda$ for all $t \leq \tau_{\text{d}}$, which can be true only if $\tau_{\text{d}} = T_W^0$ (see requirement (b) in Definition C-1.(III)), and the transversality condition (C.13) holds true, then

$$M\big(\overline{\mathrm{D}}, A\big) \leq w_{\text{init}} \quad \text{for all } A \in \mathcal{A}$$

and

$$M\big(\overline{\mathrm{D}}, 0\big) = w_{\text{init}} - \lim_{T\to\infty} \mathbb{E}\left[\mathfrak{d}_{T\wedge\tau_{\text{d}}} W_{T\wedge\tau_{\text{d}}}\right] = w_{\text{init}},$$

the second equality following from (C.13) and the fact that $\mathfrak{d}_T < e^{-rT}$ for all $T > 0$. We conclude that (C.12), (C.13) and the inequality $w_{\text{init}} \geq \bar{w}$ are sufficient conditions for a dynamic contract to be generally admissible. ∎

In light of Lemma C-2, we henceforth focus on dynamic contracts that satisfy the requirements in (C.12) and (C.13), and refer to those as admissible dynamic contracts.

Definition C-3. An *admissible dynamic contract* is a function

$$\mathrm{D} = (\mathrm{D}_C, \mathrm{D}_S, \mathrm{D}_G, \mathrm{D}_\beta) : \mathcal{A} \to \mathcal{P}_C(0) \times \mathcal{P}_S(0) \times \mathcal{P}_G(0) \times \mathcal{P}_\beta(0)$$

together with a constant $w_{\text{init}} \geq \bar{w}$ such that
(I) $(\mathrm{D}_C, \mathrm{D}_S, \mathrm{D}_G, \mathrm{D}_{\tau_{\text{d}}}, \mathrm{D}_\beta) \in \overline{\mathcal{D}}^{w_{\text{init}}}$ (see Definition C-1) where

$$\mathrm{D}_{\tau_{\text{d}}}(A) = \inf\{t \geq 0 \mid\ W_t = 0\} \in \mathcal{P}_{\tau_{\text{d}}}(A) \subseteq \mathcal{P}_{\tau_{\text{d}}}(0), \quad \text{for } A \in \mathcal{A},$$

in which expression W is the solution to the SDE (C.8) for

$$C = \mathrm{D}_C(A), \quad S = \mathrm{D}_S(A), \quad G = \mathrm{D}_G(A), \quad \beta = \mathrm{D}_\beta(A) \quad \text{and} \quad W_{0-} = w_{\text{init}};$$

(II) $\mathrm{D}_\beta(A) \geq \lambda$ for all $A \in \mathcal{A}$;
(III) the solution to the SDE (C.8) for $(C, S, G, \beta) = \big(\mathrm{D}_C(0), \mathrm{D}_S(0), \mathrm{D}_G(0), \mathrm{D}_\beta(0)\big)$ and $A = 0$ satisfies the transversality condition (C.13) for $\tau_{\text{d}} = \mathrm{D}_{\tau_{\text{d}}}(0)$.
We denote by $\mathcal{D}^{w_{\text{init}}}$ the family of all admissible dynamic contracts with initial condition $w_{\text{init}} \geq \bar{w}$. ∎

C.4 The Firm's Problem: Take Two

We now turn our attention to the firm's optimisation problem, which amounts to finding an initial condition $w_{\text{init}} \geq \bar{w}$ and a contract $\mathrm{D}^\star \in \mathcal{D}^{w_{\text{init}}}$ that maximises the firm's expected discounted profit. In view of (B.13) and the same results from credit risk theory that we used to establish Lemma C-1.(I), we can see that, given any contract $\mathrm{D} \in \mathcal{D}^{w_{\text{init}}}$, the firm's size-adjusted expected discounted profit at the start of any manager's tenure $F(\mathrm{D})$ should satisfy

$$F(\mathrm{D}) = \mathbb{E}\bigg[\int_0^{\tau_{\mathrm{d}}} \mathfrak{D}_t\Big(\mu + qG_t\big[(1+\gamma)F(\mathrm{D}) - \kappa - S_t\big]\Big)\,dt - \int_{[0,\tau_{\mathrm{d}}]\cap\mathbb{R}_+} \mathfrak{D}_t\,dC_t + \mathfrak{D}_{\tau_{\mathrm{d}}}\big[F(\mathrm{D}) - \kappa\big]\bigg], \tag{C.17}$$

where $C = \mathrm{D}_C(0)$, $S = \mathrm{D}_S(0)$, $G = \mathrm{D}_G(0)$, $\tau_{\mathrm{d}} = \mathrm{D}_{\tau_{\mathrm{d}}}(0)$, for $\mathrm{D}_{\tau_{\mathrm{d}}}$ as in Definition C-3.(I), and

$$\mathfrak{D}_t = \exp\left(-rt - \int_0^t qG_s\,ds\right). \tag{C.18}$$

To identify the optimal contract $\mathrm{D}^\star$, we first consider the following stochastic control problem.

Problem C-1. Solve the singular stochastic control problem whose value function v is defined by

$$v(w) = \sup_{(C,S,G,\beta)\in\mathcal{S}} \mathbb{E}\bigg[\int_0^{\tau_{\mathrm{d}}} \mathfrak{D}_t\Big(\mu + qG_t\big[(1+\gamma)v(w_{\mathrm{h}}) - \kappa - S_t\big]\Big)\,dt - \int_{[0,\tau_{\mathrm{d}}]\cap\mathbb{R}_+} \mathfrak{D}_t\,dC_t + \mathfrak{D}_{\tau_{\mathrm{d}}}\big[v(w_{\mathrm{h}}) - \kappa\big]\bigg], \quad \text{for } w \geq 0, \tag{C.19}$$

where $\mathfrak{D}$ is defined by (C.18), $\mathcal{S}$ is the family of all control strategies (C, S, G, β) such that

$$C \in \mathcal{P}_C(0), \quad S \in \mathcal{P}_S(0), \quad G \in \mathcal{P}_G(0), \quad \beta \in \mathcal{P}_\beta(0) \text{ with } \beta \geq \lambda,$$

the associated solution to the SDE

$$dW_t = \big[\varrho W_t + qG_t(W_t - S_t)\big]\,dt - dC_t + \sigma\beta_t\,dZ_t, \quad W_{0-} = w \geq 0, \tag{C.20}$$

satisfies the transversality condition (C.13), τ_{d} is the first hitting time of zero by W, and

$$w_{\mathrm{h}} := \bar{w} \vee w_\circ := \bar{w} \vee \arg\max_{w>0} v(w). \tag{C.21}$$

∎

The choice of w_{h} as in (C.21) reflects the idea that the firm is prepared to offer each manager a size-adjusted total discounted payoff w_{h} that may be strictly greater than the managers' reservation value $\bar{w}$ as long as this enhances the firm's total expected discounted profit (see Remark B-3).

Given the solution to this problem, the firm's optimisation problem reduces to solving the following one.

Problem C-2. Given the solution to Problem C-1, namely, the value of w_{h} and an optimal control strategy $(C^\star, S^\star, G^\star, \beta^\star) \in \mathcal{S}$, determine an admissible dynamic contract $\mathrm{D}^\star \in \mathcal{D}^{w_{\mathrm{h}}}$ such that

$$\mathrm{D}^\star_C(0) = C^\star, \quad \mathrm{D}^\star_S(0) = S^\star, \quad \mathrm{D}^\star_G(0) = G^\star \quad \text{and} \quad \mathrm{D}^\star_\beta(0) = \beta^\star.$$

We call such an admissible dynamic contract *optimal*. ∎

The firm's expected discounted profit at time 0 under an optimal contract $\mathrm{D}^\star$ is

$$F(\mathrm{D}^\star) = \sup_{\mathrm{D} \in \mathcal{D}^{w_{\mathrm{h}}}} F(\mathrm{D}) = v(w_{\mathrm{h}}).$$

D Verification Theorem and Optimal Contract

In Appendix C, the firm's contracting problem was ultimately connected to a singular stochastic control problem (Problem C-1). The next theorem expresses the solution to this problem in terms of the solution to an appropriate HJB equation. Using this result, we characterise the solution to Problem C-2, namely, we derive the optimal admissible dynamic contract (see Theorem D-2 below). The optimality properties 1–5 stated in Section 3.1 follow immediately from these results.

Theorem D-1. *Let $u : \mathbb{R}_+ \to \mathbb{R}$ be a concave C^2 function that satisfies the HJB equation*

$$\max\Big\{\frac{1}{2}\sigma^2\lambda^2 u''(w) + \varrho w u'(w) - r u(w) + \mu + q\big[w u'(w) - u(w) + (1+\gamma)u(w_{\rm h}) - \kappa\big]^+,\ -u'(w) - 1\Big\} = 0 \tag{D.1}$$

with the Wentzel-type boundary condition

$$u(0) = u(w_{\rm h}) - \kappa, \tag{D.2}$$

where $w_{\rm h} := \bar{w} \vee w_\circ := \bar{w} \vee \arg\max_{w>0} u(w)$. Define

$$w_{\rm g} = \sup\big\{w \geq 0 \mid \ w u'(w) - u(w) + (1+\gamma)u(w_{\rm h}) - \kappa \geq 0\big\} \vee 0 \tag{D.3}$$

and

$$w_{\rm c} = \inf\big\{w \geq 0 \mid \ u'(w) = -1\big\}, \tag{D.4}$$

with the usual conventions that $\sup\emptyset = -\infty$ and $\inf\emptyset = \infty$, and assume that $w_{\rm c} < \infty$. Furthermore, suppose that there exists a constant $K > 0$ such that

$$\big|u'(w)\big| \leq K \quad \text{for all } w > 0. \tag{D.5}$$

The following statements hold true:

(I) *If $w_{\rm g} < \infty$, then $w_{\rm g} < w_{\rm c}$.*

(II) *The function u identifies with the value function v defined by (C.19), namely,*

$$u(w) = v(w) \quad \text{for all } w \geq 0. \tag{D.6}$$

(III) *The solution $(C^\star, S^\star, G^\star, \beta^\star)$ to Problem C-1 is such that the identities*

$$S^\star_t = 0, \quad G^\star_t = \mathbf{1}_{[0, w_{\rm g}\wedge w_{\rm c}]}(W^\star_t), \quad \beta^\star_t = \lambda, \tag{D.7}$$

$$W^\star_t \in [0, w_{\rm c}] \quad \text{and} \quad C^\star_t = \int_{[0,t]} \mathbf{1}_{[w_{\rm c},\infty[}(W^\star_s)\, dC^\star_s \tag{D.8}$$

hold true for all $t \in [0, \tau^\star_{\rm d}]$, where $C^\star$, $W^\star$ are rigorously constructed as in the proof of Theorem D-2.

Proof. To show (I), we argue by contradiction and we assume that $w_{\mathrm{c}} \leq w_{\mathrm{g}} < \infty$. Combining the concavity of u with (D.1) and the definition (D.4) of w_{c}, we can see that $u'(w) = -1$ and $u(w) = u(w_{\mathrm{c}}) - (w - w_{\mathrm{c}})$ for all $w \geq w_{\mathrm{c}}$. These observations imply that

$$u(w) - wu'(w) = u(w) + w = u(w_{\mathrm{c}}) + w_{\mathrm{c}} \quad \text{for all } w \geq w_{\mathrm{c}}.$$

In view of these identities, the assumption that $w_{\mathrm{g}} \geq w_{\mathrm{c}}$, and the definition (D.3) of w_{g}, we obtain

$$u(w) - wu'(w) = u(w_{\mathrm{g}}) + w_{\mathrm{g}} = (1+\gamma)u(w_{\mathrm{h}}) - \kappa \quad \text{for all } w \geq w_{\mathrm{g}},$$

which contradicts (D.3).

To show (II), we fix any initial condition $w > 0$ and any admissible control strategy $(C, S, G, \beta) \in \mathcal{S}$, where $\mathcal{S}$ is defined in the statement of Problem C-1. Using (C.20), the dynamics

$$d\mathfrak{D}_t = -(r + qG_t)\mathfrak{D}_t \, dt$$

(see (C.18)), Itô's formula and the integration by parts formula, we can see that, given any time $T > 0$,

$$\begin{aligned}
&\mathfrak{D}_{T\wedge\tau_{\mathrm{d}}} u(W_{T\wedge\tau_{\mathrm{d}}}) \\
&\quad = u(w) - \int_{[0,T\wedge\tau_{\mathrm{d}}]} \mathfrak{D}_t u'(W_{t-}) \, dC_t + \sum_{0\leq t\leq T\wedge\tau_{\mathrm{d}}} \mathfrak{D}_t \big[u(W_t) - u(W_{t-}) - u'(W_{t-}) \, \Delta W_t\big] \\
&\qquad + \int_0^{T\wedge\tau_{\mathrm{d}}} \mathfrak{D}_t \left(\frac{1}{2}\sigma^2\beta_t^2 u''(W_t) + \big[\varrho W_t + qG_t(W_t - S_t)\big]u'(W_t) - \big(r + qG_t\big)u(W_t)\right) dt \\
&\qquad + \int_0^{T\wedge\tau_{\mathrm{d}}} \mathfrak{D}_t \sigma\beta_t u'(W_t) \, dZ_t.
\end{aligned}$$

In view of the fact that $\Delta W_t \equiv W_t - W_{t-} = -\Delta C_t$, we can see that

$$\begin{aligned}
&- \int_{[0,T\wedge\tau_{\mathrm{d}}]} \mathfrak{D}_t u'(W_{t-}) \, dC_t + \sum_{0\leq t\leq T\wedge\tau_{\mathrm{d}}} \mathfrak{D}_t \big[u(W_t) - u(W_{t-}) - u'(W_{t-}) \, \Delta W_t\big] \\
&\qquad = - \int_0^{T\wedge\tau_{\mathrm{d}}} \mathfrak{D}_t u'(W_t) \, dC_t^{\mathrm{c}} + \sum_{0\leq t\leq T\wedge\tau_{\mathrm{d}}} \mathfrak{D}_t \big[u(W_{t-} - \Delta C_t) - u(W_{t-})\big] \\
&\qquad = - \int_0^{T\wedge\tau_{\mathrm{d}}} \mathfrak{D}_t u'(W_t) \, dC_t^{\mathrm{c}} - \sum_{0\leq t\leq T\wedge\tau_{\mathrm{d}}} \mathfrak{D}_t \int_0^{\Delta C_t} u'(W_{t-} - \Delta C_t + x) \, dx,
\end{aligned}$$

where C^{c} is the continuous part of the process C. Combining these identities with the observation that

$$u(W_{T\wedge\tau_{\mathrm{d}}}) = u(0)\mathbf{1}_{\{\tau_{\mathrm{d}}\leq T\}} + u(W_T)\mathbf{1}_{\{T<\tau_{\mathrm{d}}\}} = \big[u(w_{\mathrm{h}}) - \kappa\big]\mathbf{1}_{\{\tau_{\mathrm{d}}\leq T\}} + u(W_T)\mathbf{1}_{\{T<\tau_{\mathrm{d}}\}},$$

which follows from (D.2), we obtain

$$\int_0^{T\wedge\tau_{\mathrm{d}}} \mathfrak{D}_t\Big(\mu + qG_t\big[(1+\gamma)u(w_{\mathrm{h}}) - \kappa - S_t\big]\Big)\,dt - \int_{[0,T\wedge\tau_{\mathrm{d}}]} \mathfrak{D}_t\,dC_t + \mathfrak{D}_{\tau_{\mathrm{d}}}\big[u(w_{\mathrm{h}}) - \kappa\big]\mathbf{1}_{\{\tau_{\mathrm{d}}\le T\}}$$
$$= u(w) - \mathfrak{D}_T u(W_T)\mathbf{1}_{\{T<\tau_{\mathrm{d}}\}}$$
$$- \int_0^{T\wedge\tau_{\mathrm{d}}} \mathfrak{D}_t\big[u'(W_t)+1\big]\,dC_t^{\mathrm{c}} - \sum_{0\le t\le T\wedge\tau_{\mathrm{d}}} \mathfrak{D}_t \int_0^{\Delta C_t} \big[u'(W_{t-} - \Delta C_t + x) + 1\big]\,dx$$
$$+ \int_0^{T\wedge\tau_{\mathrm{d}}} \mathfrak{D}_t\bigg(\frac{1}{2}\sigma^2\beta_t^2 u''(W_t) + \varrho W_t u'(W_t) - ru(W_t) + \mu$$
$$+ qG_t\Big[W_t u'(W_t) - u(W_t) + (1+\gamma)u(w_{\mathrm{h}}) - \kappa - S_t\big(u'(W_t)+1\big)\Big]\bigg)\,dt$$
$$+ \int_0^{T\wedge\tau_{\mathrm{d}}} \mathfrak{D}_t\sigma\beta_t u'(W_t)\,dZ_t.$$

The concavity of u and the fact that it satisfies the gradient constraint $u' + 1 \ge 0$ imply that

$$\sup_{\mathfrak{b}\ge\lambda}\big[\mathfrak{b}^2 u''(w)\big] = \lambda^2 u''(w) \quad \text{and} \quad \sup_{\mathfrak{s}\in[0,w]}\big[-\mathfrak{s}\big(u'(w)+1\big)\big] = 0.$$

Therefore, since u satisfies the HJB equation (D.1),

$$\int_0^{T\wedge\tau_{\mathrm{d}}} \mathfrak{D}_t\Big(\mu + qG_t\big[(1+\gamma)u(w_{\mathrm{h}}) - \kappa - S_t\big]\Big)\,dt - \int_{[0,T\wedge\tau_{\mathrm{d}}]} \mathfrak{D}_t\,dC_t + \mathfrak{D}_{\tau_{\mathrm{d}}}\big[u(w_{\mathrm{h}}) - \kappa\big]\mathbf{1}_{\{\tau_{\mathrm{d}}\le T\}}$$
$$\le u(w) - \mathfrak{D}_T u(W_T)\mathbf{1}_{\{T<\tau_{\mathrm{d}}\}} + \int_0^{T\wedge\tau_{\mathrm{d}}} \mathfrak{D}_t\sigma\beta_t u'(W_t)\,dZ_t. \qquad \text{(D.9)}$$

In view of (D.5), we can see that $\big|u(w)\big| \le \big|u(0)\big| + Kw$ for all $w \ge 0$, which, combined with the transversality condition (C.13), implies that

$$\lim_{T\to\infty} \mathbb{E}\Big[\mathfrak{D}_T\big|u(W_T)\big|\mathbf{1}_{\{T<\tau_{\mathrm{d}}\}}\Big] = 0.$$

On the other hand, we can use Itô's isometry, (C.9) and (D.5), to calculate

$$\mathbb{E}\left[\bigg(\int_0^{T\wedge\tau_{\mathrm{d}}} \mathfrak{D}_t\sigma\beta_t u'(W_t)\,dZ_t\bigg)^2\right] = \mathbb{E}\left[\int_0^{T\wedge\tau_{\mathrm{d}}} \big[\mathfrak{D}_t\sigma\beta_t u'(W_t)\big]^2\,dt\right]$$
$$\le \sigma^2 K^2\,\mathbb{E}\left[\int_0^{T\wedge\tau_{\mathrm{d}}} e^{-2rt}\beta_t^2\,dt\right]$$
$$< \infty,$$

which implies that the stochastic integral in (D.9) is a square-integrable martingale. In view of these results, we can take expectations in (D.9) and use the monotone convergence theorem to obtain

$$\begin{aligned}
&\mathbb{E}\bigg[\int_0^{\tau_\mathrm{d}} \mathfrak{D}_t\Big(\mu + qG_t\big[(1+\gamma)u(w_\mathrm{h}) - \kappa - S_t\big]\Big)\,dt - \int_{[0,\tau_\mathrm{d}]} \mathfrak{D}_t\,dC_t + \mathfrak{D}_{\tau_\mathrm{d}}\big[u(w_\mathrm{h}) - \kappa\big]\bigg] \\
&\quad = \lim_{T\to\infty} \mathbb{E}\bigg[\int_0^{T\wedge\tau_\mathrm{d}} \mathfrak{D}_t\Big(\mu + qG_t\big[(1+\gamma)u(w_\mathrm{h}) - \kappa - S_t\big]\Big)\,dt \\
&\qquad\qquad - \int_{[0,T\wedge\tau_\mathrm{d}]} \mathfrak{D}_t\,dC_t + \mathfrak{D}_{\tau_\mathrm{d}}\big[u(w_\mathrm{h}) - \kappa\big]\mathbf{1}_{\{\tau_\mathrm{d}\le T\}}\bigg] \\
&\quad \le u(w).
\end{aligned} \tag{D.10}$$

Since $(C, S, G, \beta) \in \mathcal{S}$ has been chosen arbitrarily, it follows that

$$\begin{aligned}
u(w) \ge \sup_{(C,S,G,\beta)\in\mathcal{S}} &\mathbb{E}\bigg[\int_0^{\tau_\mathrm{d}} \mathfrak{D}_t\Big(\mu + qG_t\big[(1+\gamma)u(w_\mathrm{h}) - \kappa - S_t\big]\Big)\,dt \\
&- \int_{[0,\tau_\mathrm{d}]} \mathfrak{D}_t\,dC_t + \mathfrak{D}_{\tau_\mathrm{d}}\big[u(w_\mathrm{h}) - \kappa\big]\bigg].
\end{aligned} \tag{D.11}$$

The concavity of u and the fact that this function satisfies the HJB equation (D.1) imply that

$$u'(w) = -1 \text{ for all } w \ge w_\mathrm{c}$$

and

$$wu'(w) - u(w) + (1+\gamma)u(w_\mathrm{h}) - \kappa \begin{Bmatrix} \ge 0, & \text{for all } w \in [0, w_\mathrm{g}] \\ < 0, & \text{for all } w \in \,]w_\mathrm{g}, \infty[\, \cap [0, w_\mathrm{c}] \end{Bmatrix}.$$

In view of these observations, we can check that, if $(C^\star, S^\star, G^\star, \beta^\star)$ is such that (D.7)–(D.8) hold true, then (D.9) holds with equality. We also note that this control strategy is such that the transversality condition (C.13) is satisfied because $W^\star$ takes values in the bounded interval $[0, w_\mathrm{c}]$. Following the same steps as above, we can see that (D.10) holds with equality for this control strategy, which combined with (D.11), implies that

$$\begin{aligned}
u(w) &= \mathbb{E}\bigg[\int_0^{\tau_\mathrm{d}^\star} \mathfrak{D}_t^\star\Big(\mu + qG_t^\star\big[(1+\gamma)u(w_\mathrm{h}) - \kappa\big]\Big)\,dt - \int_{[0,\tau_\mathrm{d}^\star]} \mathfrak{D}_t^\star\,dC_t^\star + \mathfrak{D}_{\tau_\mathrm{d}^\star}^\star\big[u(w_\mathrm{h}) - \kappa\big]\bigg] \\
&= \sup_{(C,S,G,\beta)\in\mathcal{S}} \mathbb{E}\bigg[\int_0^{\tau_\mathrm{d}} \mathfrak{D}_t\Big(\mu + qG_t\big[(1+\gamma)u(w_\mathrm{h}) - \kappa - S_t\big]\Big)\,dt \\
&\qquad\qquad - \int_{[0,\tau_\mathrm{d}]} \mathfrak{D}_t\,dC_t + \mathfrak{D}_{\tau_\mathrm{d}}\big[u(w_\mathrm{h}) - \kappa\big]\bigg].
\end{aligned}$$

It follows that (D.6) holds true and $(C^\star, S^\star, G^\star, \beta^\star) \in \mathcal{S}$ is optimal. ∎

The next result provides the solution to the dynamic contracting Problem C-2. The filtration $(\hat{\mathcal{F}}_t) = \big(\hat{\mathcal{F}}_t(A)\big)$ and the families $\mathcal{P}_C(A)$, $\mathcal{P}_S(A)$, $\mathcal{P}_G(A)$, $\mathcal{P}_\beta(A)$ involved in the statement of the theorem are as at the beginning of Appendix C (see (C.1)–(C.4) and (C.9) in particular).

Theorem D-2. *Suppose that the HJB equation (D.1)–(D.2) has a concave C^2 solution u such that the assumptions of Theorem D-1 are all satisfied. The following statements hold true:*

(I) *For all $A \in \mathcal{A}$, there exists a process $C = C(A) \in \mathcal{P}_C(A)$ such that, apart from a jump of size $\Delta C_0 = (w_{\mathrm{h}} - w_{\mathrm{c}})^+$ at time 0, C is continuous,*

$$W_t \in [0, w_{\mathrm{c}}] \quad \textit{and} \quad C_t = \int_{[0,t]} \mathbf{1}_{[w_{\mathrm{c}},\infty[}(W_s)\, dC_s \quad \textit{for all } t \geq 0, \tag{D.12}$$

where the $(\hat{\mathcal{F}}_t)$-adapted process W is the strong solution to the SDE

$$\begin{aligned} dW_t &= \Big[\varrho + q\mathbf{1}_{[0,w_{\mathrm{g}}\wedge w_{\mathrm{c}}]}(W_t)\Big] W_t\, dt - dC_t + \lambda\, (d\hat{Y}_t - \mu\, dt) \\ &= \Big[\varrho + q\mathbf{1}_{[0,w_{\mathrm{g}}\wedge w_{\mathrm{c}}]}(W_t)\Big] W_t\, dt - dC_t - \lambda\, dA_t + \sigma\lambda\, dZ_t, \quad W_{0-} = w_{\mathrm{h}}. \end{aligned} \tag{D.13}$$

(II) *The function $\mathrm{D}^\star = (\mathrm{D}^\star_C, \mathrm{D}^\star_S, \mathrm{D}^\star_G, \mathrm{D}^\star_\beta) : \mathcal{A} \to \mathcal{P}_C(0) \times \mathcal{P}_S(0) \times \mathcal{P}_G(0) \times \mathcal{P}_\beta(0)$ that is defined by*

$$\mathrm{D}^\star_S = 0, \quad \mathrm{D}^\star_G = \mathbf{1}_{[0,w_{\mathrm{g}}\wedge w_{\mathrm{c}}]}(W), \quad \mathrm{D}^\star_\beta = \lambda,$$

and by identifying $\mathrm{D}^\star_C(A)$ with $C(A)$ in (I) above, where W is given by (D.13), provides the solution to Problem C-2, namely, it is an optimal admissible dynamic contract.

Proof. In view of Theorem D-1 and the results asserted in (I), it is straightforward to verify that (II) is indeed true. To prove (I), we fix any $A \in \mathcal{A}$ and we recall that $\hat{Y}_t = \mu t - A_t + \sigma Z_t$. First, we assume that $w_{\mathrm{g}} = 0$, and we note that the construction corresponding to the case $w_{\mathrm{g}} = \infty$ (see also part (I) of the previous theorem) is identical if we replace ϱ by $\varrho + q$. In this case, we rewrite the SDE (D.13) in the form

$$e^{-\varrho t} W_t = w_{\mathrm{h}} - \hat{C}_t + \lambda \int_0^t e^{-\varrho s}\, (d\hat{Y}_s - \mu\, ds), \tag{D.14}$$

where

$$\hat{C}_t = \int_{[0,t]} e^{-\varrho s}\, dC_s.$$

Noting that

$$W_t \leq w_{\mathrm{c}} \quad \Leftrightarrow \quad -w_{\mathrm{h}} - \lambda \int_0^t e^{-\varrho s}\, (d\hat{Y}_s - \mu\, ds) + e^{-\varrho t} w_{\mathrm{c}} + \hat{C}_t \geq 0,$$

the analysis of Skorokhod's equation (see Lemma 6.14 in Karatzas and Shreve (1988), Chapter 3) implies that the process $\hat{C}$ defined by

$$\hat{C}_t = \sup_{\mathrm{s} \leq t} \left(w_{\mathrm{h}} + \lambda \int_0^{\mathrm{s}} e^{-\varrho s}\, (d\hat{Y}_s - \mu\, ds) - e^{-\varrho \mathrm{s}} w_{\mathrm{c}} \right)^+$$

is such that

$$W_t \in [0, w_c] \quad \text{and} \quad \hat{C}_t = \int_{[0,t]} \mathbf{1}_{[w_c,\infty[}(W_s)\, d\hat{C}_s \quad \text{for all } t \ge 0,$$

where W is the corresponding process in (D.14). If we define

$$C_t = \int_{[0,t]} e^{\varrho s}\, d\hat{C}_s,$$

then we can see that

$$C_t = \int_{[0,t]} e^{\varrho s}\, d\hat{C}_s = \int_{[0,t]} e^{\varrho s} \mathbf{1}_{[w_c,\infty[}(W_s)\, d\hat{C}_s = \int_{[0,t]} \mathbf{1}_{[w_c,\infty[}(W_s)\, dC_s.$$

By construction, C is $(\hat{\mathcal{F}}_t)$-adapted. Using Jensen's inequality, Doob's L^2-inequality and Itô's isometry, we calculate

$$\begin{aligned}\left(\mathbb{E}\left[\sup_{T\ge 0}\left|\int_0^T e^{-\varrho t}\, dZ_t\right|\right]\right)^2 &\le \mathbb{E}\left[\left(\sup_{T\ge 0}\left|\int_0^T e^{-\varrho t}\, dZ_t\right|\right)^2\right] \\ &\le 4 \sup_{T\ge 0} \mathbb{E}\left[\left(\int_0^T e^{-\varrho t}\, dZ_t\right)^2\right] = 4\int_0^\infty e^{-2\varrho t}\, dt = \frac{2}{\varrho}. \qquad \text{(D.15)}\end{aligned}$$

In view of this estimate, we can see that

$$\begin{aligned}\mathbb{E}\left[\int_{[0,\infty[} e^{-\varrho t}\, dC_t\right] &= \mathbb{E}\left[\lim_{T\to\infty} \hat{C}_T\right] \\ &\le \mathbb{E}\left[\sup_{T\ge 0}\left(w_h + \sigma\lambda \int_0^T e^{-\varrho t}\, dZ_t\right)^+\right] \\ &\le w_h + \sigma\lambda\, \mathbb{E}\left[\sup_{T\ge 0}\left|\int_0^T e^{-\varrho t}\, dZ_t\right|\right] \le w_h + \sigma\lambda\sqrt{\frac{2}{\varrho}}.\end{aligned}$$

It follows that $C \in \mathcal{P}_C(A)$, and C is the required process.

Before addressing the proof of (I) if $w_g \in\,]0, w_c[$, we first consider the existence and uniqueness of a strong solution to the SDE

$$dW_t = \left[\varrho + q\mathbf{1}_{[0,w_g]}(W_t)\right] W_t\, dt - dC_t - \lambda\, dA_t + \sigma\lambda\, dZ_t, \quad W_0 = w_h \in\,]0, w_c[, \qquad \text{(D.16)}$$

where $w_g \in\,]0, w_c[$, and $C \in \mathcal{P}_C(A)$ is continuous with $C_0 = 0$. To this end, we consider the strictly positive function

$$\begin{aligned}p_d(w) &= \exp\left(-\int_0^w \frac{2\left[\varrho + q\mathbf{1}_{[0,w_g]}(\ell)\right]\ell}{\sigma^2\lambda^2}\, d\ell\right) \\ &= \left\{\begin{array}{ll} \exp\left(-\frac{\varrho}{\sigma^2\lambda^2} w^2\right), & \text{if } w < 0 \\ \exp\left(-\frac{\varrho+q}{\sigma^2\lambda^2} w^2\right), & \text{if } w \in [0, w_g] \\ \exp\left(-\frac{\varrho}{\sigma^2\lambda^2} w^2 - \frac{q w_g^2}{\sigma^2\lambda^2}\right), & \text{if } w > w_g \end{array}\right\}\end{aligned}$$

and the strictly increasing function $p : \mathbb{R} \to]\underline{p}, \overline{p}[$ defined by

$$p(w) = \int_0^w p_{\rm d}(\ell)\, d\ell,$$

where

$$\underline{p} = \lim_{w \to -\infty} p(w) = -\frac{\sigma\lambda}{2}\sqrt{\frac{\pi}{\varrho}} \quad \text{and} \quad \overline{p} = \lim_{w \to \infty} p(w) \in]0, \infty[.$$

Using Itô's formula, we can see that, if we define $U = p(W)$, then

$$dU_t = p_{\rm d} \circ p^{-1}(U_t)\, d\Gamma_t, \quad U_0 = p(w_{\rm h}),$$

where

$$\Gamma_t = -C_t - \lambda A_t + \sigma\lambda Z_t \equiv -C_t - \lambda\mu t + \lambda\hat{Y}_t.$$

This SDE has a unique $(\hat{\mathcal{F}}_t)$-adapted strong solution up to the exit time of U from any interval $[\underline{u}, \overline{u}]$ such that $U_0 = p(w_{\rm h}) \in]\underline{u}, \overline{u}[$ and $[\underline{u}, \overline{u}] \subseteq]\underline{p}, \overline{p}[$ because Γ is a continuous $(\hat{\mathcal{F}}_t)$-semimartingale and $p_{\rm d} \circ p^{-1} :]\underline{p}, \overline{p}[\to]0, 1]$ is a locally Lipschitz function (see Theorem 6 in Protter (1992), Chapter V). It follows that the SDE (D.16) has a unique $(\hat{\mathcal{F}}_t)$-adapted strong solution up to the exit time of W from any bounded interval containing $W_0 = w_{\rm h}$. Furthermore, the expression

$$W_t = K_t \left(w_{\rm h} - \int_0^t K_s^{-1} \left(dC_s + \lambda dA_s - \sigma\lambda\, dZ_s \right) \right),$$

where

$$K_t = \exp\left(\int_0^t \left[\varrho + q\mathbf{1}_{[0, w_{\rm g}]}(W_s) \right] ds \right) \in \left[e^{\varrho t},\, e^{(\varrho + q)t} \right],$$

implies that the solution to the SDE (D.16) does not explode in finite time, namely, $\mathbb{P}\left(\sup_{t \in [0,T]} |W_t| < \infty \right) = 1$ for all $T \geq 0$.

We return to the proof of (I), now considering the case when $w_{\rm g} \in]0, w_{\rm c}[$. We first assume that $w_{\rm h} < w_{\rm c}$. To construct the required C, we determine a sequence of processes $(C^i,\ i \geq 0)$ and an increasing sequence of $(\hat{\mathcal{F}}_t)$-stopping times $(\nu^i,\ i \geq 0)$ such that, for all $i \geq 0$,

$$C^i \in \mathcal{P}_C(A) \quad \text{and} \quad C^i \text{ is continuous with } C^i_0 = 0, \tag{D.17}$$

$$W^i_t \in [0, w_{\rm c}] \quad \text{and} \quad C^i_t = \int_{[0,t]} \mathbf{1}_{[w_{\rm c}, \infty[}(W^i_s)\, dC^i_s \quad \text{for all } t \in [0, \nu^i], \tag{D.18}$$

$$W^i_{\nu^i} \mathbf{1}_{\{\nu^i < \infty\}} = \begin{Bmatrix} w_{\rm c}\mathbf{1}_{\{\nu^i < \infty\}}, & \text{if } i \text{ is even} \\ w_{\rm g}\mathbf{1}_{\{\nu^i < \infty\}}, & \text{if } i \text{ is odd} \end{Bmatrix}, \tag{D.19}$$

$$\text{and} \quad C^i_t \mathbf{1}_{\{\nu^i < \infty\}} = C^i_{\nu^i} \mathbf{1}_{\{\nu^i < \infty\}} \quad \text{for all } t \geq \nu^i, \tag{D.20}$$

where W^i is the solution to the SDE

$$dW^i_t = \Big[\varrho + q\mathbf{1}_{[0,w_{\rm g}]}(W^i_t)\Big] W^i_t\,dt - dC^i_t + \lambda\,(d\hat{Y}_t - \mu\,dt), \quad W^i_0 = w_{\rm h}. \tag{D.21}$$

To start with, we define

$$C^0 = 0 \quad \text{and} \quad \nu^0 = \inf\left\{t \geq 0 \mid \ W^0_t \geq w_{\rm c}\right\},$$

and we note that (D.17)–(D.20) hold true trivially for these choices of C^0, ν^0 and for W^0 being the solution to (D.21).

Given $i \geq 0$ even and C^i, ν^i such that (D.17)–(D.20) hold true, we consider the processes

$$\hat{C}^{i+1}_t = \mathbf{1}_{\{\nu^i \leq t\}\cap\{\nu^i<\infty\}} \sup_{\mathsf{s}\in[\nu^i,t]} \left(w_{\rm c} + \lambda \int_{\nu^i}^{\mathsf{s}} e^{-\varrho(s-\nu^i)}\,(d\hat{Y}_s - \mu\,ds) - e^{-\varrho(\mathsf{s}-\nu^i)} w_{\rm c}\right)^+,$$

$$\tilde{W}^{i+1}_t = e^{\varrho(t-\nu^i)}\left(w_{\rm c} - \hat{C}^{i+1}_t + \lambda\int_{\nu^i}^t e^{-\varrho(s-\nu^i)}\,(d\hat{Y}_s - \mu\,ds)\right)\mathbf{1}_{\{\nu^i\leq t\}\cap\{\nu^i<\infty\}},$$

and we note that $\hat{C}^{i+1}$ is a continuous increasing process such that $\hat{C}^{i+1}_t\mathbf{1}_{\{t\leq\nu^i\}} = 0$. We then define

$$\nu^{i+1} = \inf\left\{t \geq \nu^i \mid \ \tilde{W}^{i+1}_t \leq w_{\rm g}\right\} \quad \text{and} \quad C^{i+1}_t = C^i_t + \int_0^{t\wedge\nu^{i+1}} e^{\varrho(s-\nu^i)}\,d\hat{C}^{i+1}_s.$$

In view of the analysis in the first paragraph of this proof, we can see that (D.18)–(D.20) hold true for $i+1$ in place of i. In particular, the corresponding solution W^{i+1} to (D.21) is such that $W^{i+1}_t = \tilde{W}^{i+1}_t \in \,]w_{\rm g}, w_{\rm c}]$ for all $t \in [\nu^i, \nu^{i+1}[$. We shall verify that (D.17) also holds true in the penultimate paragraph of the proof.

Given $i \geq 1$ odd and C^i, ν^i such that (D.17)–(D.20) hold true, we define $C^{i+1} = C^i$ and

$$\nu^{i+1} = \inf\{t \geq \nu^i \mid \ W^i_t \geq w_{\rm c}\}.$$

It is immediate to verify that (D.17)–(D.20) hold true for $i+1$ in place of i.

In view of the observations that

$$C^{i+1}_t\mathbf{1}_{\{t\leq\nu^i\}} = C^i_t\mathbf{1}_{\{t\leq\nu^i\}} \quad \text{for all } t \geq 0 \text{ and } i \geq 0,$$

and $\lim_{i\to\infty}\nu^i = \infty$, we can see that the required process C is given by

$$C_t = \sum_{i=1}^{\infty} C^i_t\mathbf{1}_{\{\nu^{i-1}\leq t<\nu^i\}}.$$

To see the limit invoked here, we consider the SDE

$$d\overline{W}^{2i}_t = \Big[\varrho + q\mathbf{1}_{[0,w_{\rm g}]}\big(\overline{W}^{2i}_t\big)\Big]\overline{W}^{2i}_t\,dt - dC^{2i}_t - \lambda\,dA^{\nu^{2i-1}}_t + \sigma\lambda\,dZ_t,\ \overline{W}^{2i}_0 = w_{\rm h},$$

for any $i \geq 1$, where $A^{\nu^{2i-1}}_t = A_{t\wedge\nu^{2i-1}}$, and we define

$$\overline{\nu}^{2i} = \inf\left\{t \geq \nu^{2i-1} \mid \ \overline{W}^{2i}_t \geq w_{\rm c}\right\}.$$

In view of the observations that

$$\overline{W}^{2i}_{\nu^{2i-1}}\mathbf{1}_{\{\nu^{2i-1}<\infty\}} = W^{2i}_{\nu^{2i-1}}\mathbf{1}_{\{\nu^{2i-1}<\infty\}} = w_{\rm g}\mathbf{1}_{\{\nu^{2i-1}<\infty\}},$$

which follows from the fact that

$$\overline{W}^{2i}_t\mathbf{1}_{\{t\leq\nu^{2i-1}\}} = W^{2i}_t\mathbf{1}_{\{t\leq\nu^{2i-1}\}},$$

and

$$\begin{aligned}\overline{W}^{2i}_t\mathbf{1}_{\{t\geq\nu^{2i-1}\}} &= \left(w_{\rm g} + \int_{\nu^{2i-1}}^t \Big[\varrho + q\mathbf{1}_{[0,w_{\rm g}]}\big(\overline{W}^{2i}_s\big)\Big]\overline{W}^{2i}_s\,ds + \sigma\lambda(Z_t - Z_{\nu^{2i-1}})\right)\mathbf{1}_{\{t\geq\nu^{2i-1}\}}\\ &\geq W^{2i}_t\mathbf{1}_{\{t\geq\nu^{2i-1}\}},\end{aligned}$$

we can see that $\overline{\nu}^{2i} \leq \nu^{2i}$ and that the strictly positive random variables $\overline{\nu}^{2i} - \nu^{2i-1}$, $i \geq 1$, are independent and identically distributed. Combining these facts with the law of large numbers, we obtain

$$\lim_{i\to\infty}\frac{\nu^{2i}}{i} > \lim_{i\to\infty}\frac{1}{i}\sum_{k=1}^{i}(\nu^{2k} - \nu^{2k-1}) \geq \lim_{i\to\infty}\frac{1}{i}\sum_{k=1}^{i}(\overline{\nu}^{2k} - \nu^{2k-1}) = \mathbb{E}\left[\overline{\nu}^2 - \nu^1\right] > 0,$$

which implies that $\lim_{i\to\infty}\nu^i = \infty$, as claimed. Furthermore, given any $i \geq 1$,

$$\mathbb{E}\left[e^{-\varrho\nu^{2i}}\right] \leq \mathbb{E}\left[\prod_{k=1}^{i} e^{-\varrho(\nu^{2k}-\nu^{2k-1})}\right] = \left(\mathbb{E}\left[e^{-\varrho(\nu^2-\nu^1)}\right]\right)^i. \tag{D.22}$$

By construction, the processes C^i, $i \geq 0$, and C are all continuous, increasing and $(\hat{\mathcal{F}}_t)$-adapted. To show that these processes satisfy the integrability condition (B.4) and thus conclude that they belong to $\mathcal{P}_C(A)$ as well as that (D.17) holds true, we consider the probability spaces $(\Omega, \mathcal{F}, (\mathcal{F}^i_t), \mathbb{Q}^i)$, where $(\mathcal{F}^i_t)$ are the filtrations defined by $\mathcal{F}^i_t = \mathcal{F}^Z_{\nu^{2i}+t}$ and $\mathbb{Q}^i$ are the conditional probability measures $\mathbb{P}(\cdot \mid \nu^{2i} < \infty)$ that have Radon-Nikodym derivatives with respect to $\mathbb{P}$ given by

$$\frac{d\mathbb{Q}^i}{d\mathbb{P}} = \frac{1}{\mathbb{P}(\nu^{2i} < \infty)}\mathbf{1}_{\{\nu^{2i}<\infty\}}.$$

In this context, the processes Z^i defined by $Z^i_t = (Z_{\nu^{2i}+t} - Z_{\nu^{2i}})\mathbf{1}_{\{\nu^{2i}<\infty\}}$ are standard $(\mathcal{F}^i_t)$-Brownian motions that are independent of $\mathcal{F}^i_0 = \mathcal{F}^Z_{\nu^{2i}}$ (see Exercise 3.21 in Revuz and Yor (1999), Chapter IV). Furthermore,

$$\begin{aligned}&\mathbb{E}\left[\mathbf{1}_{\{\nu^{2i}<\infty\}}e^{-\varrho\nu^{2i}}\sup_{T\geq\nu^{2i}}\left|\int_{\nu^{2i}}^T e^{-\varrho(t-\nu^{2i})}\,dZ_t\right|\right]\\ &\qquad= \mathbb{P}(\nu^{2i} < \infty)\,\mathbb{E}^{\mathbb{Q}^i}\left[e^{-\varrho\nu^{2i}}\right]\mathbb{E}^{\mathbb{Q}^i}\left[\sup_{T\geq\nu^{2i}}\left|\int_{\nu^{2i}}^T e^{-\varrho(t-\nu^{2i})}\,dZ_t\right|\right]\\ &\qquad= \mathbb{E}\left[e^{-\varrho\nu^{2i}}\right]\mathbb{E}^{\mathbb{Q}^i}\left[\sup_{T\geq 0}\left|\int_0^T e^{-\varrho t}\,dZ^i_t\right|\right]\\ &\qquad\leq K\,\mathbb{E}\left[e^{-\varrho\nu^{2i}}\right],\end{aligned}$$

where $\mathbb{E}^{\mathbb{Q}^i}$ denotes expectation with respect to $\mathbb{Q}^i$ and the constant K can be determined as in (D.15). Combining such an estimate with (D.22), we calculate

$$
\begin{aligned}
&\mathbb{E}\left[\int_0^\infty e^{-\varrho t}\, dC_t\right] \\
&\qquad = \mathbb{E}\left[\sum_{i=0}^\infty \mathbf{1}_{\{\nu^{2i}<\infty\}} \int_{\nu^{2i}}^{\nu^{2i+1}} e^{-\varrho t}\, dC_t\right] \\
&\qquad = \sum_{i=0}^\infty \mathbb{E}\left[\mathbf{1}_{\{\nu^{2i}<\infty\}} e^{-\varrho\nu^{2i}} \lim_{T\to\infty} \hat{C}^{2i+1}_{T\wedge\nu^{2i+1}}\right] \\
&\qquad \leq \sum_{i=0}^\infty \left(w_{\mathrm{c}}\, \mathbb{E}\left[e^{-\varrho\nu^{2i}}\right] + \sigma\lambda\, \mathbb{E}\left[\mathbf{1}_{\{\nu^{2i}<\infty\}} e^{-\varrho\nu^{2i}} \sup_{T\geq\nu^{2i}} \left|\int_{\nu^{2i}}^T e^{-\varrho(t-\nu^{2i})}\, dZ_t\right|\right]\right) \\
&\qquad \leq (w_{\mathrm{c}} + \sigma\lambda K) \sum_{i=0}^\infty \left(\mathbb{E}\left[e^{-\varrho(\nu^2-\nu^1)}\right]\right)^i \\
&\qquad < \infty.
\end{aligned}
$$

We conclude that the processes C^i, $i \geq 0$, and C all satisfy the integrability condition (B.4) and they belong to $\mathcal{P}_C(A)$, in particular, C is the required process.

Finally, if $w_{\mathrm{g}} \in \,]0, w_{\mathrm{c}}[$ and $w_{\mathrm{h}} \equiv \bar{w} \geq w_{\mathrm{c}}$, then we can make the required construction by setting $\Delta C_0 = \bar{w} - w_{\mathrm{c}}$ and then following exactly the same arguments as above simply swapping the order of considerations associated with even and odd indices. ∎

E Preliminary Results

In this appendix, we review a range of results regarding the solvability of a second-order linear ODE on which part of our analysis of the HJB equation (D.1)–(D.2) is based, and we prove certain results that will be used repeatedly in Appendices F and G.

E.1 The General Solution to a Homogeneous ODE

The claims here follow from standard theory of linear one-dimensional diffusions (e.g., see Borodin and Salminen (2002), Chapter II). To fix ideas, we consider the process

$$d\bar{W}_t = \zeta \bar{W}_t \, dt + \sigma\lambda \, dZ_t, \quad \bar{W}_0 > 0, \tag{E.1}$$

with absorption at 0, where $\zeta > 0$ is a given constant. Given any constant $\delta > 0$, there exists a pair of C^∞ functions $\varphi, \psi : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\varphi(w_2) = \varphi(w_1) \, \mathbb{E}_{w_2}\left[e^{-\delta T_{w_1}}\right] \quad \text{for all } w_1 < w_2, \tag{E.2}$$

and

$$\psi(w_1) = \psi(w_2) \, \mathbb{E}_{w_1}\left[e^{-\delta T_{w_2}}\right] \equiv \psi(w_2) \, \mathbb{E}_{w_1}\left[e^{-\delta T_{w_2}} \mathbf{1}_{\{T_{w_2} < T_0\}}\right] \quad \text{for all } w_1 < w_2, \tag{E.3}$$

where $\mathbb{E}_{w_j}$ denotes expectation with respect to the probability measure $\mathbb{P}_{w_j}$ under which the solution to (E.1) is such that $\mathbb{P}_{w_j}(\bar{W}_0 = w_j) = 1$, for $j = 1, 2$, and T_w denotes the first hitting time of $\{w\}$, which is defined by

$$T_w = \inf\{t \geq 0 \mid \bar{W}_t = w\}, \quad \text{for } w \geq 0.$$

Note that, $\mathbb{P}_w(T_0 < \infty) > 0$ and $\mathbb{P}_w(T_\infty < \infty) = 0$ for all $w \in \,]0, \infty[$. These functions are unique, modulo multiplicative constants, and are such that

$$0 < \varphi(w) \quad \text{and} \quad \varphi'(w) < 0 \quad \text{for all } w > 0, \tag{E.4}$$

$$0 < \psi(w) \quad \text{and} \quad \psi'(w) > 0 \quad \text{for all } w > 0, \tag{E.5}$$

$$\varphi(0) = \lim_{w \downarrow 0} \varphi(w) < \infty, \quad \varphi'(0) = \lim_{w \downarrow 0} \varphi'(w) > -\infty, \tag{E.6}$$

$$\psi(0) = \lim_{w \downarrow 0} \psi(w) = 0, \quad \psi'(0) = \lim_{w \downarrow 0} \psi'(w) < \infty, \tag{E.7}$$

$$\lim_{w \to \infty} \varphi(w) = 0 \quad \text{and} \quad \lim_{w \to \infty} \psi(w) = \infty. \tag{E.8}$$

It is worth noting that (E.6)–(E.8) follow from the fact that 0 (resp., ∞) is an absorbing (resp., natural) boundary point. Furthermore, every solution to the second-order linear homogenous ODE

$$\frac{1}{2}\sigma^2\lambda^2 f''(w) + \zeta w f'(w) - \delta f(w) = 0 \tag{E.9}$$

in $]0, \infty[$ is given by

$$f(w) = A\varphi(w) + B\psi(w), \tag{E.10}$$

for some constants $A, B \in \mathbb{R}$. For future reference, we note that the fact that φ, ψ satisfy the ODE (E.9) implies that

$$\varphi = \varphi\left(\cdot\,; \frac{\zeta}{\sigma^2\lambda^2}, \frac{\delta}{\sigma^2\lambda^2}\right) \quad \text{and} \quad \psi = \psi\left(\cdot\,; \frac{\zeta}{\sigma^2\lambda^2}, \frac{\delta}{\sigma^2\lambda^2}\right),$$

namely, the two functions are parametrised by the values of $\frac{\zeta}{\sigma^2\lambda^2}$ and $\frac{\delta}{\sigma^2\lambda^2}$ only.

In the rest of our analysis, we assume that φ, ψ have been normalised through multiplication by appropriate constants so that

$$\varphi(0) = 1 \quad \text{and} \quad \psi'(0) = 1. \tag{E.11}$$

Accordingly, their Wronskian admits the expression

$$\varphi(w)\psi'(w) - \varphi'(w)\psi(w) = \exp\left(-\int_0^w \frac{2\zeta x}{\sigma^2\lambda^2}\,dx\right) = \exp\left(-\frac{\zeta}{\sigma^2\lambda^2}w^2\right). \tag{E.12}$$

E.2 Additional Results

The following results will be used repeatedly in Appendices F and G.

Lemma E-1. *The following statements hold true.*

(I) *The function φ satisfies*

$$\varphi''(0) \equiv \lim_{w\downarrow 0}\varphi''(w) = \frac{2\delta}{\sigma^2\lambda^2}, \tag{E.13}$$

$$\lim_{w\downarrow 0}\big[\varphi(w) - w\varphi'(w)\big] = 1, \quad \lim_{w\to\infty}\big[\varphi(w) - w\varphi'(w)\big] = 0, \tag{E.14}$$

$$\text{and} \quad \frac{d}{dw}\big[\varphi(w) - w\varphi'(w)\big] \equiv -w\varphi''(w) < 0 \quad \text{for all } w > 0. \tag{E.15}$$

(II) *If $\zeta > \delta$, then*

$$\psi''(0) \equiv \lim_{w\downarrow 0}\psi''(w) = 0, \quad \lim_{w\downarrow 0}\big[\psi(w) - w\psi'(w)\big] = 0, \tag{E.16}$$

$$\text{and} \quad \frac{d}{dw}\big[\psi(w) - w\psi'(w)\big] \equiv -w\psi''(w) > 0 \quad \text{for all } w > 0. \tag{E.17}$$

Furthermore,

$$\lim_{w\downarrow 0}\frac{\psi''(w)}{\varphi''(w)} = 0, \quad \left(\frac{\psi''}{\varphi''}\right)'(w) < 0 \text{ for all } w > 0, \quad \text{and} \quad \lim_{w\to\infty}\frac{\psi''(w)}{\varphi''(w)} = -\infty. \tag{E.18}$$

(III) *If $\zeta > \delta$, then*

$$\exp\left(\frac{\zeta}{\sigma^2\lambda^2}w^2\right)\psi'(w) - 1 > 0 \quad \text{for all } w > 0. \tag{E.19}$$

Proof. The properties (E.4), (E.6), (E.8) and the fact that φ satisfies the ODE (E.9) imply immediately (E.13)–(E.15).

The limits in (E.16) follow immediately once we combine (E.7) with the fact that ψ satisfies the ODE (E.9). Since $\psi' > 0$ satisfies the ODE

$$\frac{1}{2}\sigma^2\lambda^2 f'''(w) + \zeta w f''(w) + (\zeta - \delta) f'(w) = 0 \tag{E.20}$$

in $]0,\infty[$, which follows from differentiating (E.9), we can see that $\psi'''(w) < 0$ for all $w > 0$ such that $\psi''(w) = 0$. Furthermore, (E.16) and (E.20) imply that

$$\psi'''(0) \equiv \lim_{w\downarrow 0} \psi'''(w) = -(\zeta - \delta)\psi'(0) = -(\zeta - \delta) < 0.$$

This inequality and (E.16) imply that $\psi''(w) < 0$ for all $w > 0$ sufficiently small. In view of these observations, a simple contradiction argument reveals that $\psi''(w) < 0$ for all $w > 0$, and (E.17) follows.

To establish (E.18), we note that the first limit follows immediately from (E.13) and (E.16). If we define $b(w) = \frac{\psi''(w)}{\varphi''(w)}$, then we can use the fact that φ, ψ satisfy the ODEs (E.9) and (E.20) to calculate

$$b'(w) = -\frac{4\delta(\zeta - \delta)}{\sigma^4\lambda^4} \exp\left(-\frac{\zeta}{\sigma^2\lambda^2}w^2\right) \left[\varphi''(w)\right]^{-2} < 0,$$

which establishes the inequality in (E.18). On the other hand, we can differentiate (E.12) to obtain

$$\begin{aligned}\varphi(w)b(w) - \psi(w) &= -\frac{2\zeta}{\sigma^2\lambda^2} w \exp\left(-\frac{\zeta}{\sigma^2\lambda^2}w^2\right) \left[\varphi''(w)\right]^{-1} \\ &= \frac{\zeta\sigma^2\lambda^2}{2\delta(\zeta - \delta)} w\varphi''(w) b'(w).\end{aligned} \tag{E.21}$$

Furthermore, we note that, since φ satisfies (E.20),

$$\lim_{w\to\infty} w\varphi''(w) = \lim_{w\to\infty} \left[-\frac{\sigma^2\lambda^2}{2\zeta}\varphi'''(w) - \frac{\zeta - \delta}{\zeta}\varphi'(w)\right] = 0. \tag{E.22}$$

We now argue by contradiction and we assume that $\lim_{w\to\infty} b(w) > -\infty$. Such an assumption implies that $\lim_{w\to\infty}\left[\varphi(w)b(w) - \psi(w)\right] = -\infty$ thanks to (E.8). This limit and (E.21) imply that $\lim_{w\to\infty} w\varphi''(w)b'(w) = -\infty$. Combining this result with (E.22), we can see that $\lim_{w\to\infty} b'(w) = -\infty$, which contradicts the assumption that $\lim_{w\to\infty} b(w) > -\infty$, and the second limit in (E.18) follows.

Finally, (E.19) follows immediately from the calculation

$$\begin{aligned}\frac{d}{dw}\left[\exp\left(\frac{\zeta}{\sigma^2\lambda^2}w^2\right)\psi'(w) - 1\right] &= \exp\left(\frac{\zeta}{\sigma^2\lambda^2}w^2\right)\left[\psi''(w) + \frac{2\zeta}{\sigma^2\lambda^2}w\psi'(w)\right] \\ &= \frac{2\delta}{\sigma^2\lambda^2}\exp\left(\frac{\zeta}{\sigma^2\lambda^2}w^2\right)\psi(w) \\ &> 0,\end{aligned}$$

and the fact that $\psi'(0) = 1$ (see (E.11)). ∎

Lemma E-2. *Suppose that $\zeta > \delta$. There exists a strictly increasing continuous function $g : \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$w < g(w) \quad \text{and} \quad \frac{\psi'(w)}{\varphi'(w)} = \frac{\psi''\big(g(w)\big)}{\varphi''\big(g(w)\big)} \quad \text{for all } w \geq 0. \tag{E.23}$$

Furthermore, such a function is unique.

Proof. Given any $w > 0$, we use (E.9) and (E.12), to obtain

$$\varphi'(w)\psi''(w) - \varphi''(w)\psi'(w) = -\frac{2\delta}{\sigma^2\lambda^2}\big[\varphi(w)\psi'(w) - \varphi'(w)\psi(w)\big] < 0.$$

In view of (E.4) and (E.15), it follows that

$$\frac{\psi''(w)}{\varphi''(w)} > \frac{\psi'(w)}{\varphi'(w)}.$$

Combining this inequality with (E.18), we can see that, given any $w > 0$, there exists a unique $g(w) > w$ such that the identity in (E.23) holds true. We are going to prove that the function g thus defined is strictly increasing. Since all functions involved are continuous, we can therefore define $g(0) = \lim_{w \downarrow 0} g(w)$.

To show that g is strictly increasing, we differentiate the identity

$$\varphi'(w)\psi''\big(g(w)\big) - \varphi''\big(g(w)\big)\psi'(w) = 0 \tag{E.24}$$

to obtain

$$\Big[\psi'(w)\varphi'''\big(g(w)\big) - \varphi'(w)\psi'''\big(g(w)\big)\Big]g'(w) = \varphi''(w)\psi''\big(g(w)\big) - \psi''(w)\varphi''\big(g(w)\big).$$

Using the calculations

$$\psi'(w)\varphi'''\big(g(w)\big) - \varphi'(w)\psi'''\big(g(w)\big) = \frac{4\delta(\zeta-\delta)}{\sigma^4\lambda^4}\exp\left(-\frac{\zeta}{\sigma^2\lambda^2}g^2(w)\right)\frac{\varphi'(w)}{\varphi''\big(g(w)\big)}$$

$$\text{and} \quad \varphi''(w)\psi''\big(g(w)\big) - \psi''(w)\varphi''\big(g(w)\big) = \frac{2\delta}{\sigma^2\lambda^2}\exp\left(-\frac{\zeta}{\sigma^2\lambda^2}w^2\right)\frac{\varphi''\big(g(w)\big)}{\varphi'(w)},$$

which make use of (E.9), (E.12), (E.20) and (E.24), we obtain

$$g'(w) = \frac{\sigma^2\lambda^2}{2(\zeta-\delta)}\exp\left(-\frac{\zeta}{\sigma^2\lambda^2}\big(g^2(w) - w^2\big)\right)\left(\frac{\varphi''\big(g(w)\big)}{\varphi'(w)}\right)^2 > 0,$$

which establishes that g is strictly increasing. ∎

Lemma E-3. *Suppose that $\zeta > \delta$, and consider the functions $h : \mathbb{R}_+^2 \to \mathbb{R}$, $H : \mathbb{R}_+ \to \mathbb{R}$ and $J : \mathbb{R}_+ \to \mathbb{R}$ defined by*

$$\begin{aligned} h(x,y) &\equiv h\left(x, y; \frac{\zeta}{\sigma^2\lambda^2}, \frac{\delta}{\sigma^2\lambda^2}\right) \\ &= \exp\left(\frac{\zeta}{\sigma^2\lambda^2} y^2\right) \Big[\psi(x)\varphi''(y) + \big(1-\varphi(x)\big)\psi''(y)\Big], \qquad \text{(E.25)} \end{aligned}$$

$$H(x) \equiv H\left(x; \frac{\zeta}{\sigma^2\lambda^2}, \frac{\delta}{\sigma^2\lambda^2}\right) = h(x,x) \qquad \text{(E.26)}$$

$$\text{and} \quad J(x) \equiv J\left(x; \frac{\zeta}{\sigma^2\lambda^2}, \frac{\delta}{\sigma^2\lambda^2}\right) = h\big(x, g(x)\big), \qquad \text{(E.27)}$$

where g is the function defined by Lemma E-2. The following statements hold true.
(I) *The function h is such that*

$$h(x,0) = \frac{2\delta}{\sigma^2\lambda^2}\psi(x) > 0, \quad h(0,y) = 0, \qquad \text{(E.28)}$$

$$\frac{\partial^2 h(x,y)}{\partial y^2} < 0, \quad \lim_{y\to\infty} \frac{\partial^2 h(x,y)}{\partial y^2} = -\infty \quad \text{and} \quad \lim_{y\to\infty} h(x,y) = -\infty, \qquad \text{(E.29)}$$

for all $x, y > 0$,

$$\left.\frac{\partial h(x,y)}{\partial x}\right|_{y=g(x)} = 0 \quad \text{and} \quad \frac{\partial h(x,y)}{\partial y} < 0 \quad \text{for all } y \geq x > 0. \qquad \text{(E.30)}$$

(II) *The function H is strictly concave and such that*

$$H(0) = 0, \quad H'(0) > 0 \quad \text{and} \quad \lim_{x\to\infty} H(x) = -\infty.$$

(III) *The function J is such that*

$$J(0) = 0, \quad J'(x) < 0 \quad \text{and} \quad J(x) < H(x) \quad \text{for all } x > 0.$$

Proof. The identities in (E.28) follow immediately from the definition (E.25) of h, (E.5), (E.7), (E.11), (E.13) and (E.16). Differentiating h with respect to y and using the fact that the functions φ, ψ satisfy (E.20), we obtain

$$\frac{\partial h(x,y)}{\partial y} = -\frac{2(\zeta-\delta)}{\sigma^2\lambda^2} \exp\left(\frac{\zeta}{\sigma^2\lambda^2} y^2\right) \Big[\psi(x)\varphi'(y) + \big(1-\varphi(x)\big)\psi'(y)\Big]$$

$$\text{and} \quad \frac{\partial^2 h(x,y)}{\partial y^2} = -\frac{4(\zeta-\delta)\delta}{\sigma^4\lambda^4} \exp\left(\frac{\zeta}{\sigma^2\lambda^2} y^2\right) \Big[\psi(x)\varphi(y) + \big(1-\varphi(x)\big)\psi(y)\Big].$$

The properties listed in (E.29) follow from (E.4), (E.5), (E.8) and (E.11).

To prove (E.30), we use (E.12), (E.19) and the fact that $\frac{\partial^2 h(x,y)}{\partial y^2} < 0$ for all $x, y > 0$ to obtain

$$\begin{aligned}
\frac{\partial h(x,y)}{\partial y} &\leq \frac{\partial h(x,x)}{\partial y} \\
&= -\frac{2(\zeta-\delta)}{\sigma^2\lambda^2} \exp\left(\frac{\zeta}{\sigma^2\lambda^2}x^2\right) \Big[\psi(x)\varphi'(x) + \big(1-\varphi(x)\big)\psi'(x)\Big] \\
&= \frac{2(\zeta-\delta)}{\sigma^2\lambda^2} \Big[1 - \exp\left(\frac{\zeta}{\sigma^2\lambda^2}x^2\right)\psi'(x)\Big] \\
&< 0 \qquad \text{for all } y \geq x.
\end{aligned} \tag{E.31}$$

Furthermore, the calculation

$$\frac{\partial h(x,y)}{\partial x} = \exp\left(\frac{\zeta}{\sigma^2\lambda^2}y^2\right) \big[\psi'(x)\varphi''(y) - \varphi'(x)\psi''(y)\big], \tag{E.32}$$

and (E.24) imply that the identity in (E.30) holds true.

To establish (II), we note that the definition (E.26) of H yields the expressions

$$\begin{aligned}
H(x) &= \exp\left(\frac{\zeta}{\sigma^2\lambda^2}x^2\right) \Big[\psi(x)\varphi''(x) + \big(1-\varphi(x)\big)\psi''(x)\Big] \\
&= \exp\left(\frac{\zeta}{\sigma^2\lambda^2}x^2\right)\psi''(x) + \frac{2\zeta}{\sigma^2\lambda^2}x,
\end{aligned} \tag{E.33}$$

where the second equality follows from the fact that φ, ψ satisfy (E.9) and (E.12). Differentiating, we obtain

$$\begin{aligned}
H'(x) &= \frac{2\zeta}{\sigma^2\lambda^2} - \frac{2(\zeta-\delta)}{\sigma^2\lambda^2} \exp\left(\frac{\zeta}{\sigma^2\lambda^2}x^2\right)\psi'(x) \\
\text{and} \quad H''(x) &= -\frac{4\delta(\zeta-\delta)}{\sigma^4\lambda^4} \exp\left(\frac{\zeta}{\sigma^2\lambda^2}x^2\right)\psi(x).
\end{aligned}$$

It follows that H is strictly concave and $\lim_{x\to\infty} H''(x) = -\infty$, which implies that $\lim_{x\to\infty} H(x) = -\infty$. Furthermore, using (E.16) and the normalisation in (E.11), we calculate

$$H(0) = 0 \quad \text{and} \quad H'(0) = \frac{2\delta}{\sigma^2\lambda^2} > 0.$$

To prove (III), we first note that $J(0) = 0$ follows from (E.28). Combined with (E.30), the facts that g is strictly increasing and $g(x) > x$ imply that

$$J'(x) = \frac{\partial h\big(x, g(x)\big)}{\partial y} g'(x) < 0 \quad \text{and} \quad J(x) \equiv h\big(x, g(x)\big) < h(x,x) \equiv H(x).$$

∎

F The High-Growth Case

In this appendix, we consider the case arising from the verification Theorem D-1 when $w_{\mathrm{g}} = \infty$. In this context, we construct an appropriate solution to the HJB equation

$$\max\left\{\frac{1}{2}\sigma^2\lambda^2 u''(w) + (\varrho + q)wu'(w) - (r+q)u(w) \right.$$
$$\left. + \mu + q\big[(1+\gamma)u(w_{\mathrm{h}}) - \kappa\big],\ -u'(w) - 1\right\} = 0 \qquad \text{(F.1)}$$

that satisfies the Wentzel-type boundary condition

$$u(0) = u(w_{\mathrm{h}}) - \kappa, \qquad \text{(F.2)}$$

where $w_{\mathrm{h}} := \bar{w} \vee w_{\circ} := \bar{w} \vee \arg\max_{w>0} u(w)$.

To this end, we look for a concave C^2 function $u : \mathbb{R}_+ \to \mathbb{R}$ and a free-boundary point $w_{\mathrm{c}} > 0$ such that u satisfies the ODE

$$\frac{1}{2}\sigma^2\lambda^2 u''(w) + (\varrho + q)wu'(w) - (r+q)u(w) + \mu + q\big[(1+\gamma)u(w_{\mathrm{h}}) - \kappa\big] = 0 \qquad \text{(F.3)}$$

in $]0, w_{\mathrm{c}}[$, and is given by

$$u(w) = u(w_{\mathrm{c}}) - (w - w_{\mathrm{c}}), \quad \text{for } w > w_{\mathrm{c}}. \qquad \text{(F.4)}$$

For future reference, we note that (F.2) and (F.4) imply that

$$w_{\circ} = \arg\max_{w>0} u(w) \in\,]0, w_{\mathrm{c}}[. \qquad \text{(F.5)}$$

We look for a solution to (F.1)–(F.2) of the form

$$u(w) = \begin{cases} A\varphi_1(w) + B\psi_1(w) + \frac{\mu + q[(1+\gamma)u(w_{\mathrm{h}}) - \kappa]}{r+q}, & \text{if } w \leq w_{\mathrm{c}} \\ u(w_{\mathrm{c}}) - (w - w_{\mathrm{c}}), & \text{if } w > w_{\mathrm{c}} \end{cases}, \qquad \text{(F.6)}$$

for some constants $A, B \in \mathbb{R}$, where

$$\varphi_1 = \varphi_1\left(\cdot; \frac{\varrho + q}{\sigma^2\lambda^2}, \frac{r+q}{\sigma^2\lambda^2}\right) \quad \text{and} \quad \psi_1 = \psi_1\left(\cdot; \frac{\varrho + q}{\sigma^2\lambda^2}, \frac{r+q}{\sigma^2\lambda^2}\right),$$

identify with the functions φ and ψ in Appendix E for $\zeta = \varrho + q$ and $\delta = r + q$. Our analysis in this appendix will make repeated use of the functions

$$h(\cdot,\cdot) \equiv h\left(\cdot, \cdot; \frac{\varrho + q}{\sigma^2\lambda^2}, \frac{r+q}{\sigma^2\lambda^2}\right), \qquad \text{(F.7)}$$

$$H(\cdot) \equiv H\left(\cdot; \frac{\varrho + q}{\sigma^2\lambda^2}, \frac{r+q}{\sigma^2\lambda^2}\right) \quad \text{and} \quad J(\cdot) \equiv J\left(\cdot; \frac{\varrho + q}{\sigma^2\lambda^2}, \frac{r+q}{\sigma^2\lambda^2}\right) \qquad \text{(F.8)}$$

that are defined as in Lemma E-3 with $\varrho + q$, $r + q$ in place of ζ, δ and φ_1, ψ_1 in place of φ, ψ.

F.1 Analysis of the Free-Boundary Problem

To determine the constants A, B and the free-boundary point w_{c} appearing in (F.6), we require that u should be C^2 at w_{c}, which yields the system of equations

$$u'(w_{\mathrm{c}}-) \equiv A\varphi_1'(w_{\mathrm{c}}) + B\psi_1'(w_{\mathrm{c}}) = -1 \equiv u'(w_{\mathrm{c}}+) \tag{F.9}$$

and

$$u''(w_{\mathrm{c}}-) \equiv A\varphi_1''(w_{\mathrm{c}}) + B\psi_1''(w_{\mathrm{c}}) = 0 \equiv u''(w_{\mathrm{c}}+). \tag{F.10}$$

For future reference, we observe that, given any point $w_{\mathrm{c}} > 0$, the solution to (F.3) satisfying these conditions is such that

$$G(w_{\mathrm{c}}) = 0, \tag{F.11}$$

where

$$G(w) = -(r+q)u(w) - (\varrho+q)w + \mu + q\big[(1+\gamma)u(w_{\mathrm{h}}) - \kappa\big]. \tag{F.12}$$

Also, we note that the boundary condition (F.2) implies that, if $w_{\mathrm{h}} \leq w_{\mathrm{c}}$, then

$$\begin{aligned} u(0) &\equiv A + \frac{\mu + q\big[(1+\gamma)u(w_{\mathrm{h}}) - \kappa\big]}{r+q} \\ &= A\varphi_1(w_{\mathrm{h}}) + B\psi_1(w_{\mathrm{h}}) + \frac{\mu + q\big[(1+\gamma)u(w_{\mathrm{h}}) - \kappa\big]}{r+q} - \kappa \equiv u(w_{\mathrm{h}}) - \kappa \end{aligned} \tag{F.13}$$

(see also (E.7) and (E.11)). Using the fact that φ_1, ψ_1 satisfy (E.9) and (E.12) in Appendix E for $\zeta = \varrho + q$ and $\delta = r + q$, we can see that the equations (F.9)–(F.10) are equivalent to

$$\begin{aligned} A &= \exp\left(\frac{\varrho+q}{\sigma^2\lambda^2}w_{\mathrm{c}}^2\right)\left[\psi_1(w_{\mathrm{c}}) - \frac{\varrho+q}{r+q}w_{\mathrm{c}}\psi_1'(w_{\mathrm{c}})\right] \\ &= \frac{\sigma^2\lambda^2}{2(r+q)}\exp\left(\frac{\varrho+q}{\sigma^2\lambda^2}w_{\mathrm{c}}^2\right)\psi_1''(w_{\mathrm{c}}) < 0 \end{aligned} \tag{F.14}$$

and

$$\begin{aligned} B &= -\exp\left(\frac{\varrho+q}{\sigma^2\lambda^2}w_{\mathrm{c}}^2\right)\left[\varphi_1(w_{\mathrm{c}}) - \frac{\varrho+q}{r+q}w_{\mathrm{c}}\varphi_1'(w_{\mathrm{c}})\right] \\ &= -\frac{\sigma^2\lambda^2}{2(r+q)}\exp\left(\frac{\varrho+q}{\sigma^2\lambda^2}w_{\mathrm{c}}^2\right)\varphi_1''(w_{\mathrm{c}}) < 0, \end{aligned} \tag{F.15}$$

the inequalities following thanks to the results in Lemma E-1.

In view of (F.5), we can see that three possible cases can arise:

- $\bar{w} \leq w_{\circ}$, in which case $w_{\mathrm{h}} = w_{\circ}$ and $w_{\mathrm{h}} < w_{\mathrm{c}}$, or
- $w_{\circ} < \bar{w} \leq w_{\mathrm{c}}$, in which case $w_{\mathrm{h}} = \bar{w}$ and $w_{\mathrm{h}} \leq w_{\mathrm{c}}$, or
- $w_{\mathrm{c}} < \bar{w}$, in which case $w_{\mathrm{h}} = \bar{w}$ and $w_{\mathrm{h}} > w_{\mathrm{c}}$.

We first consider the case arising when it turns out that $w_\circ < \bar{w} \leq w_{\rm c}$. Substituting the expressions for A and B given by (F.14) and (F.15) in (F.13) with $\bar{w}$ in place of $w_{\rm h}$, we obtain the equation

$$h(\bar{w}, w_{\rm c}) = -\frac{2\kappa(r+q)}{\sigma^2\lambda^2}, \tag{F.16}$$

where h and its associated functions H, J are as in (F.7), (F.8).

Lemma F-1. *There exist a unique $w_{\rm c} = w_{\rm c}\left(\bar{w}, \frac{\varrho+q}{\sigma^2\lambda^2}, \frac{r+q}{\sigma^2\lambda^2}, \kappa\right) > 0$ such that (F.16) holds true. Also, there exist unique points $\bar{w}_\dagger = \bar{w}_\dagger\left(\frac{\varrho+q}{\sigma^2\lambda^2}, \frac{r+q}{\sigma^2\lambda^2}, \kappa\right)$ and $\bar{w}_\ddagger = \bar{w}_\ddagger\left(\frac{\varrho+q}{\sigma^2\lambda^2}, \frac{r+q}{\sigma^2\lambda^2}, \kappa\right)$ such that*

$$0 < \bar{w}_\dagger < \bar{w}_\ddagger, \quad J(\bar{w}_\dagger) = -\frac{2\kappa(r+q)}{\sigma^2\lambda^2} \quad \text{and} \quad H(\bar{w}_\ddagger) = -\frac{2\kappa(r+q)}{\sigma^2\lambda^2}. \tag{F.17}$$

These points are such that

$$w_{\rm c} \geq \bar{w} \;\Leftrightarrow\; \bar{w} \leq \bar{w}_\ddagger \quad \text{and} \quad w_{\rm c} = \bar{w} \;\Leftrightarrow\; \bar{w} = \bar{w}_\ddagger. \tag{F.18}$$

If the problem data is such that $\bar{w} \leq \bar{w}_\ddagger$, then the function u defined by (F.6) for A, B given by (F.14), (F.15), and $w_{\rm c}$ being the solution to (F.16), is a concave C^2 solution to the HJB equation (F.1) that satisfies the inequality

$$\left|u'(w)\right| \leq K \quad \text{for all } w \geq 0, \tag{F.19}$$

for some constant $K > 0$. Furthermore, this function u satisfies the boundary condition (F.2) as well, namely, $w_{\rm h} = \bar{w} > w_\circ$, if and only if $\bar{w} \in \,]\bar{w}_\dagger, \bar{w}_\ddagger]$.

Proof. The existence and uniqueness of a solution $w_{\rm c}$ to (F.16) follows immediately from the properties (E.28) and (E.29) of the function h that we established in Lemma E-3.(I). The existence and uniqueness of points $\bar{w}_\dagger$ and $\bar{w}_\ddagger$ such that all statements in (F.17) hold true follows immediately from the properties of the functions J and H stated in Lemma E-3.(II)-(III). It is straightforward to check that

$$w_{\rm c} \geq \bar{w} \quad \Leftrightarrow \quad h(\bar{w}, \bar{w}) \equiv H(\bar{w}) \geq -\frac{2\kappa(r+q)}{\sigma^2\lambda^2} \quad \Leftrightarrow \quad \bar{w} \leq \bar{w}_\ddagger. \tag{F.20}$$

Furthermore, we use Lemmas E-2 and E-3 to derive the equivalences

$$\begin{aligned} \bar{w} > w_\circ \quad &\Leftrightarrow \quad u'(\bar{w}) < 0 \quad \Leftrightarrow \quad \frac{\psi''(w_{\rm c})}{\varphi''(w_{\rm c})} > \frac{\psi'(\bar{w})}{\varphi'(\bar{w})} \quad \Leftrightarrow \quad w_{\rm c} < g(\bar{w}) \\ \Leftrightarrow \quad &J(\bar{w}) \equiv h\big(\bar{w}, g(\bar{w})\big) < h(\bar{w}, w_{\rm c}) = -\frac{2\kappa(r+q)}{\sigma^2\lambda^2} \quad \Leftrightarrow \quad \bar{w} > \bar{w}_\dagger. \end{aligned} \tag{F.21}$$

Therefore, $w_{\rm h} = \bar{w} > w_\circ$ and (F.20) both hold true if and only if $\bar{w} \in \,]\bar{w}_\dagger, \bar{w}_\ddagger]$.

By construction, the function u defined by (F.6) is C^2 and satisfies the boundary condition (F.2). To complete the proof, we need to show that u is concave, and that the inequalities

$$u'(w) \geq -1 \quad \text{for all } w \in [0, w_{\rm c}[\tag{F.22}$$

and

$$\frac{1}{2}\sigma^2\lambda^2 u''(w) + (\varrho + q)wu'(w) - (r+q)u(w) \\ + \mu + q\big[(1+\gamma)u(\bar{w}) - \kappa\big] \leq 0 \quad \text{for all } w > w_{\mathrm{c}} \tag{F.23}$$

hold true.

To show that (F.22) holds with strict inequality and that the restriction of u in $]0, w_{\mathrm{c}}[$ is strictly concave, we define

$$\tilde{w} = \sup\big\{w \in [0, w_{\mathrm{c}}[\;\mid\; u'(w) \leq -1\big\} \vee 0 \in [0, w_{\mathrm{c}}[, \tag{F.24}$$

with the usual convention that $\sup\emptyset = -\infty$. The inequality $\tilde{w} < w_{\mathrm{c}}$ stated here follows immediately once we combine the boundary conditions $u''(w_{\mathrm{c}}) = 0$ and $u'(w_{\mathrm{c}}) = -1$ with the observation that $\lim_{w\uparrow w_{\mathrm{c}}} u'''(w) > 0$, which is true because u' satisfies the ODE

$$\frac{1}{2}\sigma^2\lambda^2 u'''(w) + (\varrho + q)wu''(w) + (\varrho - r)u'(w) = 0$$

in $]0, w_{\mathrm{c}}[$. The fact that u satisfies the ODE (F.3) in $]0, w_{\mathrm{c}}[$ also implies that

$$\frac{1}{2}\sigma^2\lambda^2 u''(w) + (\varrho + q)w\big[u'(w) + 1\big] + G(w) = 0, \tag{F.25}$$

where G is defined by (F.12). In view of the assumption that $\varrho > r$ (see condition (12)) and the definition of $\tilde{w}$, we can see that

$$G'(w) = -(r+q)\big[u'(w) + 1\big] - (\varrho - r) < 0 \quad \text{for all } w \in [\tilde{w}, w_{\mathrm{c}}],$$

which, combined with (F.11), implies that $G(w) > 0$ for all $w \in [\tilde{w}, w_{\mathrm{c}}[$. This observation, the definition (F.24) of $\tilde{w}$, and (F.25) imply that

$$u''(w) < 0 \quad \text{for all } w \in [\tilde{w}, w_{\mathrm{c}}[.$$

This result and the boundary condition $u'(w_{\mathrm{c}}) = -1$ imply that $u'(w) > -1$ for all $w \in [\tilde{w}, w_{\mathrm{c}}[$. Combining this inequality with the definition of $\tilde{w}$ and the continuity of u', we can see that $\tilde{w} = 0$. Furthermore, (F.22) holds true with strict inequality and the restriction of u in $]0, w_{\mathrm{c}}[$ is strictly concave.

Using (F.6), we can see that (F.23) is equivalent to

$$-(\varrho + q)\big[w_{\mathrm{c}} + (w - w_{\mathrm{c}})\big] - (r+q)\big[u(w_{\mathrm{c}}) - (w - w_{\mathrm{c}})\big] \\ + \mu + q\big[(1+\gamma)u(\bar{w}) - \kappa\big] \leq 0 \quad \text{for all } w > w_{\mathrm{c}}.$$

In view of (F.11), we note that this inequality is equivalent to $-(\varrho - r)(w - w_{\mathrm{c}}) \leq 0$, which holds true because $\varrho > r$.

Finally, we note that (E.6)–(E.7) and the fact that $A, B \in \mathbb{R}$ imply that $\lim_{w\downarrow 0}\big|u'(w)\big| = \big|A\varphi'(0) + B\psi'(0)\big| < \infty$. Combining this observation with the continuity of u' and the fact that $u'(w) = -1$ for all $w \geq w_{\mathrm{c}}$, we can see that (F.19) holds true. ∎

We now turn to the case arising when it turns out that $\bar{w} \le w_\circ = w_\mathrm{h}$. In this case, C^2-continuity of the function u defined by (F.6) at w_c implies that the parameters A and B should again be given by (F.14) and (F.15). On the other hand, the boundary condition (F.13) should now be considered for $w_\mathrm{h} = w_\circ$. As a result, we now obtain equation

$$h(w_\circ, w_\mathrm{c}) = -\frac{2\kappa(r+q)}{\sigma^2\lambda^2}. \tag{F.26}$$

in place of (F.16). Furthermore, (F.5) yields the equation

$$u'(w_\circ) \equiv A\varphi_1'(w_\circ) + B\psi_1'(w_\circ) = 0. \tag{F.27}$$

Substituting the expressions for A, B from (F.14), (F.15) in (F.27), we obtain

$$\frac{\psi''(w_\mathrm{c})}{\varphi''(w_\mathrm{c})} = \frac{\psi'(w_\circ)}{\varphi'(w_\circ)}, \tag{F.28}$$

which implies that

$$w_\mathrm{c} = g(w_\circ) > w_\circ, \tag{F.29}$$

where g is defined in Lemma E-2. Substituting this expression for w_c in (F.26) we see that, if it exists, then $w_\circ$ must satisfy

$$J(w_\circ) \equiv h\big(w_\circ, g(w_\circ)\big) = -\frac{2\kappa(r+q)}{\sigma^2\lambda^2}. \tag{F.30}$$

Lemma F-2. *Equation (F.30) has a unique solution $w_\circ$ that identifies with the point $\bar{w}_\dagger$ in Lemma F-1. If the problem data is such that $\bar{w} \le \bar{w}_\dagger$, then the function u defined by (F.6) for A, B given by (F.14), (F.15), and w_c being given by (F.29), is a concave C^2 solution to the HJB equation (F.1) that satisfies the boundary condition (F.2) with $w_\mathrm{h} = w_\circ \ge \bar{w}$ as well as the inequality*

$$\big|u'(w)\big| \le K \quad \text{for all } w \ge 0, \tag{F.31}$$

for some constant $K > 0$.

Proof. The claims about $w_\circ$ follow immediately from Lemma F-1 and a straightforward comparison of (F.17) and (F.30). Also, the equivalences in (F.21) imply that $w_\mathrm{h} = w_\circ \ge \bar{w}$ if and only if $\bar{w} \le \bar{w}_\dagger$. The rest of the proof is the same as in the proof of Lemma F-1. ∎

Finally, we consider the case arising when it turns out that $w_\mathrm{c} < \bar{w} = w_\mathrm{h}$. In this case, C^2-continuity of the function u defined by (F.6) at w_c implies that the parameters A and B should again be given by (F.14) and (F.15). On the other hand, (F.6) and the inequality $w_\mathrm{c} < \bar{w}$ imply that

$$\begin{aligned} u(w_\mathrm{h}) &= u(\bar{w}) = u(w_\mathrm{c}) - (\bar{w} - w_\mathrm{c}) \\ &= A\varphi_1(w_\mathrm{c}) + B\psi_1(w_\mathrm{c}) + \frac{\mu + q[(1+\gamma)u(\bar{w}) - \kappa]}{r+q} - (\bar{w} - w_\mathrm{c}). \end{aligned}$$

Using this expression and (F.14)–(F.15), we can see that the boundary condition (F.2) yields the equation

$$\hat{h}(w_{\mathrm{c}}, \bar{w}) = -\frac{2\kappa(r+q)}{\sigma^2\lambda^2}, \tag{F.32}$$

where

$$\hat{h}(w, \bar{w}) = H(w) + \frac{2(r+q)}{\sigma^2\lambda^2}(\bar{w} - w),$$

with H being as in (F.8).

Lemma F-3. *There exists a unique point $w_{\mathrm{c}} = w_{\mathrm{c}}\left(\bar{w}, \frac{\varrho+q}{\sigma^2\lambda^2}, \frac{r+q}{\sigma^2\lambda^2}, \kappa\right) > 0$ such that (F.32) holds true. Also, $w_{\mathrm{c}} < \bar{w}$ if and only if $\bar{w} > \bar{w}_{\ddagger}$, where the point $\bar{w}_{\ddagger} = \bar{w}_{\ddagger}\left(\frac{\varrho+q}{\sigma^2\lambda^2}, \frac{r+q}{\sigma^2\lambda^2}, \kappa\right) > 0$ is as in Lemma F-1. Furthermore, if the problem data is such that $\bar{w} > \bar{w}_{\ddagger}$, then the function u defined by (F.6) for A, B given by (F.14), (F.15), and w_{c} being the solution of (F.32), is a concave C^2 solution to the HJB equation (F.1) that satisfies the boundary condition (F.2) with $w_{\mathrm{h}} = \bar{w} > w_{\circ}$ as well as the inequality*

$$\left|u'(w)\right| \leq K \quad \text{for all } w \geq 0,$$

for some constant $K > 0$.

Proof. In view of the properties of the function H that we established in Lemma E-3.(II), we can see that $\hat{h}(\cdot, \bar{w})$ is strictly concave, $\hat{h}(0, \bar{w}) = \frac{2(r+q)}{\sigma^2\lambda^2}\bar{w} > 0$ and $\lim_{w\to\infty} \hat{h}(w, \bar{w}) = -\infty$. It follows that there exists a unique solution $w_{\mathrm{c}} > 0$ to the equation (F.32). Furthermore, this solution is strictly less that $\bar{w}$ if and only if $\hat{h}(\bar{w}, \bar{w}) = H(\bar{w}) < -\frac{2\kappa(r+q)}{\sigma^2\lambda^2}$, which is equivalent to $\bar{w} > \bar{w}_{\ddagger}$. The rest of the proof is exactly the same as in the proof of Lemma F-1. ∎

The following result provides a necessary and sufficient condition for the function u to identify with the value function.

Lemma F-4. *The concave C^2 function u studied in Lemma F-1, Lemma F-2 or Lemma F-3, depending on whether $\bar{w} \in \,]\bar{w}_{\dagger}, \bar{w}_{\ddagger}]$, $\bar{w} \in \,]0, \bar{w}_{\dagger}]$ or $\bar{w} > \bar{w}_{\ddagger}$, where $0 < \bar{w}_{\dagger} < \bar{w}_{\ddagger}$ are as in Lemma F-1, identifies with the value function v if and only if the problem data is such that the inequality*

$$\gamma(\mu + r\kappa) \geq (r + \gamma\varrho)w_{\mathrm{c}} - \frac{\sigma^2\lambda^2 r(1+\gamma)}{2(r+q)}H(w_{\mathrm{c}}) \tag{F.33}$$

holds true, where H is as in (F.8) and w_{c} is given by (F.16), (F.29) or (F.32), depending on the case.

Proof. By construction, the function u will satisfy the HJB equation (D.1)–(D.2) in Theorem D-1 if and only if

$$\frac{1}{2}\sigma^2\lambda^2 u''(w) + \varrho w u'(w) - r u(w) + \mu \leq 0 \quad \text{for all } w \in \,]0, w_{\mathrm{c}}[. \tag{F.34}$$

Since u satisfies the ODE (F.3), we can see that (F.34) holds true if and only if

$$-wu'(w) + u(w) - (1+\gamma)u(w_{\mathrm{h}}) + \kappa \le 0 \quad \text{for all } w \in \left]0, w_{\mathrm{c}}\right[.$$

Furthermore, the concavity of u and (F.9) imply that this inequality is equivalent to

$$u(w_{\mathrm{c}}) \le -w_{\mathrm{c}} + (1+\gamma)u(w_{\mathrm{h}}) - \kappa, \tag{F.35}$$

which, in view of (F.11), is equivalent to

$$w_{\mathrm{c}} \ge \frac{\mu + r\kappa - r(1+\gamma)u(w_{\mathrm{h}})}{\varrho - r}. \tag{F.36}$$

Using the identities

$$u(w_{\mathrm{h}}) = u(0) + \kappa = A + \frac{\mu + q\big[(1+\gamma)u(w_{\mathrm{h}}) - \kappa\big]}{r+q} + \kappa$$

to derive an expression for $u(w_{\mathrm{h}})$, and substituting for A using (F.14) and the definition of H in (E.33), we calculate

$$\begin{aligned} u(w_{\mathrm{h}}) &= \frac{r+q}{r-q\gamma}A + \frac{\mu + r\kappa}{r-q\gamma} \\ &= \frac{\sigma^2\lambda^2}{2(r-q\gamma)}H(w_{\mathrm{c}}) - \frac{\varrho+q}{r-q\gamma} + \frac{\mu+r\kappa}{r-q\gamma} \end{aligned}$$

(note that this result is valid in the context of either one of Lemmas F-1–F-3). It is then a matter of simple algebraic manipulation to derive the equivalence of (F.36) and (F.33).

In view of the results derived in Lemmas F-1–F-3, we conclude that u satisfies all of the requirements of Theorem D-1, and therefore $u = v$, if and only if the problem's parameters are such that (F.33) holds true. ∎

F.2 Proof of Proposition 4

Apart from (i) and (iv), all claims follow immediately from Lemmas F-1–F-4. To prove statement (i), we first recall that the set of all permissible parameter values is

$$\Big\{(r,\varrho,\mu,\sigma,q,\gamma,\lambda,\kappa,\bar{w}) \in \mathbb{R}^9 \mid r,\varrho,\sigma,q,\gamma,\kappa > 0,\ \lambda \in (0,1], \\ \varrho > r > q\gamma \text{ and } \frac{\gamma\mu}{r} > \kappa + (1+\gamma)\bar{w}\Big\}$$

(see Conditions (12)–(14)). We next fix any values of $r,\varrho,\sigma,q,\gamma,\kappa > 0$ and $\lambda \in (0,1]$ such that $\varrho > r > q\gamma$, and we note that these determine the value of $\bar{w}_\ddagger = \bar{w}_\ddagger\big(\frac{\varrho+q}{\sigma^2\lambda^2}, \frac{r+q}{\sigma^2\lambda^2}, \kappa\big)$ defined in Lemma F-1 (see (F.17)). Furthermore, we consider the inequality

$$\frac{\gamma\mu}{r} > \kappa + (1+\gamma)\bar{w} \tag{F.37}$$

(see Condition (14)), as well as the inequality

$$\frac{\gamma\mu}{r} \geq \ell(\bar{w}), \tag{F.38}$$

where

$$\begin{aligned}
\ell(\bar{w}) &= \ell(\bar{w};\, r, \varrho, \sigma, q, \gamma, \lambda, \kappa) \\
&= \kappa + (1+\gamma)\bar{w} + \frac{(\varrho - r)\gamma}{r} w_{\mathrm{c}} \\
&\quad + (1+\gamma)(w_{\mathrm{c}} - \bar{w}) - \frac{\sigma^2\lambda^2 r(1+\gamma)}{2(r+q)} \left[H(w_{\mathrm{c}}) + \frac{2\kappa(r+q)}{\sigma^2\lambda^2} \right],
\end{aligned}$$

which is equivalent to (F.33) (recall that $w_{\mathrm{c}} = w_{\mathrm{c}}\left(\bar{w}, \frac{\varrho+q}{\sigma^2\lambda^2}, \frac{r+q}{\sigma^2\lambda^2}, \kappa\right)$). In view of Lemma F-4, the result will follow if we show that the set of values of $\mu, \bar{w} > 0$ for which (F.37)–(F.38) both hold true contains an open subset of $\mathbb{R}^2$. To this end, we use (F.17) and the identity $w_{\mathrm{c}}\left(\bar{w}_{\ddagger}, \frac{\varrho+q}{\sigma^2\lambda^2}, \frac{r+q}{\sigma^2\lambda^2}, \kappa\right) = \bar{w}_{\ddagger}$ (see (F.18)) to calculate

$$\ell(\bar{w}_{\ddagger}) = \kappa + (1+\gamma)\bar{w}_{\ddagger} + \frac{(\varrho - r)\gamma}{r} \bar{w}_{\ddagger}. \tag{F.39}$$

The continuity of the functions H and $w_{\mathrm{c}}\left(\cdot, \frac{\varrho+q}{\sigma^2\lambda^2}, \frac{r+q}{\sigma^2\lambda^2}, \kappa\right)$ implies that there exists $\varepsilon_1 \in \left]0, \bar{w}_{\ddagger}\right]$ such that

$$\ell(\bar{w}) < \ell(\bar{w}_{\ddagger}) + 1 \quad \text{for all } \bar{w} \in \left]\bar{w}_{\ddagger} - \varepsilon_1, \bar{w}_{\ddagger} + \varepsilon_1\right[. \tag{F.40}$$

If we define $\varepsilon = \varepsilon_1 \wedge (1+\gamma)^{-1}$, then

$$\begin{aligned}
\kappa + (1+\gamma)\bar{w} &< \kappa + (1+\gamma)\bar{w}_{\ddagger} + (1+\gamma)\varepsilon \\
&\leq \kappa + (1+\gamma)\bar{w}_{\ddagger} + 1 \\
&< \ell(\bar{w}_{\ddagger}) + 1 \quad \text{for all } \bar{w} \in \left]\bar{w}_{\ddagger} - \varepsilon, \bar{w}_{\ddagger} + \varepsilon\right[.
\end{aligned}$$

It follows that, given any point $(\mu, \bar{w})$ in the open set $\left]r\big(\ell(\bar{w}_{\ddagger}) + 1\big)/\gamma, \infty\right[\times \left]\bar{w}_{\ddagger} - \varepsilon, \bar{w}_{\ddagger} + \varepsilon\right[$, the inequalities (F.37)–(F.38) both hold true, and the proof of statement (i) is complete.

Finally, statement (iv) follows immediately once we combine (F.17) and (F.18) in Lemma F-1 with the claim associated with (F.33) in Lemma F-4. In particular, we note for future reference that, if $\bar{w} = \bar{w}_{\ddagger}$, then the equivalence of (F.33) and (F.35) implies that

$$(1+\gamma)u(\bar{w}) - \kappa - w_{\mathrm{c}} - u(w_{\mathrm{c}}) \geq 0 \quad \Leftrightarrow \quad \gamma\mu \geq r\kappa + (r + \gamma\varrho)\bar{w} \tag{F.41}$$

and

$$(1+\gamma)u(\bar{w}) - \kappa - w_{\mathrm{c}} - u(w_{\mathrm{c}}) = 0 \quad \Leftrightarrow \quad \gamma\mu = r\kappa + (r + \gamma\varrho)\bar{w}. \tag{F.42}$$

F.3 Proof of Proposition 5

We fix any initial permissible values r_0, ϱ_0, μ_0, σ_0, q_0, γ_0, λ_0, κ_0, $\bar{w}_0$ of the parameters r, ϱ, μ, σ, q, γ, λ, κ, $\bar{w}$ such that

$$\bar{w}_0 = \bar{w}_{\ddagger}\left(\frac{\varrho_0 + q_0}{\sigma_0^2\lambda_0^2}, \frac{r_0 + q_0}{\sigma_0^2\lambda_0^2}, \kappa_0\right) \quad \text{and} \quad \gamma_0\mu_0 = r_0\kappa_0 + (r_0 + \gamma_0\varrho_0)\bar{w}_0. \tag{F.43}$$

It is immediate to see that a drop in μ causes (35) to fail. Moreover, the second equality in (F.43) is equivalent to

$$\gamma_0(\mu_0 - \varrho_0 \bar{w}_0) = r_0(\kappa_0 + \bar{w}_0) > 0,$$

from which it is immediate to see that a drop in γ also causes (35) to fail.

Using p to stand for either λ, σ, κ or q, we define

$$\Theta(p) = (1+\gamma)u(\bar{w};p) - \kappa - w_{\mathrm{c}}(p) - u\big(w_{\mathrm{c}}(p);p\big),$$

where we write $u(\cdot;p)$ and $w_{\mathrm{c}}(p)$ to stress the dependence on p of the solution u and w_{c} to the free-boundary problem. We also use the notation

$$u' = \frac{\partial u}{\partial w}, \quad u'' = \frac{\partial^2 u}{\partial w^2}, \quad u_p = \frac{\partial u}{\partial p}, \quad u'_p = \frac{\partial^2 u}{\partial p\, \partial w} \quad \text{and} \quad u''_p = \frac{\partial^3 u}{\partial p\, \partial w^2}.$$

In view of the identity $u'\big(w_{\mathrm{c}}(p);p\big) = -1$, we can see that

$$\begin{aligned}\Theta'(p) &= (1+\gamma)u_p(\bar{w};p) - \frac{\partial \kappa}{\partial p} - w'_{\mathrm{c}}(p) - \big[w'_{\mathrm{c}}(p)u'\big(w_{\mathrm{c}}(p);p\big) + u_p\big(w_{\mathrm{c}}(p);p\big)\big] \\ &= (1+\gamma)u_p(\bar{w};p) - \frac{\partial \kappa}{\partial p} - u_p\big(w_{\mathrm{c}}(p);p\big).\end{aligned}$$

This calculation and the identity $w_{\mathrm{c}}(p_0) = \bar{w}_0$ (see also (F.18)), where p_0 stands for either λ_0, σ_0, κ_0 or q_0, imply that

$$\Theta'(p_0) = \gamma u_p(\bar{w}_0;p_0) - \frac{\partial \kappa}{\partial p}. \tag{F.44}$$

Furthermore, we can see that (F.42) implies that

$$\Theta(p_0) = 0. \tag{F.45}$$

To proceed further, we note that, for $\bar{w} \geq \bar{w}_\dagger$, the C^2 function u satisfies the ODE (F.3) in $]0, w_{\mathrm{c}}[$ with $\bar{w}$ in place of w_{h}, as well as the boundary conditions

$$u(0) = u(\bar{w}) - \kappa \quad \text{and} \quad u'(w_{\mathrm{c}}) = -1 \tag{F.46}$$

(see Proposition 4.(ii)). Differentiating with respect to λ, we can see that u_λ satisfies the ODE

$$\frac{1}{2}\sigma^2\lambda^2 u''_\lambda(w) + (\varrho + q)w u'_\lambda(w) - (r+q)u_\lambda(w) + \sigma^2\lambda u''(w) + q(1+\gamma)u_\lambda(\bar{w}) = 0$$

in $]0, w_{\mathrm{c}}[$, with boundary conditions

$$u_\lambda(0) = u_\lambda(\bar{w}) \quad \text{and} \quad u'_\lambda(w_{\mathrm{c}}) = 0.$$

In view of these expressions, a Feynman-Kac type of formula implies that

$$\begin{aligned}u_\lambda(w) &= \sigma^2\lambda\, \mathbb{E}\left[\int_0^\tau e^{-(r+q)t} u''(\tilde{W}_t)\, dt\right] + q(1+\gamma)\, \mathbb{E}\left[\int_0^\tau e^{-(r+q)t}\, dt\right] u_\lambda(\bar{w}) + Q u_\lambda(\bar{w}) \\ &= \sigma^2\lambda\, \mathbb{E}\left[\int_0^\tau e^{-(r+q)t} u''(\tilde{W}_t)\, dt\right] + \left(\frac{q(1+\gamma)}{r+q}(1-Q) + Q\right) u_\lambda(\bar{w}),\end{aligned} \tag{F.47}$$

where $\tilde{W}$ is the solution to the SDE

$$d\tilde{W}_t = (\varrho + q)\tilde{W}_t\, dt - dC_t + \sigma\lambda\, dZ_t, \quad \tilde{W}_0 = w, \tag{F.48}$$

with C reflecting $\tilde{W}$ in w_c in the negative direction, τ is the first hitting time of zero by $\tilde{W}$, and

$$Q = \mathbb{E}\left[e^{-(r+q)\tau}\right]. \tag{F.49}$$

Evaluating the left-hand side of (F.47) at $w = \bar{w}$ and rearranging terms, we obtain

$$u_\lambda(\bar{w}) = \frac{(r+q)\sigma^2\lambda^2}{(1-Q)(r-q\gamma)}\,\mathbb{E}\left[\int_0^\tau e^{-(r+q)t}u''(\tilde{W}_t)\, dt\right] < 0, \tag{F.50}$$

the inequality following thanks to the strict concavity of u in $]0, w_c[$ and the fact that $Q \in\,]0,1[$. Combining this result with (F.44)–(F.45) for p standing for λ and the equivalence stated in (F.41), we can see that, if $\lambda_0 < 1$, then $\Theta(\lambda) < 0$ and (35) fails for all $\lambda > \lambda_0$ sufficiently close to λ_0. Using exactly the same arguments, we can prove the required result involving σ.

Differentiating (F.3) and (F.46) with respect to κ, we can see that u_κ satisfies the ODE

$$\frac{1}{2}\sigma^2\lambda^2 u_\kappa''(w) + (\varrho+q)wu_\kappa'(w) - (r+q)u_\kappa(w) + q(1+\gamma)u_\kappa(\bar{w}) - q = 0$$

in $]0, w_c[$, with boundary conditions

$$u_\kappa(0) = u_\kappa(\bar{w}) - 1 \quad \text{and} \quad u_\kappa'(w_c) = 0.$$

It follows that

$$\begin{aligned} u_\kappa(w) &= q\big[(1+\gamma)u_\kappa(\bar{w}) - 1\big]\,\mathbb{E}\left[\int_0^\tau e^{-(r+q)t}\, dt\right] + Q\big[u_\kappa(\bar{w}) - 1\big] \\ &= q\big[(1+\gamma)u_\kappa(\bar{w}) - 1\big]\frac{1-Q}{r+q} + Q\big[u_\kappa(\bar{w}) - 1\big], \end{aligned}$$

where $\tilde{W}$, τ and Q are as in (F.48)–(F.49). Evaluating the left-hand side of this expression at $w = \bar{w}$ and rearranging terms, we obtain

$$(1-Q)\frac{r-q\gamma}{r+q}u_\kappa(\bar{w}) = -Q - \frac{1-Q}{r+q} < 0,$$

which implies $u_\kappa(\bar{w}) < 0$. Combining this inequality with (F.44)–(F.45) for p standing for κ and the equivalence stated in (F.41) we can see that $\Theta(\kappa) < 0$ and (35) fails for all $\kappa > \kappa_0$ sufficiently close to κ_0.

Finally, differentiating (F.3) and (F.46) with respect to q, we can see that u_q satisfies the ODE

$$\begin{gathered} \frac{1}{2}\sigma^2\lambda^2 u_q''(w) + (\varrho+q)wu_q'(w) - (r+q)u_q(w) \\ + q(1+\gamma)u_q(\bar{w}) + (1+\gamma)u(\bar{w}) - \kappa + wu'(w) - u(w) = 0 \end{gathered}$$

in $]0, w_{\mathrm{c}}[$, with boundary conditions

$$u_q(0) = u_q(\bar{w}) \quad \text{and} \quad u_q'(w_{\mathrm{c}}) = 0.$$

Using the same arguments as above, we obtain

$$(1 - Q)\frac{r - q\gamma}{r + q} u_q(\bar{w}) = \mathbb{E}\left[\int_0^\tau e^{-(r+q)t} f(\tilde{W}_t)\, dt\right], \tag{F.51}$$

where $\tilde{W}$, τ and Q are as in (F.48)–(F.49), and

$$f(w) = (1 + \gamma)u(\bar{w}) - \kappa + wu'(w) - u(w). \tag{F.52}$$

The concavity of u implies that the function f is decreasing in w. On the other hand, the identity $u'(w_{\mathrm{c}}) = -1$ and (F.42) imply that $f(w_{\mathrm{c}}) = 0$ if the parameter values are such that (F.43) holds true. Therefore, the right-hand side of (F.51) is positive and $u_q(\bar{w}) > 0$ for such parameter values. This inequality and (F.44)–(F.45) for p standing for q, together with the equivalence stated in (F.41) imply that $\Theta(q) < 0$ and (35) fails for all $q < q_0$ sufficiently close to q_0.

F.4 Proof of Proposition 6

Fix any $(r, \varrho, \mu, \sigma, q, \gamma, \lambda, \kappa, \bar{w})$ in the interior of the set of permissible parameter values for which the firm is of a high-growth type. Statement (i) follows from (F.16), (F.29)–(F.30) and (F.32).

To establish statement (ii), we further assume that the parameters are initially such that $\bar{w} = \bar{w}_\ddagger\left(\frac{\varrho+q}{\sigma^2\lambda^2}, \frac{r+q}{\sigma^2\lambda^2}, \kappa\right)$, so that $w_{\mathrm{c}} = w_{\mathrm{h}} = \bar{w}$ (see Proposition 4). To establish the sensitivity of w_{c} with respect to λ, we consider equation (F.11) that w_{c} satisfies, namely,

$$-(r + q)u(w_{\mathrm{c}}) - (\varrho + q)w_{\mathrm{c}} + \mu + q\big[(1 + \gamma)u(\bar{w}) - \kappa\big] = 0. \tag{F.53}$$

Differentiating with respect to λ and using the same notation as the one introduced at the beginning of Section F.3, we obtain

$$w_{\mathrm{c}}'(\lambda) = -\frac{(r + q)u_\lambda(w_{\mathrm{c}}) - q(1 + \gamma)u_\lambda(\bar{w})}{\varrho - r} = -\frac{r - q\gamma}{\varrho - r} u_\lambda(\bar{w}) > 0,$$

the second identity being true because $w_{\mathrm{c}} = \bar{w} = \bar{w}_\dagger$ (see also (F.17)), and the strict inequality following from (F.50) and the permissibility conditions $\varrho > r > q\gamma$.

The sensitivity of w_{c} with respect to σ is the same as the one with respect to λ because w_{c} depends on either of these two parameters via the product $\sigma\lambda$.

To establish the sensitivity of w_{c} with respect to q and complete the proof, we differentiate (F.53) with respect to q to obtain

$$\begin{aligned}(\varrho - r)w_{\mathrm{c}}'(q) &= -\Big((r + q)u_q(w_{\mathrm{c}}) - q(1 + \gamma)u_q(\bar{w}) - \big[(1 + \gamma)u(\bar{w}) - \kappa - w_{\mathrm{c}} - u(w_{\mathrm{c}})\big]\Big) \\ &= -\Big((r - q\gamma)u_q(\bar{w}) - \big[(1 + \gamma)u(\bar{w}) - \kappa - w_{\mathrm{c}} - u(w_{\mathrm{c}})\big]\Big),\end{aligned}$$

where the second identity follows from the fact that $w_{\mathrm{c}} = \bar{w} = \bar{w}_{\dagger}$. Therefore,

$$w_{\mathrm{c}}'(q) = -\frac{(r - q\gamma)u_q(\bar{w}) - f(w_{\mathrm{c}})}{\varrho - r},$$

where f is defined by (F.52). Combining the fact that f is decreasing in w, which follows from the concavity of u, with (F.51) and the definition (F.49) of Q, we can see that

$$(r - q\gamma)u_q(\bar{w}) > \frac{r+q}{1-Q}\,\mathbb{E}\left[\int_0^{\tau} e^{-(r+q)t} f(w_{\mathrm{c}})\,dt\right] = f(w_{\mathrm{c}}),$$

and the desired inequality $w_{\mathrm{c}}'(q) < 0$ follows.

F.5 The "No-Growth" Case

We close this appendix by considering the "no-growth" configuration that arises from the verification Theorem D-1 when

$$-wu'(w) + u(w) - (1+\gamma)u(w_{\mathrm{h}}) + \kappa > 0 \quad \text{for all } w \in [0, w_{\mathrm{c}}], \tag{F.54}$$

in which case the point w_{g} defined by (D.3) is equal to zero. In such a configuration, the value function v should identify with a concave solution u to the HJB equation

$$\max\left\{\frac{1}{2}\sigma^2\lambda^2 u''(w) + \varrho w u'(w) - ru(w) + \mu,\ -u'(w) - 1\right\} = 0 \tag{F.55}$$

that satisfies (F.54) as well as the boundary condition

$$u(0) = u(w_{\mathrm{h}}) - \kappa, \tag{F.56}$$

where $w_{\mathrm{h}} := \bar{w} \vee \arg\max_{w>0} u(w)$. The concavity of the solution to (F.55)–(F.56) implies that (F.54) is satisfied if and only if the inequality is true for $w = 0$. Therefore, in view of the fact that $\lim_{w\downarrow 0}\left|u'(w)\right| < \infty$ and the boundary condition (F.56), we can see that, if the solution to (F.55)–(F.56) is such that (F.54) holds, then $u(w_{\mathrm{h}}) < 0$.

G The Low-Growth Case

In this appendix, we consider the case arising from the verification Theorem D-1 when $0 < w_{\mathrm{g}} < w_{\mathrm{c}}$. In this context, we address the problem of constructing a solution to the HJB equation (D.1)–(D.2) such that

$$wu'(w) - u(w) + (1+\gamma)u(w_{\mathrm{h}}) - \kappa \begin{Bmatrix} > 0, & \text{if } w \in [0, w_{\mathrm{g}}[\\ < 0, & \text{if } w \in \,]w_{\mathrm{g}}, w_{\mathrm{c}}] \end{Bmatrix}. \tag{G.1}$$

To this end, we look for a concave C^2 function $u : \mathbb{R}_+ \to \mathbb{R}$ and for strictly positive free-boundary points $w_{\mathrm{g}} < w_{\mathrm{c}}$ such that u satisfies the ODE

$$\frac{1}{2}\sigma^2\lambda^2 u''(w) + (\varrho + q)wu'(w) - (r+q)u(w) + \mu + q\big[(1+\gamma)u(w_{\mathrm{h}}) - \kappa\big] = 0 \tag{G.2}$$

in $]0, w_{\mathrm{g}}[$, the ODE

$$\frac{1}{2}\sigma^2\lambda^2 u''(w) + \varrho wu'(w) - ru(w) + \mu = 0 \tag{G.3}$$

in $]w_{\mathrm{g}}, w_{\mathrm{c}}[$, is given by

$$u(w) = u(w_{\mathrm{c}}) - (w - w_{\mathrm{c}}), \quad \text{for } w > w_{\mathrm{c}}, \tag{G.4}$$

and satisfies the Wentzel-type boundary condition

$$u(0) = u(w_{\mathrm{h}}) - \kappa, \tag{G.5}$$

where $w_{\mathrm{h}} := \bar{w} \vee w_\circ := \bar{w} \vee \arg\max_{w>0} u(w)$. We also note that (G.4) and (G.5) imply that

$$w_\circ = \arg\max_{w>0} u(w) \in \,]0, w_{\mathrm{c}}[. \tag{G.6}$$

G.1 Analysis of the Free-Boundary Problem

If it exists, then the solution to the free-boundary problem (G.2)–(G.5) is of the form

$$u(w) = \begin{Bmatrix} A_1\varphi_1(w) + B_1\psi_1(w) + \frac{\mu}{r+q} + \frac{q}{r+q}V_{\mathrm{g}}, & \text{if } w \in [0, w_{\mathrm{g}}] \\ A_2\varphi_2(w) + B_2\psi_2(w) + \frac{\mu}{r}, & \text{if } w \in \,]w_{\mathrm{g}}, w_{\mathrm{c}}] \\ u(w_{\mathrm{c}}) - (w - w_{\mathrm{c}}), & \text{if } w \in \,]w_{\mathrm{c}}, \infty[\end{Bmatrix}, \tag{G.7}$$

where

$$V_{\mathrm{g}} = (1+\gamma)u(w_{\mathrm{h}}) - \kappa, \tag{G.8}$$

for some constants $A_1, B_1, A_2, B_2 \in \mathbb{R}$, where φ_1 and ψ_1 identify with the functions φ and ψ in Appendix E for $\zeta = \varrho + q$ and $\delta = r + q$, while φ_2 and ψ_2 identify with the functions φ and ψ in Appendix E for $\zeta = \varrho$ and $\delta = r$.

To determine the four parameters A_1, B_1, A_2, B_2 and the two free-boundary points w_{g}, w_{c}, we note that C^1 continuity of u at w_{g} implies

$$u(w_{\mathrm{g}}-) \equiv A_1\varphi_1(w_{\mathrm{g}}) + B_1\psi_1(w_{\mathrm{g}}) + \frac{\mu}{r+q} + \frac{q}{r+q}V_{\mathrm{g}} \\ = A_2\varphi_2(w_{\mathrm{g}}) + B_2\psi_2(w_{\mathrm{g}}) + \frac{\mu}{r} \equiv u(w_{\mathrm{g}}+) \tag{G.9}$$

and

$$u'(w_{\mathrm{g}}-) \equiv A_1\varphi_1'(w_{\mathrm{g}}) + B_1\psi_1'(w_{\mathrm{g}}) = A_2\varphi_2'(w_{\mathrm{g}}) + B_2\psi_2'(w_{\mathrm{g}}) \equiv u'(w_{\mathrm{g}}+). \tag{G.10}$$

Furthermore, C^2 continuity of u at w_{g} and the fact that u satisfies (G.2)–(G.3) imply that

$$w_{\mathrm{g}}u'(w_{\mathrm{g}}) - u(w_{\mathrm{g}}) + V_{\mathrm{g}} = 0. \tag{G.11}$$

This equation is equivalent to $u'(w_{\mathrm{g}}) = \big[u(w_{\mathrm{g}}) - V_{\mathrm{g}}\big]/w_{\mathrm{g}}$, and can therefore be viewed as a "tangency condition" at w_{g}. C^2 continuity of u at w_{c} gives rise to the equations

$$u'(w_{\mathrm{c}}-) \equiv A_2\varphi_2'(w_{\mathrm{c}}) + B_2\psi_2'(w_{\mathrm{c}}) = -1 \equiv u'(w_{\mathrm{c}}+) \tag{G.12}$$

and

$$u''(w_{\mathrm{c}}-) \equiv A_2\varphi_2''(w_{\mathrm{c}}) + B_2\psi_2''(w_{\mathrm{c}}) = 0 \equiv u''(w_{\mathrm{c}}+). \tag{G.13}$$

For future reference, we observe that, given any point $w_{\mathrm{c}} > 0$, the solution to (G.3) satisfying (G.12)–(G.13) is such that

$$-\varrho w_{\mathrm{c}} - ru(w_{\mathrm{c}}) + \mu = 0. \tag{G.14}$$

Finally, the boundary condition (G.5) implies that

$$u(0) \equiv A_1 + \frac{\mu}{r+q} + \frac{q}{r+q}V_{\mathrm{g}} = u(w_{\mathrm{h}}) - \kappa \tag{G.15}$$

(see also (E.7) and (E.11)).

Using the fact that φ_2, ψ_2 satisfy (E.9) and (E.12) in Appendix E for $\zeta = \varrho$ and $\delta = r$, we can see that the equations (G.12)–(G.13) are equivalent to

$$A_2 \equiv A_2(w_{\mathrm{c}}) = \exp\left(\frac{\varrho}{\sigma^2\lambda^2}w_{\mathrm{c}}^2\right)\left[\psi_2(w_{\mathrm{c}}) - \frac{\varrho}{r}w_{\mathrm{c}}\psi_2'(w_{\mathrm{c}})\right] \\ = \frac{\sigma^2\lambda^2}{2r}\exp\left(\frac{\varrho}{\sigma^2\lambda^2}w_{\mathrm{c}}^2\right)\psi_2''(w_{\mathrm{c}}) < 0 \tag{G.16}$$

and

$$B_2 \equiv B_2(w_{\mathrm{c}}) = -\exp\left(\frac{\varrho}{\sigma^2\lambda^2}w_{\mathrm{c}}^2\right)\left[\varphi_2(w_{\mathrm{c}}) - \frac{\varrho}{r}w_{\mathrm{c}}\varphi_2'(w_{\mathrm{c}})\right] \\ = -\frac{\sigma^2\lambda^2}{2r}\exp\left(\frac{\varrho}{\sigma^2\lambda^2}w_{\mathrm{c}}^2\right)\varphi_2''(w_{\mathrm{c}}) < 0, \tag{G.17}$$

the inequalities following thanks to the results in Lemma E-1. Also, we can verify that (G.9)–(G.10) are equivalent to

$$A_1 \equiv A_1(w_{\mathrm{g}}, w_{\mathrm{c}}, V_{\mathrm{g}}) = Q_1(w_{\mathrm{g}}, w_{\mathrm{c}}) - \frac{q}{r+q} \exp\left(\frac{\varrho + q}{\sigma^2\lambda^2} w_{\mathrm{g}}^2\right) \psi_1'(w_{\mathrm{g}}) V_{\mathrm{g}} \tag{G.18}$$

and

$$B_1 \equiv A_1(w_{\mathrm{g}}, w_{\mathrm{c}}, V_{\mathrm{g}}) = Q_2(w_{\mathrm{g}}, w_{\mathrm{c}}) + \frac{q}{r+q} \exp\left(\frac{\varrho + q}{\sigma^2\lambda^2} w_{\mathrm{g}}^2\right) \varphi_1'(w_{\mathrm{g}}) V_{\mathrm{g}}, \tag{G.19}$$

where

$$\begin{aligned} Q_1(w_{\mathrm{g}}, w_{\mathrm{c}}) = {} & \frac{\sigma^2\lambda^2}{2r} \exp\left(\frac{\varrho + q}{\sigma^2\lambda^2} w_{\mathrm{g}}^2\right) \exp\left(\frac{\varrho}{\sigma^2\lambda^2} w_{\mathrm{c}}^2\right) \\ & \times \Big(\big[\varphi_2(w_{\mathrm{g}})\psi_1'(w_{\mathrm{g}}) - \varphi_2'(w_{\mathrm{g}})\psi_1(w_{\mathrm{g}})\big] \psi_2''(w_{\mathrm{c}}) \\ & \qquad + \big[\psi_1(w_{\mathrm{g}})\psi_2'(w_{\mathrm{g}}) - \psi_1'(w_{\mathrm{g}})\psi_2(w_{\mathrm{g}})\big] \varphi_2''(w_{\mathrm{c}}) \Big) \\ & + \frac{\mu q}{r(r+q)} \exp\left(\frac{\varrho + q}{\sigma^2\lambda^2} w_{\mathrm{g}}^2\right) \psi_1'(w_{\mathrm{g}}) \end{aligned} \tag{G.20}$$

and

$$\begin{aligned} Q_2(w_{\mathrm{g}}, w_{\mathrm{c}}) = {} & \frac{\sigma^2\lambda^2}{2r} \exp\left(\frac{\varrho + q}{\sigma^2\lambda^2} w_{\mathrm{g}}^2\right) \exp\left(\frac{\varrho}{\sigma^2\lambda^2} w_{\mathrm{c}}^2\right) \\ & \times \Big(\big[\varphi_1(w_{\mathrm{g}})\varphi_2'(w_{\mathrm{g}}) - \varphi_1'(w_{\mathrm{g}})\varphi_2(w_{\mathrm{g}})\big] \psi_2''(w_{\mathrm{c}}) \\ & \qquad - \big[\varphi_1(w_{\mathrm{g}})\psi_2'(w_{\mathrm{g}}) - \varphi_1'(w_{\mathrm{g}})\psi_2(w_{\mathrm{g}})\big] \varphi_2''(w_{\mathrm{c}}) \Big) \\ & - \frac{\mu q}{r(r+q)} \exp\left(\frac{\varrho + q}{\sigma^2\lambda^2} w_{\mathrm{g}}^2\right) \varphi_1'(w_{\mathrm{g}}). \end{aligned} \tag{G.21}$$

The tangency condition (G.11) gives rise to the equation

$$\begin{aligned} \lim_{w \downarrow w_{\mathrm{g}}} \big[u(w) - wu'(w)\big] - \frac{\mu}{r} & \equiv A_2\big[\varphi_2(w_{\mathrm{g}}) - w_{\mathrm{g}}\varphi_2'(w_{\mathrm{g}})\big] + B_2\big[\psi_2(w_{\mathrm{g}}) - w_{\mathrm{g}}\psi_2'(w_{\mathrm{g}})\big] \\ & = V_{\mathrm{g}} - \frac{\mu}{r}. \end{aligned}$$

Substituting for A_2 and B_2 from (G.16)–(G.17), we obtain the identity

$$\begin{aligned} \frac{\sigma^2\lambda^2}{2r} \exp\left(\frac{\varrho}{\sigma^2\lambda^2} w_{\mathrm{c}}^2\right) & \Big(\big[\varphi_2(w_{\mathrm{g}}) - w_{\mathrm{g}}\varphi_2'(w_{\mathrm{g}})\big] \psi_2''(w_{\mathrm{c}}) \\ & - \big[\psi_2(w_{\mathrm{g}}) - w_{\mathrm{g}}\psi_2'(w_{\mathrm{g}})\big] \varphi_2''(w_{\mathrm{c}}) \Big) = V_{\mathrm{g}} - \frac{\mu}{r}. \end{aligned} \tag{G.22}$$

On the other hand, the boundary condition (G.15), combined with (G.8) and (G.18), yields the equation

$$Q_1(w_g, w_c) = -\frac{\mu}{r+q} - \frac{\gamma\kappa}{1+\gamma} + \frac{1}{r+q}\left[q \exp\left(\frac{\varrho+q}{\sigma^2\lambda^2}w_g^2\right)\psi_1'(w_g) + \frac{r-q\gamma}{1+\gamma}\right]V_g. \quad \text{(G.23)}$$

Similarly to the high-growth case that we have studied in Appendix F, any of the possibilities

$$\begin{aligned} &\bar{w} < w_\circ \text{ and } \big(w_\circ \le w_g < w_c \text{ or } w_g < w_\circ\big) \\ \text{or} \quad &w_\circ \le \bar{w} \le w_c \text{ and } \big(\bar{w} \le w_g < w_c \text{ or } w_g < \bar{w}\big) \\ \text{or} \quad &\bar{w} > w_c > w_g \end{aligned}$$

may hold true, where $w_\circ = \arg\max_{w>0} u(w) \in \,]0, w_c[$ (see also (G.6)). In view of (G.7)–(G.8) and (G.18)–(G.19), the case $w_\circ \le \bar{w} \le w_g < w_c$ is associated with the expressions

$$V_g = \frac{(r+q)\left[\varphi_1(\bar{w})Q_1(w_g,w_c) + \psi_1(\bar{w})Q_2(w_g,w_c) + \frac{\mu}{r+q} - \frac{\kappa}{1+\gamma}\right]}{\frac{r-q\gamma}{1+\gamma} + q\exp\left(\frac{\varrho+q}{\sigma^2\lambda^2}w_g^2\right)\left[\varphi_1(\bar{w})\psi_1'(w_g) - \psi_1(\bar{w})\varphi_1'(w_g)\right]} \quad \text{(G.24)}$$

$$\text{and} \quad A_1(w_g,w_c,V_g)\varphi_1'(\bar{w}) + B_1(w_g,w_c,V_g)\psi_1'(\bar{w}) \le 0. \quad \text{(G.25)}$$

In view of (G.7)–(G.8) and (G.16)–(G.17), the case $w_\circ \le \bar{w}$ and $w_g < \bar{w} \le w_c$ is associated with the expressions

$$V_g = (1+\gamma)\left[A_2(w_c)\varphi_2(\bar{w}) + B_2(w_c)\psi_2(\bar{w}) + \frac{\mu}{r}\right] - \kappa \quad \text{(G.26)}$$

$$\text{and} \quad A_2(w_c)\varphi_2'(\bar{w}) + B_2(w_c)\psi_2'(\bar{w}) \le 0, \quad \text{(G.27)}$$

while the case $\bar{w} > w_c > w_g$ is associated with the expression

$$V_g = (1+\gamma)\left[A_2(w_c)\varphi_2(w_c) + B_2(w_c)\psi_2(w_c) + \frac{\mu}{r} - (\bar{w} - w_c)\right] - \kappa. \quad \text{(G.28)}$$

Similarly, the case $\bar{w} < w_\circ \le w_g < w_c$ is associated with the identities

$$V_g = \frac{(r+q)\left[\varphi_1(w_\circ)Q_1(w_g,w_c) + \psi_1(w_\circ)Q_2(w_g,w_c) + \frac{\mu}{r+q} - \frac{\kappa}{1+\gamma}\right]}{\frac{r-q\gamma}{1+\gamma} + q\exp\left(\frac{\varrho+q}{\sigma^2\lambda^2}w_g^2\right)\left[\varphi_1(w_\circ)\psi_1'(w_g) - \psi_1(w_\circ)\varphi_1'(w_g)\right]} \quad \text{(G.29)}$$

$$\text{and} \quad A_1(w_g,w_c,V_g)\varphi_1'(w_\circ) + B_1(w_g,w_c,V_g)\psi_1'(w_\circ) = 0, \quad \text{(G.30)}$$

while the case $\bar{w} < w_\circ$ and $w_g < w_\circ$ is associated with the identities

$$V_g = (1+\gamma)\left[A_2(w_c)\varphi_2(w_\circ) + B_2(w_c)\psi_2(w_\circ) + \frac{\mu}{r}\right] - \kappa \quad \text{(G.31)}$$

$$\text{and} \quad A_2(w_c)\varphi_2'(w_\circ) + B_2(w_c)\psi_2'(w_\circ) = 0, \quad \text{(G.32)}$$

We are thus faced with the following problem.

Problem G-0. Determine necessary and sufficient conditions on the permissible values of the parameters r, ϱ, μ, σ, q, γ, λ, κ, $\bar{w}$ such that

(I) the system of equations (G.22)–(G.24) has a solution (w_g, w_c, V_g) such that $\bar{w} \leq w_g < w_c$ and (G.25) holds true;

(II) the system of equations (G.22)–(G.23), (G.26) has a solution (w_g, w_c, V_g) such that $0 < w_g < \bar{w} \leq w_c$ and (G.25) holds true;

(III) the system of equations (G.22)–(G.23), (G.28) has a solution (w_g, w_c, V_g) such that $w_g < w_c < \bar{w}$.

(IV) the system of equations (G.22)–(G.23), (G.29)–(G.30) has a solution $(w_\circ, w_g, w_c, V_g)$ such that $\bar{w} < w_\circ \leq w_g < w_c$.

(V) the system of equations (G.22)–(G.23), (G.31)–(G.32) has a solution $(w_\circ, w_g, w_c, V_g)$ such that $\bar{w} < w_\circ$ and $w_g < w_\circ$. ∎

Problem G-0 is substantially more challenging than the one we solved in Appendix F.1. Deriving necessary and sufficient conditions under which each of these five systems has an appropriate solution is a most challenging exercise indeed. Instead of attempting to solve this, we have opted for a less ambitious approach: we show that the low-growth configuration *can* arise, namely, there exists a set of permissible parameter values of strictly positive Lebesgue measure in which the HJB equation (D.1)–(D.2) has a solution satisfying (G.1). To this end, we will need the following result.

Proposition G-1. *Assume that there exists a C^2 function $u : \mathbb{R}_+ \to \mathbb{R}$ whose restriction in $[0, w_c]$ is strictly concave that satisfies the free-boundary problem (G.2)–(G.5) for some free-boundary points $0 < w_g < w_c$. The following statements hold true:*

(I) *u is given by (G.7) for A_1, B_1, A_2 and B_2 being defined by (G.16)–(G.19) (see also (G.8)).*

(II) *u satisfies the HJB equation (D.1)–(D.2) as well as (G.1).*

(III) *u identifies with the value function v. Furthermore, w_c and w_g identify with the corresponding thresholds in Properties 2 and 5, respectively.*

Proof. Statement (I) follows immediately from the analysis at the beginning of this section, while Statement (III) follows from (II) and Theorem D-1.

To establish statement (II), we first note that that (G.1) holds true thanks to the identity (G.11) and the strict concavity of the restriction of u in $]0, w_c[$. We will show that u satisfies the HJB (D.1) if we prove that the inequalities

$$u'(w) \geq -1 \quad \text{for all } w \in [0, w_c[, \tag{G.33}$$

$$\frac{1}{2}\sigma^2\lambda^2 u''(w) + \varrho w u'(w) - r u(w) + \mu \leq 0 \quad \text{for all } w \in]0, w_g[\cup]w_c, \infty[\tag{G.34}$$

and

$$\frac{1}{2}\sigma^2\lambda^2 u''(w) + (\varrho + q) w u'(w) - (r + q) u(w) + \mu + q V_g \leq 0 \quad \text{for all } w > w_g \tag{G.35}$$

hold true. The inequality (G.33) follows immediately from the concavity of u and the fact that $u'(w_c) = -1$. To prove (G.34), we first note that, inside the interval $]0, w_g[$, u satisfies the ODE (G.2), which can be rewritten

$$\frac{1}{2}\sigma^2\lambda^2 u''(w) + \varrho w u'(w) - r u(w) + \mu + q\big[w u'(w) - u(w) + V_g\big] = 0.$$

In view of this identity, (G.11) and the concavity of u we can see that (G.34) holds true inside the interval $]0, w_g[$. Inside the interval $]w_c, \infty[$, (G.34) is equivalent to

$$-\varrho w - r\big[u(w_c) - (w - w_c)\big] + \mu \leq 0 \quad \Leftrightarrow \quad -\varrho w_c - r u(w_c) - (\varrho - r)(w - w_c) + \mu \leq 0$$

(see (G.7)). Using (G.14), we see that this inequality is equivalent to $-(\varrho - r)(w - w_c) \leq 0$, which is true because $\varrho > r$.

To establish (G.35), we first note that the inequality is equivalent to

$$\frac{1}{2}\sigma^2\lambda^2 u''(w) + \varrho w u'(w) - r u(w) + \mu + q\big[w u'(w) - u(w) + V_g\big] \leq 0.$$

Combining the fact that u satisfies the ODE (G.3) inside $]w_g, w_c[$ with the fact that $wu'(w) - u(w) + V_g < 0$ for all $w \in\,]w_g, w_c[$ (see (G.1)), we see that (G.35) is true inside $]w_g, w_c[$. Finally, inside $]w_c, \infty[$, (G.35) is equivalent to

$$-(\varrho + q)w - (r + q)\big[u(w_c) - (w - w_c)\big] + \mu + qV_g \leq 0$$

(see (G.7)). In view of (G.14), we see that this inequality is equivalent to

$$-(\varrho - r)(w - w_c) + q\big[-w_c - u(w_c) + V_g\big] \leq 0,$$

which is true thanks to (G.1), (G.12), and the fact that $\varrho > r$. ∎

G.2 Auxiliary Problems

We now study a pair of auxiliary problems on which the analysis of our main construction in Section G.3 relies.

Problem G-1. Given permissible values for the parameters r, ϱ, μ, σ, q, λ and constants w_g, V_g, s such that

$$w_g > 0 \quad \text{and} \quad 0 < V_g < \frac{\mu}{r}, \tag{G.36}$$

find a function $u_1 : [0, w_g] \to \mathbb{R}$ that satisfies the ODE

$$\frac{1}{2}\sigma^2\lambda^2 u_1''(w) + (\varrho + q)w u_1'(w) - (r + q)u_1(w) + \mu + qV_g = 0, \tag{G.37}$$

with boundary conditions

$$u_1(w_g) = V_g + s w_g \quad \text{and} \quad u_1'(w_g) = s. \tag{G.38}$$

∎

Problem G-2. Given permissible values for the parameters r, ϱ, μ, σ, λ and strictly positive constants w_{g}, V_{g}, find a free-boundary point $w_{\mathrm{c}} > w_{\mathrm{g}}$ and a function $u_2 : [0, w_{\mathrm{c}}] \to \mathbb{R}$ such that u_2 satisfies the ODE

$$\frac{1}{2}\sigma^2\lambda^2 u_2''(w) + \varrho w u_2'(w) - r u_2(w) + \mu = 0 \tag{G.39}$$

and the conditions

$$u_2'(w_{\mathrm{c}}) = -1, \quad u_2''(w_{\mathrm{c}}) = 0 \quad \text{and} \quad u_2(w_{\mathrm{g}}) - w_{\mathrm{g}} u_2'(w_{\mathrm{g}}) = V_{\mathrm{g}}. \tag{G.40}$$

■

The next results are concerned with properties of the solutions to these problems.

Lemma G-1. *Problem G-1 has a unique solution. This solution is such that*

$$u_1(0) = \exp\left(\frac{\varrho + q}{\sigma^2\lambda^2} w_{\mathrm{g}}^2\right)\left[\frac{rV_{\mathrm{g}} - \mu}{r + q}\psi_1'(w_{\mathrm{g}}) - s\big[\psi_1(w_{\mathrm{g}}) - w_{\mathrm{g}}\psi_1'(w_{\mathrm{g}})\big]\right] + \frac{\mu + qV_{\mathrm{g}}}{r + q}. \tag{G.41}$$

Furthermore, if $s \geq 0$, then the function u_1 is strictly increasing and strictly concave, and $u_1(0) < V_{\mathrm{g}}$.

Proof. If it exists, the solution of Problem G-1 is of the form

$$u_1(w) = A_1\varphi_1(w) + B_1\psi_1(w) + \frac{\mu + qV_{\mathrm{g}}}{r + q}, \quad \text{for } w \in [0, w_{\mathrm{g}}].$$

The two boundary conditions at w_{g} provide a system of two linear equations for A_1 and B_1, which has a unique solution because its determinant is non-zero (see (E.12)). It follows that Problem G-1 has a unique solution. In particular,

$$A_1 = \exp\left(\frac{\varrho + q}{\sigma^2\lambda^2} w_{\mathrm{g}}^2\right)\left[\frac{rV_{\mathrm{g}} - \mu}{r + q}\psi_1'(w_{\mathrm{g}}) - s\big[\psi_1(w_{\mathrm{g}}) - w_{\mathrm{g}}\psi_1'(w_{\mathrm{g}})\big]\right],$$

and (G.41) follows because $u_1(0) = A_1 + \frac{\mu + qV_{\mathrm{g}}}{r+q}$ (see (E.7) and (E.11)).

In the rest of the proof, we assume that $s \geq 0$. To show that u_1 is strictly concave, we define

$$\hat{w} = \sup\big\{w \in [0, w_{\mathrm{g}}[\;\big|\; u_1'(w) \leq s\big\} \vee 0 \in [0, w_{\mathrm{g}}[, \tag{G.42}$$

with the usual convention that $\sup\emptyset = -\infty$. Here, the inequality $\hat{w} < w_{\mathrm{g}}$ follows from the boundary condition $u_1'(w_{\mathrm{g}}) = s$ and the observation that (G.36)–(G.38) imply that

$$\lim_{w \uparrow w_{\mathrm{g}}} u_1''(w_{\mathrm{g}}) = -\frac{2}{\sigma^2\lambda^2}\big[(\varrho - r)sw_{\mathrm{g}} + \mu - rV_{\mathrm{g}}\big] < 0.$$

In view of (G.37), the strict concavity of u_1 is equivalent to the inequality

$$(\varrho + q)w\big[u_1'(w) - s\big] - (r + q)u_1(w) + (\varrho + q)sw + \mu + qV_{\mathrm{g}} > 0 \tag{G.43}$$

holding true in $]0, w_{\rm g}[$. In the next paragraph, we show that

$$-(r+q)u_1(w) + (\varrho+q)sw + \mu + qV_{\rm g} > 0 \quad \text{for all } w \in [\hat{w}, w_{\rm g}[. \tag{G.44}$$

In view of the definition (G.42) of $\hat{w}$, this inequality implies that (G.43) is true for all $w \in [\hat{w}, w_{\rm g}[$, therefore

$$u_1''(w) < 0 \quad \text{for all } w \in [\hat{w}, w_{\rm g}[.$$

This result and the fact that $u_1'(w_{\rm g}) = s$ imply that $u_1'(w) > s$ for all $w \in [\hat{w}, w_{\rm g}[$. It follows that $\hat{w} = 0$, and u_1 is strictly concave.

To show (G.44), we note that this is equivalent to

$$u_1(w) < \frac{\varrho+q}{r+q}sw + \frac{\mu+qV_{\rm g}}{r+q} \quad \text{for all } w \in [\hat{w}, w_{\rm g}[.$$

Combining the boundary condition $u_1(w_{\rm g}) = V_{\rm g} + sw_{\rm g}$ with the fact that $u_1'(w) > s$ for all $w \in [\hat{w}, w_{\rm g}[$, we can see that $u_1(w) < V_{\rm g} + sw$ for all $w \in [\hat{w}, w_{\rm g}[$. It follows that a sufficient condition for (G.44) to be true is given by

$$\mu - rV_{\rm g} + (\varrho - r)sw > 0,$$

which holds true under our assumptions.

Finally, we note that the strict concavity of u_1 and the boundary condition $u_1'(w_{\rm g}) = s \geq 0$ imply that u_1 is strictly increasing in $[0, w_{\rm g}]$. Furthermore, the concavity of u_1 and (G.38) imply that

$$u_1(0) = u_1(w_{\rm g}) - \int_0^{w_{\rm g}} u_1'(w)\,dw < V_{\rm g} + sw_{\rm g} - s\int_0^{w_{\rm g}} dw = V_{\rm g}.$$

∎

Lemma G-2. *Problem G-2 has a solution if and only if the inequality*

$$\frac{\varrho-r}{r}w_{\rm g} < \frac{\mu}{r} - V_{\rm g} \tag{G.45}$$

is true, in which case the solution is unique, and the following statements hold true:
(I) *The function u_2 is strictly concave and $u_2(0) < V_{\rm g}$. Furthermore, $u_2(0) > 0$ if*

$$1 - \frac{r}{\mu}V_{\rm g} < \varphi_2(w_{\rm g}) - w_{\rm g}\varphi_2'(w_{\rm g}). \tag{G.46}$$

(II) *There exists $\delta_\star = \delta_\star(r, \varrho, \sigma, \lambda, w_{\rm g}) > \frac{\varrho-r}{r}w_{\rm g}$ such that*

$$u_2'(w_{\rm g}) > 0 \quad \Leftrightarrow \quad \frac{\mu}{r} - V_{\rm g} > \delta_\star \tag{G.47}$$

$$\text{and} \quad u_2'(w_{\rm g}) = 0 \quad \Leftrightarrow \quad \frac{\mu}{r} - V_{\rm g} = \delta_\star. \tag{G.48}$$

Furthermore,

$$\frac{\partial \delta_\star(r, \varrho, \sigma, \lambda, w_{\rm g})}{\partial w_{\rm g}} > 0. \tag{G.49}$$

Proof. By inspection, if Problem G-2 has a solution, then it is of the form

$$u_2(w) = A_2\varphi_2(w) + B_2\psi_2(w) + \frac{\mu}{r}, \quad \text{for } w \in [0, w_c].$$

The two boundary conditions at w_c imply that A_2 and B_2 are given by (G.16)–(G.17). Furthermore, the tangency condition $u_2(w_g) - w_g u_2'(w_g) = V_g$ implies that, if it exists, the free-boundary point w_c should satisfy (G.22), namely,

$$\ell(w_c) = V_g - \frac{\mu}{r}, \tag{G.50}$$

where

$$\ell(w) \equiv \ell(w; r, \varrho, \sigma, \lambda, w_g) := \frac{\sigma^2\lambda^2}{2r} \exp\left(\frac{\varrho}{\sigma^2\lambda^2}w^2\right) \Big(\left[\varphi_2(w_g) - w_g\varphi_2'(w_g)\right]\psi_2''(w) - \left[\psi_2(w_g) - w_g\psi_2'(w_g)\right]\varphi_2''(w) \Big).$$

Differentiating the function ℓ and using the ODEs satisfied by φ_2, ψ_2 and their first derivatives, we obtain

$$\ell'(w) = -\frac{\varrho - r}{r} \exp\left(\frac{\varrho}{\sigma^2\lambda^2}w^2\right) \Big(\big(\varphi_2(w_g) - w_g\varphi_2'(w_g)\big)\psi_2'(w) - \big(\psi_2(w_g) - w_g\psi_2'(w_g)\big)\varphi_2'(w) \Big)$$

and

$$\ell''(w) = -\frac{2(\varrho - r)}{\sigma^2\lambda^2} \exp\left(\frac{\varrho}{\sigma^2\lambda^2}w^2\right) \Big(\big(\varphi_2(w_g) - w_g\varphi_2'(w_g)\big)\psi_2(w) - \big(\psi_2(w_g) - w_g\psi_2'(w_g)\big)\varphi_2(w) \Big).$$

An inspection of these expressions reveals that

$$\ell(w) < 0 \quad \text{and} \quad \ell'(w) < 0 \quad \text{for all } w > 0, \tag{G.51}$$

thanks to the inequalities established in Lemma E-1. Furthermore, the inequalities in Lemma E-1 imply that $\lim_{w\to\infty} \ell''(w) < 0$, which, combined with (G.51), implies that $\lim_{w\to\infty} \ell(w) = -\infty$. It follows that (G.50) has a solution $w_c > w_g$ if and only if $\ell(w_g) > V_g - \frac{\mu}{r}$. This inequality is equivalent to (G.45) thanks to the calculations

$$\begin{aligned}\ell(w_g) &= -\frac{\varrho - r}{r} w_g \exp\left(\frac{\varrho}{\sigma^2\lambda^2}w_g^2\right) \left[\varphi_2(w_g)\psi_2'(w_g) - \varphi_2'(w_g)\psi_2(w_g)\right] \\ &= -\frac{\varrho - r}{r} w_g,\end{aligned} \tag{G.52}$$

where we have used the ODE satisfied by φ_2, ψ_2 for the first identity, and (E.12) for the second identity.

To proceed further, we assume that Problem G-2 has a solution, namely, (G.45) holds true. To establish the strict concavity of u_2, we define

$$\tilde{w} = \sup\{w \in [0, w_c[\mid u_2'(w) \leq -1\} \vee 0 \in [0, w_c[, \tag{G.53}$$

with the usual convention that $\sup\emptyset = -\infty$. The inequality $\tilde{w} < w_c$ stated here follows immediately once we combine the boundary conditions $u_2'(w_c) = -1$ and $u_2''(w_c) = 0$ with the observation that $\lim_{w\uparrow w_c} u_2'''(w) > 0$, which is true because u_2' satisfies the ODE

$$\frac{1}{2}\sigma^2\lambda^2 u_2'''(w) + \varrho w u_2''(w) + \varrho u_2'(w) = 0$$

in $]0, w_c[$. The fact that u_2 satisfies the ODE (G.39) in $]0, w_c[$ also implies that

$$\frac{1}{2}\sigma^2\lambda^2 u_2''(w) + \varrho w\big[u_2'(w) + 1\big] + F(w) = 0 \tag{G.54}$$

in $]0, w_c[$, where

$$F(w) = -\varrho w - r u_2(w) + \mu.$$

In view of the assumption that $\varrho > r$ (see condition (12)) and the definition of $\tilde{w}$, we can see that

$$F'(w) = -r\big[u_2'(w) + 1\big] - (\varrho - r) < 0 \quad \text{for all } w \in [\tilde{w}, w_c],$$

which, combined with (G.14), implies that $F(w) > 0$ for all $w \in [\tilde{w}, w_c[$. This observation, the definition (G.53) of $\tilde{w}$, and (G.54) imply that

$$u_2''(w) < 0 \quad \text{for all } w \in [\tilde{w}, w_c[. \tag{G.55}$$

This result and the boundary condition $u_2'(w_c) = -1$ imply that $u_2'(w) > -1$ for all $w \in [\tilde{w}, w_c[$. Combining this inequality with the definition of $\tilde{w}$ and the continuity of u_2', we can see that $\tilde{w} = 0$. In view of (G.55), it follows that u_2 is strictly concave on $]0, w_c[$.

To complete the proof of statement (I), we first note that the concavity of u_2 and (G.40) imply that

$$u_2(0) = u_2(w_g) - \int_0^{w_g} u_2'(w)\,dw < V_g + w_g u_2'(w_g) - u_2'(w_g)\int_0^{w_g} dw = V_g.$$

We also note that, in view of (G.16) and the definition of the function ℓ, (G.50) is equivalent to

$$A_2\left(\varphi_2(w_g) - w_g\varphi_2'(w_g) - \big[\psi_2(w_g) - w_g\psi_2'(w_g)\big]\frac{\varphi_2''(w_c)}{\psi_2''(w_c)}\right) = V_g - \frac{\mu}{r}.$$

Therefore

$$u_2(0) = A_2 + \frac{\mu}{r} = \left(1 - \frac{1 - \frac{rV_g}{\mu}}{\varphi_2(w_g) - w_g\varphi_2'(w_g) - \big[\psi_2(w_g) - w_g\psi_2'(w_g)\big]\frac{\varphi_2''(w_c)}{\psi_2''(w_c)}}\right)\frac{\mu}{r}$$

(see (E.7) and (E.11)). In view of (E.14)–(E.18) in Lemma E-1, we can see that

$$0 < \varphi_2(w_{\mathrm{g}}) - w_{\mathrm{g}}\varphi_2'(w_{\mathrm{g}}) < \varphi_2(w_{\mathrm{g}}) - w_{\mathrm{g}}\varphi_2'(w_{\mathrm{g}}) - \big[\psi_2(w_{\mathrm{g}}) - w_{\mathrm{g}}\psi_2'(w_{\mathrm{g}})\big]\frac{\varphi_2''(w_{\mathrm{c}})}{\psi_2''(w_{\mathrm{c}})},$$

and the sufficient condition (G.46) ensures that the inequality $u_2(0) > 0$ holds true.

To complete the proof, we still need to establish (II). To this end, we note that the expressions for A_2, B_2 given by (G.16), (G.17) and the inequalities (E.4) and (E.15) yield

$$\begin{aligned} u_2'(w_{\mathrm{g}}) \equiv A_2(w_{\mathrm{c}})\varphi_2'(w_{\mathrm{g}}) + B_2(w_{\mathrm{c}})\psi_2'(w_{\mathrm{g}}) > 0 \quad &\Leftrightarrow \quad \psi_2''(w_{\mathrm{c}})\varphi_2'(w_{\mathrm{g}}) - \varphi_2''(w_{\mathrm{c}})\psi_2'(w_{\mathrm{g}}) > 0 \\ &\Leftrightarrow \quad \frac{\psi_2''(w_{\mathrm{c}})}{\varphi_2''(w_{\mathrm{c}})} < \frac{\psi_2'(w_{\mathrm{g}})}{\varphi'(w_{\mathrm{g}})} \end{aligned} \tag{G.56}$$

and

$$u_2'(w_{\mathrm{g}}) \equiv A_2(w_{\mathrm{c}})\varphi_2'(w_{\mathrm{g}}) + B_2(w_{\mathrm{c}})\psi_2'(w_{\mathrm{g}}) = 0 \quad \Leftrightarrow \quad \frac{\psi_2''(w_{\mathrm{c}})}{\varphi_2''(w_{\mathrm{c}})} = \frac{\psi_2'(w_{\mathrm{g}})}{\varphi'(w_{\mathrm{g}})}. \tag{G.57}$$

The point $w_{\mathrm{c}}^* \equiv w_{\mathrm{c}}^*(r, \varrho, \sigma, \lambda, w_{\mathrm{g}}) := g(w_{\mathrm{g}}) > w_{\mathrm{g}}$, where g is as in Lemma E-2, is such that

$$\frac{\psi_2''(w_{\mathrm{c}}^*)}{\varphi_2''(w_{\mathrm{c}}^*)} = \frac{\psi_2'(w_{\mathrm{g}})}{\varphi_2'(w_{\mathrm{g}})}.$$

In view of (E.18), we can see that (G.56) (resp., (G.57)) is satisfied if and only if $w_{\mathrm{c}} > w_{\mathrm{c}}^*$ (resp., $w_{\mathrm{c}} = w_{\mathrm{c}}^*$). Furthermore, an inspection of (G.50)–(G.51) reveals that $w_{\mathrm{c}} > w_{\mathrm{c}}^*$ (resp., $w_{\mathrm{c}} = w_{\mathrm{c}}^*$) if and only if $V_{\mathrm{g}} - \frac{\mu}{r} < \ell(w_{\mathrm{c}}^*)$ (resp., $V_{\mathrm{g}} - \frac{\mu}{r} = \ell(w_{\mathrm{c}}^*)$). By construction, the point $\delta_\star := -\ell(w_{\mathrm{c}}^*) > 0$ is determined uniquely by r, ϱ, σ, λ and w_{g}. In particular, it does not depend on μ or V_{g}. Also, combining the inequality $w_{\mathrm{c}}^* > w_{\mathrm{g}}$ with (G.51) and (G.52), we obtain

$$\delta_\star = -\ell(w_{\mathrm{c}}^*) > -\ell(w_{\mathrm{g}}) = \frac{\varrho - r}{r} w_{\mathrm{g}}.$$

Finally, (G.49) follows from the calculations

$$\begin{aligned} \frac{\partial \delta_\star(r, \varrho, \sigma, \lambda, w_{\mathrm{g}})}{\partial w_{\mathrm{g}}} &= -\ell'(w_{\mathrm{c}}^*)\frac{\partial w_{\mathrm{c}}^*(r, \varrho, \sigma, \lambda, w_{\mathrm{g}})}{\partial w_{\mathrm{g}}} \\ &= -\ell'\big(g(w_{\mathrm{g}})\big)g'(w_{\mathrm{g}}), \\ &> 0, \end{aligned}$$

where the strictly increasing function g is defined in Lemma E-2 and we have also used (G.51). ∎

Lemma G-3. *Fix any permissible values for the parameters r, ϱ, μ, σ, q, λ, and let w_{g}, V_{g} be strictly positive constants such that (G.45) and (G.47) or (G.48) hold true, namely,*

$$\delta_\star \le \frac{\mu}{r} - V_{\mathrm{g}},$$

where $\delta_\star = \delta_\star(r, \varrho, \sigma, \lambda, w_\mathrm{g}) > 0$ *is as in Lemma G-2.(II). Given the solution* w_c *and* u_2 *to Problem G-2, if* u_1 *is the solution to Problem G-1 for* $s = u_2'(w_\mathrm{g}) \geq 0$ *and we define*

$$u(w) = \begin{cases} u_1(w), & \text{if } w \in [0, w_\mathrm{g}[\\ u_2(w), & \text{if } w \in [w_\mathrm{g}, w_\mathrm{c}] \\ u_2(w_\mathrm{c}) - (w - w_\mathrm{c}), & \text{if } w > w_\mathrm{c} \end{cases}, \tag{G.58}$$

then u *is* C^2*, strictly increasing in* $[0, w_\mathrm{g}]$*, strictly concave in* $[0, w_\mathrm{c}]$*, and such that* $u(0) < V_\mathrm{g}$*. Furthermore, if there exists* γ*,* κ *and* $\bar{w}$ *such that*

$$u(w_\mathrm{h}) - \kappa = u(0) \quad \text{and} \quad (1 + \gamma)u(w_\mathrm{h}) - \kappa = V_\mathrm{g}, \tag{G.59}$$

which are equivalent to

$$\kappa = u(w_\mathrm{h}) - u(0) \quad \text{and} \quad \gamma = \frac{V_\mathrm{g} - u(0)}{u(w_\mathrm{h})}, \tag{G.60}$$

where $w_\mathrm{h} := \bar{w} \vee w_\circ := \bar{w} \vee \arg\max_{w>0} u(w)$*, then* u *is a solution to the free-boundary problem (G.2)–(G.5) for the given values of the parameters* r*,* ϱ*,* μ*,* σ*,* q*,* γ*,* λ*,* κ*,* $\bar{w}$*.*

Proof. In view of (G.37)–(G.39), the C^2 continuity of u at w_g follows immediately from the fact that $s = u_2'(w_\mathrm{g})$. The results derived in Lemmas G-1 and G-2 imply the strict concavity of u in $[0, w_\mathrm{c}]$ as well as the facts that u is strictly increasing on $[0, w_\mathrm{g}]$ and the inequality $u(0) < V_\mathrm{g}$. Finally, it is immediate to verify that u satisfies the free-boundary problem (G.2)–(G.5) if (G.59) hold true. ∎

G.3 Proof of Proposition 7

In view of Proposition G-1, we need to show that the set of permissible parameter values for which the free-boundary problem (G.2)–(G.5) has a C^2 strictly concave solution has non-empty interior in $\mathbb{R}^9$. To prove that this is indeed the case, we rely on Lemma G-3. Our constructive argument proceeds in several steps.

Step 1. We consider any

$$\varrho > r > 0, \quad \sigma > 0, \quad \lambda \in]0, 1], \quad \tilde{w}_\mathrm{g} > 0,$$

and

$$\mu > \frac{r\delta_\star(r, \varrho, \sigma, \lambda, \tilde{w}_\mathrm{g})}{\varphi_2(\tilde{w}_\mathrm{g}) - \tilde{w}_\mathrm{g}\varphi_2'(\tilde{w}_\mathrm{g})} > 0,$$

where $\delta_\star \equiv \delta_\star(r, \varrho, \sigma, \lambda, \tilde{w}_\mathrm{g}) > 0$ is as in Lemma G-2. We also define

$$\tilde{V}_\mathrm{g} \equiv \tilde{V}_\mathrm{g}(r, \varrho, \mu, \sigma, \lambda, \tilde{w}_\mathrm{g}) = \frac{\mu}{r} - \delta_\star(r, \varrho, \sigma, \lambda, \tilde{w}_\mathrm{g}) > 0. \tag{G.61}$$

Note that the strict positivity of μ and $\tilde{V}_\mathrm{g}$ follows from (E.14)–(E.15). For any such choices of r, ϱ, μ, σ, λ, $\tilde{w}_\mathrm{g}$ and for $\tilde{V}_\mathrm{g}$ given by (G.61), inequalities (G.45) and (G.46) are both satisfied. Therefore, Lemma G-2 implies that the solution u_2 to Problem G-2, with $\tilde{w}_g$, $\tilde{V}_g$ in place of w_g, V_g, exists, is unique, and satisfies

$$0 < u_2(0) < \tilde{V}_g = u_2(\tilde{w}_g) \quad \text{and} \quad u_2'(\tilde{w}_g) = 0, \tag{G.62}$$

while

$$w_c = g(\tilde{w}_g),$$

where g is defined in Lemma E-2 (see also the analysis of (G.57)). Furthermore, using the fact that u_2 satisfies the ODE (G.39) with boundary conditions $u_2(\tilde{w}_g) = \tilde{V}_g$ and $u_2'(\tilde{w}_g) = 0$, we can show that

$$u_2(w) = -\exp\left(\frac{\varrho}{\sigma^2\lambda^2}\tilde{w}_g^2\right)\left[\psi_2'(\tilde{w}_g)\varphi_2(w) - \varphi_2'(\tilde{w}_g)\psi_2(w)\right]\delta_\star(\tilde{w}_g) + \frac{\mu}{r}. \tag{G.63}$$

Step 2. Given any $w_g \in \,]0, \tilde{w}_g]$ and $q > 0$, we consider the solution u_1 to Problem G-1 with boundary conditions at w_g given by

$$V_g = u_2(w_g) - w_g u_2'(w_g) < \tilde{V}_g \quad \text{and} \quad s = u_2'(w_g) > 0, \tag{G.64}$$

which we studied in Lemma G-1. In view of the ODEs (G.37), (G.39), which u_1, u_2 satisfy, and the identities

$$u_1(w_g) = u_2(w_g) \quad \text{and} \quad u_1'(w_g) = u_2'(w_g),$$

we can see that the function $q \mapsto u_1(0;q)$ is continuous and $\lim_{q\downarrow 0} u_1(0;q) = u_2(0)$. It follows that, given any $\epsilon > 0$, there exists $q_\star(\epsilon) = q_\star(\epsilon; r, \varrho, \mu, \sigma, \lambda, w_g, \tilde{w}_g) > 0$ such that

$$u_1(0;q) \in \,]u_2(0) - \epsilon, u_2(0) + \epsilon[\quad \text{for all } q \in \,]0, q_\star(\epsilon)[. \tag{G.65}$$

Step 3. In the context of the previous two steps (in particular, see (G.61), (G.62), (G.64) and (G.65)), we can see that the continuity of all functions involved implies that there exist constants

$$0 < \varepsilon < C_1 < C_2$$

and an open set $\mathcal{O}_0 \subseteq \mathbb{R}^8$ such that

$$0 < q < r\frac{C_1 - \varepsilon}{C_2 - C_1 + \varepsilon}, \quad C_1 < \tilde{w}_g, \quad C_1 + \varepsilon < V_g < \tilde{V}_g < C_2,$$
$$u_1(0) \in \,]C_1 - \varepsilon, C_1 + \varepsilon[\quad \text{and} \quad 0 < w_g < \tilde{w}_g \quad \text{for all } (r, \varrho, \mu, \sigma, q, \lambda, w_g, \tilde{w}_g) \in \mathcal{O}_0. \tag{G.66}$$

Given any point in $\mathcal{O}_0$, the function u that is defined as in (G.58) by pasting u_1 and u_2 at w_g

is C^2, strictly increasing in $[0, \tilde{w}_g]$, strictly concave in $[0, w_c]$, and such that
$$0 < C_1 - \varepsilon < u(0) < C_1 + \varepsilon < V_g < \tilde{V}_g = u(\tilde{w}_g) < C_2 \text{ and } u'(\tilde{w}_g) = 0. \tag{G.67}$$

It is straightforward to see that

$$w_\circ := \arg\max_{w>0} u(w) = \tilde{w}_{\rm g},$$

which implies that

$$u(w_\circ) = \tilde{V}_{\rm g} > u(0) > 0.$$

Step 4. Noting that $w_{\rm h} := \bar{w} \vee w_\circ = \tilde{w}_{\rm g}$ for all $\bar{w} \leq \tilde{w}_{\rm g}$, we can see that, for

$$\kappa = u(w_\circ) - u(0) = \tilde{V}_{\rm g} - u(0) > 0 \quad \text{and} \quad \gamma = \frac{V_{\rm g} - u(0)}{u(w_\circ)} = \frac{V_{\rm g} - u(0)}{\tilde{V}_{\rm g}} > 0, \tag{G.68}$$

the function u constructed in Step 3 satisfies (G.60). Therefore, u provides a solution to the free-boundary problem (G.2)–(G.5) for all values of the parameters r, ϱ, μ, σ, q, λ in the relevant projection of $\mathcal{O}_0$, for $\bar{w} \leq \tilde{w}_{\rm g}$, and for κ, γ given by (G.68).

Step 5. We now consider whether the implied values for κ and γ given by (G.68) satisfy conditions (13) and (14). In the current context, these conditions are equivalent to

$$\frac{V_{\rm g} - u(0)}{\tilde{V}_{\rm g}} < \frac{r}{q} \tag{G.69}$$

and

$$\frac{V_{\rm g} - u(0)}{\tilde{V}_{\rm g}} > \frac{\kappa + \bar{w}}{\frac{\mu}{r} - \bar{w}} \equiv \frac{\tilde{V}_{\rm g} - u(0) + \bar{w}}{\frac{\mu}{r} - \bar{w}}, \tag{G.70}$$

respectively. Using the fact that $\tilde{V}_{\rm g} > u(0)$, we can see that (G.69), and therefore Condition (13), holds true if

$$q < r\frac{u(0)}{V_{\rm g} - u(0)},$$

which follows immediately from (G.66). On the other hand, inequality (G.70) holds true in the limit as $\bar{w} \downarrow 0$ and $w_{\rm g} \uparrow \tilde{w}_{\rm g}$ because $V_{\rm g} \uparrow \tilde{V}_{\rm g} < \frac{\mu}{r}$ as $w_{\rm g} \uparrow \tilde{w}_{\rm g}$. Therefore, Condition (14) is satisfied as long as $\bar{w} \in \,]0, C_1[$ is sufficiently small and $w_{\rm g}$ is sufficiently close to $\tilde{w}_{\rm g}$.

Step 6. In view of the conclusion of the previous step and the continuity of all functions involved, there exists an open set $\mathcal{O}_1$ of points $(r, \varrho, \mu, \sigma, q, \lambda, \bar{w}, w_{\rm g}, \tilde{w}_{\rm g}) \in \mathbb{R}^9$ such that $(r, \varrho, \mu, \sigma, q, \lambda, w_{\rm g}, \tilde{w}_{\rm g}) \in \mathcal{O}_0$ and

$$\begin{gathered}\text{the function } u \text{ defined as in Step 3 satisfies (G.67)}\\ \text{and solves the free-boundary problem (G.2)–(G.5) for } \kappa,\ \gamma \text{ given by (G.68),}\\ \text{while Conditions (13) and (14) hold true.}\end{gathered} \tag{G.71}$$

Step 7. Given any point in $\mathcal{O}_1$, we denote $\boldsymbol{p} = (r, \varrho, \mu, \sigma, q, \lambda, \bar{w})$, and we note that

$$\kappa = \kappa(\boldsymbol{p}, w_{\rm g}, \tilde{w}_{\rm g}), \quad \gamma = \gamma(\boldsymbol{p}, w_{\rm g}, \tilde{w}_{\rm g}), \quad \delta_\star = \delta_\star(\boldsymbol{p}, \tilde{w}_{\rm g}),$$
$$u_1 = u_1(\cdot; \boldsymbol{p}, w_{\rm g}, \tilde{w}_{\rm g}), \quad u_2 = u_2(\cdot; \boldsymbol{p}, \tilde{w}_{\rm g}) \quad \text{and} \quad V_{\rm g} = V_{\rm g}(\boldsymbol{p}, w_{\rm g}, \tilde{w}_{\rm g}).$$

To simplify the notation, we drop the dependence on $\boldsymbol{p}$ throughout this step.

In view of (G.61), we can see that (G.68) yield the expressions

$$\kappa(w_{\rm g}, \tilde{w}_{\rm g}) = \frac{\mu}{r} - \delta_\star(\tilde{w}_{\rm g}) - u_1(0; w_{\rm g}, \tilde{w}_{\rm g})$$
$$\text{and} \quad \gamma(w_{\rm g}, \tilde{w}_{\rm g}) = \frac{V_{\rm g}(w_{\rm g}, \tilde{w}_{\rm g}) - u_1(0; w_{\rm g}, \tilde{w}_{\rm g})}{\frac{\mu}{r} - \delta_\star(\tilde{w}_{\rm g})}.$$

We consider the Jacobian determinant of κ, γ as functions of $w_{\rm g}$, $\tilde{w}_{\rm g}$, which is defined by

$$\mathrm{J}(w_{\rm g}, \tilde{w}_{\rm g}) := \begin{vmatrix} \frac{\partial \kappa(w_{\rm g}, \tilde{w}_{\rm g})}{\partial w_{\rm g}} & \frac{\partial \kappa(w_{\rm g}, \tilde{w}_{\rm g})}{\partial \tilde{w}_{\rm g}} \\ \frac{\partial \gamma(w_{\rm g}, \tilde{w}_{\rm g})}{\partial w_{\rm g}} & \frac{\partial \gamma(w_{\rm g}, \tilde{w}_{\rm g})}{\partial \tilde{w}_{\rm g}} \end{vmatrix},$$

and we compute

$$\begin{aligned} \mathrm{J}(w_{\rm g}, \tilde{w}_{\rm g}) = \frac{1}{\tilde{V}_{\rm g}(\tilde{w}_{\rm g})} \Bigg[& \frac{\partial V_{\rm g}(w_{\rm g}, \tilde{w}_{\rm g})}{\partial w_{\rm g}} \frac{\partial u_1(0; w_{\rm g}, \tilde{w}_{\rm g})}{\partial \tilde{w}_{\rm g}} - \frac{\partial V_{\rm g}(w_{\rm g}; \tilde{w}_{\rm g})}{\partial \tilde{w}_{\rm g}} \frac{\partial u_1(0; w_{\rm g}, \tilde{w}_{\rm g})}{\partial w_{\rm g}} \\ & - \gamma(w_{\rm g}, \tilde{w}_{\rm g}) \frac{\partial u_1(0; w_{\rm g}, \tilde{w}_{\rm g})}{\partial w_{\rm g}} \delta'_\star(\tilde{w}_{\rm g}) + \frac{\partial V_{\rm g}(w_{\rm g}, \tilde{w}_{\rm g})}{\partial w_{\rm g}} \delta'_\star(\tilde{w}_{\rm g}) \\ & - \frac{\partial u_1(0; w_{\rm g}, \tilde{w}_{\rm g})}{\partial w_{\rm g}} \delta'_\star(\tilde{w}_{\rm g}) \Bigg]. \end{aligned} \tag{G.72}$$

Recalling the definition

$$V_{\rm g}(w_{\rm g}, \tilde{w}_{\rm g}) = u_2(w_{\rm g}; \tilde{w}_{\rm g}) - w_{\rm g} u_2'(w_{\rm g}; \tilde{w}_{\rm g}), \tag{G.73}$$

we compute

$$\frac{\partial V_{\rm g}(w_{\rm g}, \tilde{w}_{\rm g})}{\partial w_{\rm g}} = -w_{\rm g} u_2''(w_{\rm g}; \tilde{w}_{\rm g}) \tag{G.74}$$

$$\text{and} \quad \frac{\partial V_{\rm g}(w_{\rm g}, \tilde{w}_{\rm g})}{\partial \tilde{w}_{\rm g}} = \frac{\partial u_2(w_{\rm g}; \tilde{w}_{\rm g})}{\partial \tilde{w}_{\rm g}} - w_{\rm g} \frac{\partial u_2'(w_{\rm g}; \tilde{w}_{\rm g})}{\partial \tilde{w}_{\rm g}}. \tag{G.75}$$

Here, as well as in what follows, we use the notation

$$u_2'(w_{\rm g}; \tilde{w}_{\rm g}) = \frac{\partial u_2(w_{\rm g}; \tilde{w}_{\rm g})}{\partial w_{\rm g}} \quad \text{and} \quad u_2''(w_{\rm g}; \tilde{w}_{\rm g}) = \frac{\partial^2 u_2(w_{\rm g}; \tilde{w}_{\rm g})}{\partial w_{\rm g}^2}.$$

Differentiating the expression

$$u_2(w_{\rm g}; \tilde{w}_{\rm g}) = -\exp\left(\frac{\varrho}{\sigma^2 \lambda^2} \tilde{w}_{\rm g}^2\right) \left[\psi_2'(\tilde{w}_{\rm g}) \varphi_2(w_{\rm g}) - \varphi_2'(\tilde{w}_{\rm g}) \psi_2(w_{\rm g})\right] \delta_\star(\tilde{w}_{\rm g}) + \frac{\mu}{r}$$

(see (G.63)), we obtain

$$\frac{\partial u_2(w_g;\tilde{w}_g)}{\partial \tilde{w}_g} = -\exp\left(\frac{\varrho}{\sigma^2\lambda^2}\tilde{w}_g^2\right)\left[\frac{2r}{\sigma^2\lambda^2}\big[\varphi_2(w_g)\psi_2(\tilde{w}_g) - \varphi_2(\tilde{w}_g)\psi_2(w_g)\big]\delta_\star(\tilde{w}_g) + \big[\varphi_2(w_g)\psi_2'(\tilde{w}_g) - \varphi_2'(\tilde{w}_g)\psi_2(w_g)\big]\delta_\star'(\tilde{w}_g)\right]$$

and

$$\frac{\partial u_2'(w_g;\tilde{w}_g)}{\partial \tilde{w}_g} = -\exp\left(\frac{\varrho}{\sigma^2\lambda^2}\tilde{w}_g^2\right)\left[\frac{2r}{\sigma^2\lambda^2}\big[\varphi_2'(w_g)\psi_2(\tilde{w}_g) - \varphi_2(\tilde{w}_g)\psi_2'(w_g)\big]\delta_\star(\tilde{w}_g) + \big[\varphi_2'(w_g)\psi_2'(\tilde{w}_g) - \varphi_2'(\tilde{w}_g)\psi_2'(w_g)\big]\delta_\star'(\tilde{w}_g)\right].$$

In particular, we note that

$$\left.\frac{\partial u_2(w_g;\tilde{w}_g)}{\partial \tilde{w}_g}\right|_{w_g=\tilde{w}_g} = -\delta_\star'(\tilde{w}_g) \quad \text{and} \quad \left.\frac{\partial u_2'(w_g;\tilde{w}_g)}{\partial \tilde{w}_g}\right|_{w_g=\tilde{w}_g} = \frac{2r}{\sigma^2\lambda^2}\delta_\star(\tilde{w}_g). \tag{G.76}$$

Combining these identities with (G.75), we can see that

$$\left.\frac{\partial V_g(w_g,\tilde{w}_g)}{\partial \tilde{w}_g}\right|_{w_g=\tilde{w}_g} = -\delta_\star'(\tilde{w}_g) - \frac{2r}{\sigma^2\lambda^2}\tilde{w}_g\delta_\star(\tilde{w}_g). \tag{G.77}$$

Differentiating the expression (see (G.41))

$$u_1(0;w_g,\tilde{w}_g) = \exp\left(\frac{\varrho+q}{\sigma^2\lambda^2}w_g^2\right)\left[\frac{rV_g(w_g,\tilde{w}_g)-\mu}{r+q}\psi_1'(w_g) - s(w_g,\tilde{w}_g)\big[\psi_1(w_g) - w_g\psi_1'(w_g)\big]\right] + \frac{\mu+qV_g(w_g,\tilde{w}_g)}{r+q},$$

with V_g given by (G.73) and $s(w_g,\tilde{w}_g) = u_2'(w_g;\tilde{w}_g)$, we calculate

$$\frac{\partial u_1(0;w_g,\tilde{w}_g)}{\partial w_g} = \frac{q}{r+q}\left[\exp\left(\frac{\varrho+q}{\sigma^2\lambda^2}w_g^2\right)\psi_1'(w_g) - 1\right]w_g u_2''(w_g;\tilde{w}_g). \tag{G.78}$$

Differentiating the same expression with respect to $\tilde{w}_g$, and using (G.76), (G.77) to evaluate the partial derivatives at $w_g = \tilde{w}_g$, we compute

$$\begin{aligned}\left.\frac{\partial u_1(0;w_g,\tilde{w}_g)}{\partial \tilde{w}_g}\right|_{w_g=\tilde{w}_g} = & -\exp\left(\frac{\varrho+q}{\sigma^2\lambda^2}\tilde{w}_g^2\right)\left[\frac{r}{r+q}\psi_1'(\tilde{w}_g)\delta_\star'(\tilde{w}_g) + \frac{2r}{\sigma^2\lambda^2}\psi_1(\tilde{w}_g)\delta_\star(\tilde{w}_g)\right] \\ & + \frac{q}{r+q}\frac{2r}{\sigma^2\lambda^2}\left[\exp\left(\frac{\varrho+q}{\sigma^2\lambda^2}\tilde{w}_g^2\right)\psi_1'(\tilde{w}_g) - 1\right]\tilde{w}_g\delta_\star(\tilde{w}_g) \\ & - \frac{q}{r+q}\delta_\star'(\tilde{w}_g).\end{aligned} \tag{G.79}$$

In view of (G.74) and (G.77)–(G.79), the Jacobian determinant given by (G.72) evaluated at $w_{\rm g} = \tilde{w}_{\rm g}$ admits the expression

$$\begin{aligned} \mathrm{J}(\tilde{w}_{\rm g}, \tilde{w}_{\rm g}) = \Bigg[& \frac{2r}{\sigma^2 \lambda^2} \exp\left(\frac{\varrho + q}{\sigma^2 \lambda^2} \tilde{w}_{\rm g}^2\right) \psi_1(\tilde{w}_{\rm g}) \delta_\star(\tilde{w}_{\rm g}) \\ & + \frac{r - q\gamma(\tilde{w}_{\rm g}, \tilde{w}_{\rm g})}{r + q} \left[\exp\left(\frac{\varrho + q}{\sigma^2 \lambda^2} \tilde{w}_{\rm g}^2\right) \psi_1'(\tilde{w}_{\rm g}) - 1\right] \delta_\star'(\tilde{w}_{\rm g}) \Bigg] \frac{\tilde{w}_{\rm g} u_2''(\tilde{w}_{\rm g}; \tilde{w}_{\rm g})}{\tilde{V}_{\rm g}(\tilde{w}_{\rm g})} \\ < 0. & \end{aligned} \tag{G.80}$$

The inequality here follows from (E.19), (G.49), the strict concavity of the restriction of u_2 in $]0, w_{\rm c}[\ni \tilde{w}_{\rm g}$ and the fact that Condition (13), namely, the inequality $r > q\gamma$, holds true for every point in $\mathcal{O}_1$ (see (G.71)).

Step 8. By continuity, (G.80) implies that there exists an open set $\mathcal{O}_2 \subseteq \mathcal{O}_1$ of points $(r, \varrho, \mu, \sigma, q, \lambda, \bar{w}, w_{\rm g}, \tilde{w}_{\rm g}) \in \mathbb{R}^9$ such that $\mathrm{J}(\boldsymbol{p}, w_{\rm g}, \tilde{w}_{\rm g}) < 0$. It follows that $\kappa(\boldsymbol{p}, w_{\rm g}, \tilde{w}_{\rm g})$ and $\gamma(\boldsymbol{p}, w_{\rm g}, \tilde{w}_{\rm g})$ as functions of $w_{\rm g}$ and $\tilde{w}_{\rm g}$ are invertible for each $\boldsymbol{p}$ in the appropriate projection of $\mathcal{O}_2$ in $\mathbb{R}^7$. Combining this observation with (G.71), we conclude that there exists an open set of permissible parameter values in $\mathbb{R}^9$ such that, for any value of $(r, \varrho, \mu, \sigma, q, \gamma, \lambda, \kappa, \bar{w})$ in this set, there exist points $0 < w_{\rm g} < w_{\rm c}$ as well as a concave C^2 function u such that (G.2)–(G.5) all hold true.

H Data Sources and Variable Construction

Data Sources. Our sample relies on information on CEO turnover and CEO compensation as reported in the widely used Standard and Poors ExecuComp database from January 1992 to December 2014. Accounting information comes from the Compustat Industrial Annual files, and stock price and stock return information comes from the monthly CRSP tapes. The dataset is at annual frequency (on a calendar year basis), although our measure of past performance is constructed using stock return data at monthly frequency.

CEO Episodes and Turnover. The starting point of the construction of the data set is to identify CEO episodes, which track the tenure of a given manager as CEO of a given firm. Using the information available in ExecuComp, we define the first year of a CEO episode as the first year in which the CEO is reported as being in charge of the firm. The variable *TenureYear* is set at 1 in the year of his appointment and is incremented for each calendar year he remains in office. A turnover event is recorded in the year that ExecuComp reports the CEO leaves office. In cases where ExecuComp does not report a date leaving office but a new CEO is reported for the same firm in a subsequent year, a turnover event is recorded in the last year of the old CEO's reported tenure. The variable *Turnover* is a binary variable which equals 1 for a CEO episode in the year of a turnover event and zero otherwise. For all CEO episodes completed within our sample period, the variable *TotTenure* equals the total length of the episode, in years.

Compensation. We define the variable *TotPay* as the total annual compensation as recorded in the ExecuComp variable `tdc1`. This includes salary, cash bonus, retirement benefits, stock, and stock options in the year they are awarded.

Average Q and Growth-Related Proxies. For a given CEO episode, the variable *QInit* is equal to the 'average Q' of each firm in the year before the CEO was appointed. Average Q or simply Q is defined as the ratio of the market value of assets divided by the book value of assets (`at`). The market value of assets is equal to total assets (`at`), plus the product of common stock holdings (`csho`) times the closing stock price at the end of the fiscal year (`prcc_c`), minus the book value of common equity (`ceq`). Consistent with Almeida and Campello (2007), we set as missing those values of Q above 10. Our proxy for the arrival of a growth opportunity in year t, *RatioQ*, is equal to the ratio of the firm's average Q in year $t-1$ to *QInit*. Our results are robust to using, in the construction of *QInit* and *RatioQ*, the arithmetic mean of the average Q of all firms in the same 4-digit SIC industry group rather than the firm's own Q.

Cumulative Abnormal Returns. In the regressions reported in the main text, we control for past performance in year t using the 2-year annualized cumulative abnormal stock returns between January of year $t-2$ and December of year $t-1$, which we denote by CAR. The results presented in the paper are robust to computing the performance measure between January of year $t-1$ and December of year t, and to the use of shorter or longer window lengths for the measurement of past performance. To construct our CAR variable, we use monthly return data from CRSP to obtain abnormal returns at monthly frequency, compute compounded cumulative abnormal returns over 24 months, and annualize. To obtain monthly abnormal returns, we compute Dimson betas using rolling regressions over a 60-month time window, where the explanatory variables in the regressions include the current, lagged and forward values of the return on the market portfolio proxied by the CRSP value-weighted index.

Other Controls. In a given year t, the variable *lnAssets* equals the logarithm of the total assets of the firm as reported in Compustat (`at`) for year $t-1$. The variable *ROA*, or return on assets, equals the ratio of earnings (`ib`) over total assets (`at`) in year $t-1$. All variables are winsorized at the 1% level.

Compensation Duration. The variable *PayDuration* is computed according to the duration formula (40) displayed in the main text. The sample of CEO episodes for which this variable is computed comprises: (i) all completed episodes for which we observe annual compensation from year 1 of the CEO's tenure until he leaves post; and (ii) episodes in which the CEO is still in office by the end of our sample period and has been managing the firm for at least 13 years with no interruption in reported compensation (thirteen years corresponds to the 90% percentile of the variable *TotTenure* in our sample). The results presented in Section 4.3 of the paper are qualitatively unaffected when the latter group of episodes is removed from the sample.