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Sharing the proceeds from a hierarchical venture*

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Abstract

We consider the problem of distributing the proceeds generated from a joint venture in which the participating agents are hierarchically organized. We characterize a family of allocation rules ranging from the so-called *zero-transfer* rule (which awards agents in the hierarchy their individually generated revenues) and the *full-transfer* rule (which awards all the proceeds to the agent at the top of the hierarchy). The intermediate rule of the family imposes a sequence of transfers along the hierarchy consistent with the so-called *MIT strategy*, recently singled out as an optimal social mobilization mechanism. Our benchmark model refers to the case of linear hierarchies, but we also extend the analysis to the case in which hierarchies convey a general tree structure.

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1 Introduction

Agents often organize themselves into hierarchies when involved in joint ventures (e.g., Mookherjee, 2006). It is argued that workplace structures that are rich in sequentiality are desirable from the point of view of incentives (e.g., Winter, 2010). Ownership or power structures generate natural hierarchies with related chains of command and responsibility. As argued by Demange (2004), hierarchies yield stable cooperation structures when it comes to allocating resources.

Hierarchies may also relate to recruitment and communication channels as, for instance, in multi-level marketing (e.g., Emek et al., 2011), or social mobilization systems (e.g., Pickard et al., 2011). In those cases, the problem is to allocate a reward for executing a task, or to recruit new members.

In this paper, we consider a group of agents involved in a joint venture generating collective proceeds. The group is structured in several layers, each reflecting a different degree of responsibility, command, or even seniority. Thus, an agent located at a given layer is in command of (or, at least, held accountable for) all agents located at a lower layer. In such a hierarchy, agents are characterized by their degree of responsibility (location in the hierarchy), and the individual revenue they produce for the joint venture. Based on that information, the issue is how to allocate the overall produced revenue among the agents.

Our stylized model is flexible enough to accommodate various forms of organizations that are frequent in different professional sectors. For instance, as Galanter and Palay (1990) put it, "*Law firms are hierarchical. The working groups that serve clients consist of senior and junior lawyers. The latter are hired on the basis of their qualifications directly from prestigious law schools. The work of these junior lawyers is supervised and reviewed by seniors. Training is imparted to young lawyers in the course of a prolonged apprenticeship, normally ended either by promotion to partnership or by departure from the firm*". Similar patterns emerge in some physicians' practice arrangements (e.g., Kletke et al., 1996; Grytten et al., 2009) as well as with renowned architectural practices (e.g., Cuff, 1992; Winch and Schneider, 1993).

Two focal, and somewhat polar, allocation rules can be considered for the stylized setting described above. On the one hand, the *No Transfer rule*, in which each agent keeps her share (thus, ignoring the command structure conveyed by the hierarchy). On the other hand, the *Full Transfer rule*, in which the agent at the top of the hierarchy (the *boss*, or *venture capitalist*) gets all the proceeds (thus, ignoring individual contributions to the joint proceeds). Our main contribution in this paper is to characterize a family of rules reflecting a compromise between the two polar rules, in which certain *upward transfers* are allowed. The family of (transfer) rules we obtain are close in spirit to the so-called *geometric (incentive tree) mechanisms* of Pickard et al., (2011) and Lv and Moscibroda (2013). An *incentive tree* models the participation of people in crowdsourcing or human tasking systems. An incentive tree mechanism is an algorithm that determines how much reward each individual participant receives based on all the participants' contributions, as well as the structure of the solicitation tree. In geometric (incentive tree) mechanisms, a certain fraction α "bubbles-up" from one agent to the immediate superior, a fraction α^2 bubbles up to the immediate superior of the immediate superior, and so forth. In our case, a transfer rule imposes that the lowest-ranked agent gets a share λ of her revenue, her immediate superior gets a share λ of her revenue, and of any remaining 'surplus' from the lowest-ranked agent, and so forth. We further show that, interpreting λ as the propensity of getting a subordinate for any agent in the hierarchy, the agent at the top of the hierarchy (the boss) maximizes her revenue by setting $\lambda = 0.5$. This corresponds, precisely, to what Pickard et al., (2011) dub the *MIT strategy*.

Our contribution is also related to the sizable literature on fair division in networks (see, for instance, Hougaard (2009), or Thomson (2014), and the references cited therein). This literature mostly organizes itself into two strands.

On the one hand, the strand in which the networks give rise to cooperative games and where the structure of the network is exploited in order to define fair allocation among agents connected in the graph. The canonical case is that of cost sharing within a rooted tree, which can be traced back to Claus and Kleitman (1973) and Bird (1976). For fixed trees, the so-called Bird rule, which can be seen as a counterpart to the no-transfer rule described above, and the so-called serial rules, which convey a different form of transfers to the ones described above, are prominent. A particular case is the so-called *airport problem*, which can be traced back to Littlechild and Owen (1973) and Littlechild and Thompson (1977).[1] In airport problems, the runway cost has to be shared among different types of airplanes with a linear graph representing the runway. The rules highlighted in our work will also be reminiscent of some of the rules considered for airport problems. Actually, as we shall see later, one of the axioms we consider in this paper (lower-agent consistency) is similar to the so-called first agent consistency axiom of Potters and Sudhölter (1999). A related problem, which also has received considerable attention within this literature, is the so-called *minimum cost spanning tree problem* in which the issue is to (fairly) share the cost of designing the optimal network serving a group of agents.

[1] See Thomson (2013) for a recent survey.

Focal rules have been proposed for such a problem (e.g., Bergantiños and Vidal-Puga, 2007; Bogomolnaia and Moulin, 2010) and, as we shall see later, the rules we highlight in this paper are also reminiscent to some of them. A common feature for all these models, however, is that the cheapest connection (minimal distance) to the root becomes a crucial element, as it represents the stand-alone option for the agents. This is not the case in our model where the crucial feature is the combination of the agents' revenue and the location in the hierarchy.

On the other hand, there is a strand of the literature where networks restrict cooperative games. Myerson (1977, 1980) pioneered this approach by using graphs to represent permission structures in cooperative games. A central result within this approach is that if agents are allowed to cooperate in tree structures, the original TU-game need only be superadditive to guarantee that the graph-restricted game has a non-empty core (see also Demange, 2004). Closer to our approach is the literature on TU-games with precedence structure (e.g., Faigle and Kern, 1992; Grabisch and Sudhölter, 2012, 2014). Therein, the set of players has a hierarchical structure, and a coalition is feasible if, for each player in the coalition, all the players preceding her in the hierarchy are also members of the coalition. Compared to this strand of the literature we do not have a predefined cooperative game where the hierarchies are restricting cooperation. Instead, we relate fairness directly to the network structure.

The rest of the paper is organized as follows. In Section 2, we introduce the canonical model in which the hierarchy can be expressed as a rooted line. The main results for that model are collected in Section 3. Section 4 generalizes the analysis to the case of branch hierarchies (not necessarily linear) and shows how the characterization results generalize to such a setting with minimal adjustments of the axioms. Section 5 further links the transfer rules we characterize to incentive tree mechanisms. Finally, Section 6 concludes.

2 The canonical model

There exists a set of potential **agents**, identified with the set of natural numbers. Let $\mathcal{M}$ be the class of finite subsets of natural numbers, with generic element M. Each set $M \in \mathcal{M}$ will represent a **linear hierarchy**, with the convention that lower numbers in M refer to lower positions in the hierarchy. For instance, if $M = \{1, \ldots, m\}$, then 1 is representing the agent with the lowest rank in the hierarchy, whereas m is representing the agent with the highest rank. In an ownership structure, m would be interpreted as the *boss*, or the *venture capitalist*.

Agents in each linear hierarchy will be involved in a joint venture to which all of them contribute. Formally, for each $i \in M$, let $r_i \in \mathbb{R}_{++}$ be the **revenue** that agent i generates, and $r \equiv (r_i)_{i \in M}$ the profile of revenues.[2]

A *linear hierarchy revenue sharing problem*, or simply, a **problem** is a duplet consisting of a linear hierarchy $M \in \mathcal{M}$ and a profile of revenues $r \in \mathbb{R}^{|M|}_{++}$. Let $\mathcal{R}^M$ be the set of problems involving the hierarchy M and $\mathcal{R} \equiv \bigcup_{M \in \mathcal{M}} \mathcal{R}^M$.

Given a problem $(M, r) \in \mathcal{R}$, an **allocation** is a vector $x \in \mathbb{R}^{|M|}$ satisfying the following two conditions:

(i) for each $i \in M$, $0 \leq x_i \leq \sum_{j \leq i} r_j$, and

(ii) $\sum_{i \in M} x_i = \sum_{i \in M} r_i$.

Condition (i), which we refer to as **boundedness**, sets that agents can neither get a negative payment, nor a higher payment than the aggregate revenue generated by their subordinates in the hierarchy (including the agent herself). Condition (ii), which we call **balance**, sets that the total revenue is fully allocated among the agents in the hierarchy.

An **allocation rule** is a mapping ϕ assigning to each problem $(M, r) \in \mathcal{R}$ an allocation $\phi(M, r)$. We assume from the outset that rules are **anonymous**, i.e., for each problem $(M, r) \in \mathcal{R}$, and for each strictly monotonic bijective function $g : M \to M'$, $\phi_{g(i)}(M', r') = \phi_i(M, r)$, where $r'_{g(i)} = r_i$, for each $i \in M$. Thus, in what follows, with the exception for Section 5, we assume, without loss of generality, that $M = \{1, \ldots, m\}$.

Two (polar) examples of rules are those capturing the minimal and maximal possible revenue transfers from subordinates to their superiors in the hierarchy.

More precisely, the first one imposes that each agent in the linear hierarchy keeps her own revenue and transfers nothing to her superiors. Formally,

No-Transfer rule, $\phi^{\mathbf{NT}}$: For each $(M, r) \in \mathcal{R}$,

$$\phi^{NT}(M, r) = r.$$

Its polar rule imposes that the boss receives all revenues. Formally,

Full-Transfer rule, $\phi^{\mathbf{FT}}$: For each $(M, r) \in \mathcal{R}$,

$$\phi^{FT}(M, r) = \left(0, \ldots, 0, \sum_{i \in M} r_i\right).$$

In between the two extreme rules presented above a vast number of rules can be imagined. Instead of endorsing a specific rule directly, we take an axiomatic approach and propose first

[2] For each $M \in \mathcal{M}$, each $S \subseteq M$, and each $z \in \mathbb{R}^m$, let $z_S \equiv (z_i)_{i \in S}$. For each $i \in M$, let $z_{-i} \equiv z_{M \setminus \{i\}}$.

several axioms reflecting principles that we find normatively appealing in the context of these problems. Ultimately, our goal will be to single out rules as a result of combining those axioms.

We start with the principle of consistency, an operational notion that has played an instrumental role in axiomatic analyses of diverse problems, and for which normative underpins have also been provided (e.g., Thomson, 2012). The principle refers to the way in which rules react to agents leaving the scene with their awarded amounts. Here we concentrate on a minimalistic version of the principle referring only to the case in which the agent with the lowest rank leaves the hierarchy after the allocation took place. It seems natural to assume that subordinates refer to their immediate superiors in the linear hierarchy to terminate their relationship. Thus, we assume that, after leaving, a new problem arises in which the agent with the second-lowest rank in the original problem becomes the lowest-ranked agent, but now also generating the eventual revenue that the leaving agent generated in the original problem and did not take in the allocation. The next axiom states that the solution of the new problem agrees with the solution of the original problem for all the standing agents in the hierarchy.[3] Formally,

Lowest Rank Consistency: For each $(M, r) \in \mathcal{R}$,

$$\phi_{M\setminus\{1\}}(M, r) = \phi\left(M \setminus \{1\}, (r_2 + r_1 - \phi_1(M, r), r_{M\setminus\{1,2\}})\right).$$

The next two properties focus on the opposite edge of the hierarchy.

The first one says that the revenue generated by the highest-ranked agent (i.e., the boss) is irrelevant for the allocation of all the subordinates. A plausible rationale for this axiom is that, in an ownership structure, the boss is the indisputable owner of her own revenue. Formally,

Highest Rank Revenue Independence: For each $(M, r) \in \mathcal{R}$, and each $\hat{r}_m \in \mathbb{R}_{++}$,

$$\phi_{M\setminus\{m\}}(M, r) = \phi_{M\setminus\{m\}}\left(M, (r_{-m}, \hat{r}_m)\right).$$

The second one avoids certain strategic manipulations of the allocation by the highest-ranked agent. More precisely, it says that the boss cannot benefit from splitting her revenue into two amounts represented by two agents ranked highest in the new hierarchy.[4] Formally,

[3]As mentioned in the introduction, this axiom is reminiscent of the so-called "first agent consistency" axiom proposed by Potters and Sudhölter (1999) for airport problems.

[4]Axioms of this sort have been widely explored in various models of resource allocation (e.g., Ju et al., 2007). Note that our axiom only requires "splitting-proofness" in a specific situation, which makes it weaker than the standard counterpart axioms in such a literature.

Highest Rank Splitting Neutrality: For each $(M, r) \in \mathcal{R}$, let $(M', r') \in \mathcal{R}$ be such that $M' = M \cup \{k\}$, $k > m$, $r_m = r'_k + r'_m$, and $r'_{M\setminus\{m\}} = r_{M\setminus\{m\}}$. Then,

$$\phi_{M\setminus\{m\}}(M', r') = \phi_{M\setminus\{m\}}(M, r).$$

Finally, we consider a technical property stating that if revenues are scaled by a factor α, so is the solution. In particular, the axiom says that the currency in which we measure revenue is irrelevant for the allocation process.[5]

Scale Invariance: For each $(M, r) \in \mathcal{R}$, and each $\alpha > 0$,

$$\phi(M, \alpha r) = \alpha\phi(M, r).$$

3 The main results

It is not difficult to show that the no-transfer rule and the full-transfer rule presented above satisfy the list of axioms introduced in the previous section. In this section, we identify all the remaining existing rules satisfying these axioms. To do so, it is worth noting first that the no-transfer and the full-transfer rules are extreme in an obvious sense, which suggests that the set of rules satisfying the axioms should consist of all rules resulting from a compromise between the no-transfer and the full-transfer rules. It turns out that this compromise can be described as follows:

Suppose the lowest-ranked agent gets a share $\lambda \in [0, 1]$ of her revenue, her immediate superior gets a share λ of her revenue, as well as any remaining 'surplus' from the lowest-ranked agent, etc., and the highest-ranked agent gets the residual. Hence, if $M = \{1, \ldots, m\}$, payment shares are determined recursively as

$$x_i^\lambda = \lambda r_i + (1 - \lambda)(x_{i-1}^\lambda), \tag{1}$$

for each $i \in M$, with the notational convention that $x_0^\lambda = 0$.

Note that we may rewrite (1) in the closed-form expression

$$x_i^\lambda = \lambda \left(r_i + (1 - \lambda)r_{i-1} + \cdots + (1 - \lambda)^{i-1} r_1\right),$$

[5] This axiom appears frequently in axiomatic studies of resource allocation (e.g., Friedman and Moulin, 1999; Hougaard and Tvede, 2015).

for $i = 1, \ldots, m-1$ and

$$x_m^\lambda = r_m + (1-\lambda)r_{m-1} + \cdots + (1-\lambda)^{m-1}r_1.$$

Denote the corresponding family of allocation rules, so defined, which we call **transfer rules**, by $\{\phi^\lambda\}_{\lambda\in[\mathbf{0},\mathbf{1}]}$. It is straightforward to see that ϕ^1 is the no-transfer allocation rule, defined by (2), whereas ϕ^0 is the full-transfer allocation rule, defined by (2).

Example 1: Consider the problem $(\{1,2,3\},(12,6,12))$, i.e., a strict hierarchy made of three agents, 1, 2, and 3, in which agent 1 generates a revenue of 12, agent 2 a revenue of 6, and agent 3 a revenue of 12. Figure 1 below illustrates the situation.

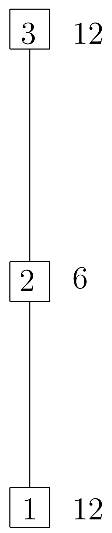

Figure 1: A linear hierarchy.

It is straightforward to see that the no-transfer rule selects the allocation $(12, 6, 12)$ for this example, whereas the full-transfer rule selects the allocation $(0, 0, 30)$. In general, the transfer rules select the allocations

$$(12\lambda, (18-12\lambda)\lambda, 30(1-\lambda)+12\lambda^2),$$

for each $\lambda \in [0,1]$. In particular, for $\lambda = 0.5$, the corresponding transfer rule selects the allocation $(6, 6, 18)$. Thus, in such a case, agent 2 receives the same as agent 1, despite the fact that agent 1 is generating twice the revenue.

Our main result is the following.

Theorem 1 *A rule ϕ satisfies Lowest Rank Consistency, Highest Rank Revenue Independence, Highest Rank Splitting Neutrality, and Scale Invariance if and only if it is a transfer rule, i.e.,* $\phi \in \{\phi^\lambda\}_{\lambda\in[0,1]}$.

Proof: It is not difficult to see that the transfer rules satisfy all the axioms in the statement of the theorem. As an illustration, we show that they satisfy *Lowest Rank Consistency.* To do so, let $\lambda \in [0,1]$ and $(M,r) \in \mathcal{R}$ be given. For each $i \in M$, let $x_i = \phi_i^\lambda(M,r)$ and $\widetilde{x}_i = \phi_i^\lambda\left(M \setminus \{1\}, (r_2 + r_1 - x_1, r_{M\setminus\{1,2\}})\right)$. Then, $\widetilde{x}_2 = \lambda(r_2 + r_1 - x_1) = x_2$. For each $j \neq m$, $\widetilde{x}_j = \lambda r_j + (1-\lambda)\widetilde{x}_{j-1}$. Thus, by induction, $\widetilde{x}_j = x_j$ and $\widetilde{x}_m = r_m + r_{m-1} - x_1 - \sum_{k=2}^{n-1} \widetilde{x}_k = x_m$.

We now suppose that ϕ is a rule satisfying all the axioms in the statement of the theorem. First, let $M = \{1\}$ and $r = r_1$. By balance, $\phi_1(M,r) = r_1 = \phi_1^\lambda(M,r)$, for each $\lambda \in [0,1]$. Next, add a superior agent 2 with revenue r_2. Let $M' = \{1,2\}$ and $r' = (r_1, r_2)$. Then, by boundedness, $\phi_1(M',r') \in [0, r_1]$, so $\phi_1(M',r') = \lambda r_1 = \phi_1^\lambda(M',r')$ for some $\lambda \in [0,1]$. By *Highest Rank Revenue Independence*, λ is independent of r_2. Moreover, λ is independent of r_1. To see this, suppose, by contradiction, that we have $\tilde{r} = (\tilde{r_1}, \tilde{r_2})$ with $r_2 = \tilde{r_2}$ and $\phi_1(M',r') = \lambda r_1$ and $\phi_1(M',\tilde{r}) = \widetilde{\lambda}\widetilde{r}_1$ with $\lambda \neq \widetilde{\lambda}$. Then, by *Scale Invariance*, $\phi_1(M', \frac{\widetilde{r}_1}{r_1} r) = \frac{\widetilde{r}_1}{r_1}\lambda r_1 = \lambda\widetilde{r}_1 \neq \widetilde{\lambda}\widetilde{r}_1$, contradicting that λ is independent of r_2. Now, by balance, $\phi_2(M',r') = r_2 - r_1 - \phi_1(M',r') = \phi_2^\lambda(M',r')$.

Next, suppose there is λ such that $\phi = \phi^\lambda$ for all problems with up to k agents, $k \geq 2$. Now, consider the problem (M^k, r^k) with $M^k = \{1, \ldots, k\}$ and $r^k = \{r_1, \ldots, r_k\}$ and add an agent $k+1$. By *Highest Rank Revenue Independence*, and *Highest Rank Splitting Neutrality*, $\phi_i(M^{k+1}, r^{k+1}) = \phi_i(M^k, r^k) = \phi_i^\lambda(M^k, r^k)$ for all $i \leq k-1$. By *Lowest Rank Consistency*, $\phi_k(M^{k+1}, r^{k+1}) = \phi_k(M^{k+1} \setminus \{1\}, r_2^{k+1} + r_1^{k+1} - \phi_1(M^{k+1}, r^{k+1}), r_{M^{k+1}\setminus\{1,2\}})$ and thus, by the induction hypothesis, $\phi_k(M^{k+1}, r^{k+1}) = \phi_k^\lambda(M^{k+1}, r^{k+1})$. Finally, by balance,

$$\phi_{k+1}(M^{k+1}, r^{k+1}) = r_{k+1} - \sum_{i=1}^{k} \phi_i^\lambda(M^{k+1}, r^{k+1}) = \phi_{k+1}^\lambda(M^{k+1}, r^{k+1}). \qquad \blacksquare$$

Theorem 1 is tight. In order to show that, let us consider the following rules.

- The classical *serial rule* (e.g., Moulin and Shenker, 1992) imposes that each agent's revenue is split equally among her superiors and herself. In Example 1, it would yield the allocation $(4, 7, 19)$. The serial rule violates *Highest Rank Splitting Neutrality*, while satisfying all the remaining axioms at the statement of Theorem 1.

- Another natural rule is the one in which all agents keep a fraction λ of their own revenue and the boss receives the residual. In Example 1, it would yield the allocation $(6, 3, 21)$, for the case with $\lambda = 0.5$. This rule violates *Lowest Rank Consistency*, while satisfying all the remaining axioms at the statement of Theorem 1.

- The hybrid rule obtained while using the full-transfer rule if $\sum_{j=2}^{m} r_j < r_1$, and the zero-transfer rule otherwise is another well-defined rule for our setting. This rule violates *Highest Rank Revenue Independence*, while satisfying all the remaining axioms at the statement of Theorem 1.

- Finally, a rule defined as a transfer rule, but in which λ depends on the sum of the revenues, violates *Scale Invariance*, while satisfying all the remaining axioms at the statement of Theorem 1.

In what follows, we complement the above characterization result by adding two new axioms, which allows us to single out the extreme members of the family of transfer rules.

The following axiom states that agents producing higher revenues should be awarded more. Formally,

Revenue Order Preservation: For each $(M, r) \in \mathcal{R}$, and each pair $i, j \in M$ such that $r_i \geq r_j$, $\phi_i(M, r) \;\geq\; \phi_j(M, r)$.

The zero-transfer rule is the only transfer rule satisfying the previous axiom. More interestingly, and as shown by the next result, the rule is characterized by such an axiom in combination with *Highest Rank Revenue Independence*.

Theorem 2 *A rule satisfies Highest Rank Revenue Independence and Revenue Order Preservation if and only if it is the zero-transfer rule.*

Proof: We concentrate on the non-trivial implication, i.e., let ϕ be a rule satisfying *Highest Rank Revenue Independence* and *Revenue Order Preservation.* Let $(M, r) \in \mathcal{R}$ be given. We claim first that $\sum_{j=1}^{m-1} \phi_j(M, r) \leq \sum_{j=1}^{m-1} r_j$. By contradiction, assume otherwise. Then, by *Highest Rank Revenue Independence* we can vary r_m without affecting the shares of the other agents ($i = 1, \ldots, m-1$). Thus, let $r_m < \sum_{j=1}^{m-1} \phi_j(M, r) - \sum_{j=1}^{m-1} r_j$, which contradicts either boundedness or balance.

As $\sum_{j=1}^{m-1} \phi_j(M, r) \leq \sum_{j=1}^{m-1} r_j$, balance implies that $\phi_m(M, r) \geq r_m$. Thus, letting $r_m = r_i$ for any $i = 1, \ldots, m-1$ we get, by *Revenue Order Preservation*, that $\phi_i(M, r) = \phi_m(M, r) \geq r_i$. Now, balance gives $\phi_i(M, r) = r_i$ for all $i \in M$. ■

The next axiom states that agents located higher in the hierarchy should be awarded more. Formally,

Hierarchical Order Preservation: For each $(M,r) \in \mathcal{R}$, and each pair $i,j \in M$, where $i \geq j$, $\phi_i(M,r) \ \geq \ \phi_j(M,r)$.

The full-transfer rule is the only transfer rule satisfying the previous axiom. More interestingly, and as shown by the next result, the rule is characterized by such an axiom in combination with *Highest Rank Splitting Neutrality.*

Theorem 3 *A rule satisfies Highest Rank Splitting Neutrality and Hierarchical Order Preservation if and only if it is the full-transfer rule.*

Proof: We concentrate on the non-trivial implication, i.e., let ϕ be a rule satisfying *Highest Rank Splitting Neutrality* and *Hierarchical Order Preservation.* By contradiction, suppose that there exists a problem $(M,r) \in \mathcal{R}$ and an agent $i \neq m$, such that $\phi_i(M,r) = \epsilon > 0$.

Consider a new problem (M',r'), where $M' = \{1,\ldots,m+x\}$, $r'_i = r_i$ for all $i < m$ and $\sum_{j=m}^{m+x} r'_j = r_m$. By *Highest Rank Splitting Neutrality* $\phi_i(M',r') = \phi_i(M,r)$ for all $i < m$. Now, choose $x > \frac{\sum_{j=1}^{m+x} \phi_j(M',r')}{\epsilon}$. By *Hierarchical Order Preservation*, $\phi_j(M',r') \geq \epsilon$ for all $j = m,\ldots,m+x$, which contradicts balance. ■

4 Branch hierarchies

In this section, we extend the linear-hierarchy case considered above to account for *branch hierarchies*, i.e., situations in which a given agent can have more than one immediate subordinate.

We represent a branch hierarchy as a tree, where each agent is connected to the (unique) boss via a unique rank path consisting of all her superiors (see Figure 1).

A *branch hierarchy revenue sharing problem*, or simply, a **b-problem** is a triple (M,r,s), where N is a non-empty finite set of agents, r is a revenue profile specifying the revenue of each agent in N, and s is a function mapping each agent $i \in N$ to her immediate superior agent $j = s(i)$, with the convention that $s(i) = i$ if i is the boss, such that the graph induced by s has no cycles. Let $\mathcal{B}$ denote the set of b-problems.

Given a b-problem (M,r,s), a **b-allocation** is a vector $x \in \mathbb{R}^{|M|}$ satisfying the counterpart conditions of **boundedness**, and **balance** in this setting. A **b-allocation rule** is a mapping β assigning to each problem (M,r,s) an allocation $\beta(M,r,s) = x$. We also impose from the outset, as in the linear case, that rules are **anonymous**, i.e., for each strictly monotonic bijective function $g : M \to M'$, $\beta_{g(i)}(M',r',s') = \beta_i(M,r,s)$, where $r'_{g(i)} = r_i$, and $s'_{g(i)} = s_i$ for each $i \in M$.

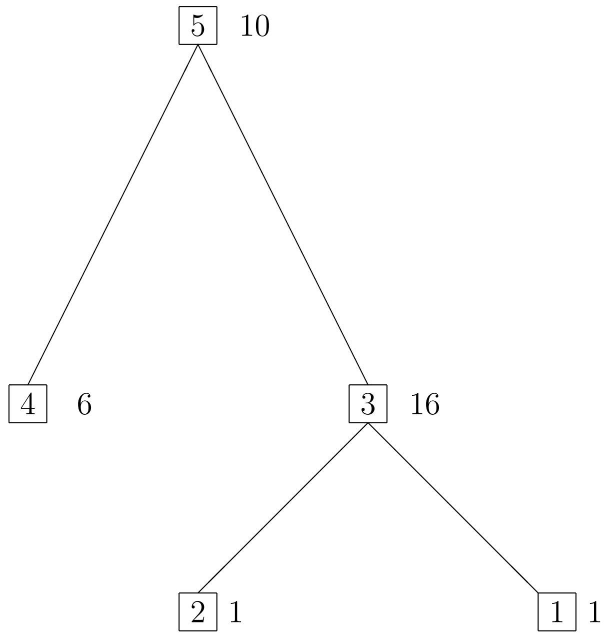

Figure 2: A branch hierarchy. This figure illustrates a branch hierarchy involving five agents, with agent 5 denoting the boss, agents 3 and 4 her direct subordinates and agants 1 and 2 being the subordinates of agent 3. Each of the two agents at the third layer generate a revenue of 1. Agent 4 yields a revenue of 6, whereas agent 3 yields a revenue of 16. Finally, agent 5 yields a revenue of 10. In summary, the hierarchy so illustrated is $(M, r, s) = (\{1, 2, 3, 4, 5\}, (1, 1, 16, 6, 10), s)$, where $s(1) = s(2) = 3$, $s(3) = s(4) = s(5) = 5$.

The transfer rules have a simple generalization to branch hierarchies. Formally, let i be an agent at the bottom of the hierarchy, somewhere in the tree. Then,

$$x_i^\lambda = \lambda r_i.$$

Her immediate superior $s(i)$ gets

$$x_{s(i)}^\lambda = \lambda \left(\sum_{j \in M:\ i=s(j)} (1-\lambda) r_j + r_{s(i)} \right),$$

and so forth. Denote the corresponding family of allocation rules, which we call **b-transfer rules**, by $\{\beta^\lambda\}_{\lambda \in [\mathbf{0},\mathbf{1}]}$.

Our axioms from the linear hierarchy model also have a natural extension to the branch hierarchy model. Formally,

b-Lowest Rank Consistency: For each $(M, r, s) \in \mathcal{B}$, and each $i \in N$ without subordinates,

$$\beta_{M\setminus\{i\}}(M, r, s) = \beta\left(M \setminus \{i\}, (r_{s(i)} + r_i - \beta_i(M, r, s), r_{M\setminus\{i,s(i)\}}), s_{M\setminus\{i\}}\right).$$

b-Highest Rank Revenue Independence: For each $(M, r, s) \in \mathcal{B}$ and each $\hat{r}_i \in \mathbb{R}_{++}$, if i is the boss, then

$$\beta_{M\setminus\{i\}}(M, r, s) = \beta_{M\setminus\{i\}}\left(M, (r_{-i}, \hat{r}_i), s\right).$$

b-Highest Rank Splitting Neutrality: For each $(M, r, s) \in \mathcal{B}$ where agent i is the boss, let (M', r', s'), be such that $M' = M \cup \{k\}$, $s(i) = k$, $r_i = r'_k + r'_i$, and $r'_{M\setminus\{i\}} = r_{M\setminus\{i\}}$. Then,

$$\beta_{M\setminus\{i\}}(M', r', s') = \beta_{M\setminus\{i\}}\left(M, r, s\right).$$

b-Scale Invariance: For each $(M, r, s) \in \mathcal{B}$, and each $\alpha > 0$,

$$\beta(M, \alpha r, s) \; = \alpha\beta(M, r, s).$$

With these extended axioms in place we can now extend Theorem 1 to branch hierarchies.

Theorem 4 *A b-rule β satisfies b-Lowest Rank Consistency, b-Highest Rank Revenue Independence, b-Highest Rank Splitting Neutrality, and b-Scale Invariance if and only if it is a b-transfer rule, i.e., $\beta \in \{\beta^\lambda\}_{\lambda\in[0,1]}$.*

Proof: It is not difficult to see that the b-transfer rules satisfy all the axioms at the statement of the theorem. Conversely, let β be a rule satisfying all the axioms at the statement of the theorem. Let $(M, r, s) \in \mathcal{B}$. We distinguish two cases.

Case 1: $(M, r) \in \mathcal{R}$.

In this case, the branch hierarchy $(M, r, s) \in \mathcal{B}$ consists of a line, and thus we use the abbreviated notation $(M, r) \in \mathcal{R}$. Then, by Theorem 1, there exists $\lambda \in [0, 1]$, such that $\beta(M, r) = \beta^\lambda(M, r)$.

Case 2: $(M, r, s) \in \mathcal{B} \setminus \mathcal{R}$.

Let i denote an agent without subordinates in the branch hierarchy (M, r, s). By boundedness, $x_i = \beta_i(M, r, s) = \delta r_i$ for some $\delta \in [0, 1]$. Iteratively, we can apply b-Lowest Rank Consistency to all agents not located on the direct path of superiors from i to the boss, in order to reduce the branch hierarchy to a line. For each iteration i, the payment remains unchanged and we ultimately end up with the line $\delta = \lambda$, which concludes the proof.

This argument can be repeated for any agent without subordinates. By using b-Lowest Rank Consistency again, we can derive the payment of immediate superiors $s(i)$. The proof easily concludes from here. ■

5 Further insights

It is interesting to note that the $\lambda-$transfer rule arising when $\lambda = 0.5$ has a close relation to what Pickard et al., (2011) call the *MIT strategy*. More precisely, Pickard et al., (2011) consider optimal social mobilization mechanisms to solve a task (e.g. finding an object). In order to solve the task, a principal may recruit agents, who may recruit agents themselves, and so forth, so that a tree-structured recruitment relation is generated. If the task is solved (e.g., the object is found), the principal gets paid an amount B, and she may transfer payments along the tree according to a given mechanism. The so-called *MIT strategy* is the mechanism arising with the following payment scheme, for a linear recruitment graph: the agent who solves the task keeps $B/2$, then his recruiter gets $B/4$, the recruiter's recruiter gets $B/8$, and, so forth. Pickard et al., (2011) show that this mechanism is never in deficit, i.e., the residual from B, after obeying this payment scheme, is always non-negative.

The MIT strategy corresponds exactly to the $\lambda-$transfer rule with $\lambda = 0.5$, in a situation where the revenue of the lowest ranked agent is B and all other agents have revenue 0, with the distinction that the boss gets to keep the residual (due to the balance condition of our rules).

We provide in this section additional rationale to single out such a rule, as an *optimal* rule within the family of $\lambda-$transfer rule. In order to do that, suppose that the boss is in charge of selecting a rule, within the $\lambda-$transfer rules. We show next that selecting the rule corresponding to $\lambda = 0.5$ constitutes the optimal choice, if the parameter λ reflects agents' propensity to join hierarchy.

More precisely, consider the case where the boss aims to create a linear hierarchy, in which proceeds will be shared according to a $\lambda-$transfer rule. Assume that λ is the probability that any agent in the hierarchy gets a subordinate. That is, if the boss selects the full transfer rule ($\lambda = 0$) the probability of having agents to join the hierarchy as subordinates is 0, as all their revenues are transferred to the boss. Likewise, using the no transfer rule ($\lambda = 1$) the probability is 1, as agents get to keep their own revenue anyway.

Because we may face unlimited hierarchies, we modify our previous notational convention, so that the boss is now denoted as agent 0 while subordinated agents are indexed by negative numbers $-1, -2, \ldots$, where -1 is the subordinate of the boss. In this way we can formulate the boss' choice of λ as an expected revenue maximization problem.

Proposition 1 *If λ denotes the probability that an agent within a linear hierarchy gets a subordinate, the boss' optimal choice within the family of λ-transfer rules is obtained at $\lambda = 0.5$.*

Proof: As we know, the boss' total revenue obtained from a linear hierarchy, when using a λ-transfer rule, is given by

$$\sum_{t=1}^{\infty}(1-\lambda)^t r_{-t} + r_0.$$

Now, if the boss aims to maximize total revenue in expected terms, when λ denotes the probability that an agent within a linear hierarchy gets a subordinate, the following problem should be solved:

$$\max_{\lambda} \sum_{t=1}^{\infty}((1-\lambda)\lambda)^t r_{-t}.$$

It is straightforward to see that the previous problem is equivalent to the following one:

$$\max_{\lambda} f(\lambda) = (1-\lambda)\lambda,$$

whose solution is $\lambda = 0.5$. ■

As an illustration, note that when every agent has revenue equal to 1, the expected transfer from subordinates to the boss at optimal $\lambda = 0.5$ is $\sum_{t=1}^{\infty}(1/4)^t = 1/3$.

Notice that, for branch hierarchies, the maximization problem is the same and $\lambda = 0.5$ remains the unique optimal solution, when no agent has more than 3 subordinates. With 4 or more subordinates for all agents, the expected revenue becomes infinite. However, for any branch hierarchy with a finite limit (length) $\lambda = 0.5$ also remains the unique optimal solution, as the expected revenue can be expressed as the sum of expected revenues from each layer in the branch hierarchy.

6 Conclusion

We have presented in this paper a stylized model to analyze the problem of sharing the collective proceeds generated from a joint venture, in which participating agents, who are hierarchically organized, contribute with (possibly different) individual revenues to the collective proceeds. Our model is flexible enough to accommodate several forms of professional organizations and practices in real life.

We characterize a family of allocation rules for our model, ranging from the rule ignoring the command structure conveyed by the hierarchy, to the rule ignoring individual contributions to the joint proceeds. The rules convey a compromise between those two polar rules, allowing for certain upward transfers in the command structure. The family is characterized by four independent axioms (Lowest Rank Consistency, Highest Rank Revenue Independence, Highest Rank Splitting Neutrality, and Scale Invariance). When an additional axiom modeling order preservation (with respect to either individual revenues, or the command structure) is added, each of the two polar rules mentioned above can be singled out within the family.

The intermediate member of our family, obtained when the compromise between the polar rules is balanced, is a translation to our context of the so-called MIT strategy, which has shown to be an optimal mechanism for social mobilization. We also show that the rule is optimal, within our family, if the aim is to maximize the expected revenues of the venture capitalist, i.e., the agent at the top of the hierarchy, and the process to get subordinates is probabilistic.

The previous results are obtained for the benchmark model referring to the case of linear hierarchies. Nevertheless, we also extend the analysis to the case in which hierarchies convey a general tree structure and provide therein a counterpart (generalized) characterization for the family of *transfer* rules.

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