The Hard Problem and the Tyranny of the Loser

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Abstract

A *Hard Problem* is a collective choice problem in which the only feasible alternatives apart from the status quo consist of a welfare gain to some people (the *Winners*) and a welfare loss to the others (the *Losers*). These problems are typical in a number of settings, such as climate action, anti-trust regulation, and tax design. We study how to make collective choices when faced with Hard Problems. We find that requiring a relatively weak fairness condition, which we call *Expansion Solidarity*, necessarily leads to a dictatorship of the Losers, no matter how small their number. Even one single Loser must be given the power to veto any departure from the status quo, regardless of the number of Winners, how large the gains, or how small the loss.

JEL: D63; D70.

Keywords: Pareto improvements, hard choices, solidarity, compromises, maximin difference.

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1 Introduction

Pareto improvements, where at least someone is made better off without even a single person made worse off, are not feasible in many important settings. Take climate action. Specifically, the issue of whether, and to what extent, the current generation ought to bear the cost of fighting climate change for the sake of future generations. Future generations cannot compensate the current one, so net of any altruistic motives, actions to fight climate change would lead to a welfare loss for (at least some members of) the current generation and a welfare gain for future ones, relative to a status quo of taking no action. Actions that do not harm even a single living person are impossible. The only feasible alternatives other than the status quo lead to a welfare loss for some (possibly very few) *Losers*, despite a welfare gain for some (possibly very many) *Winners*. We call such collective choice problems *Hard Problems*.

On top of settings where Pareto improvements are physically unfeasible, Hard Problems arise in policy decisions where Pareto-improving compensations from Winners to Losers are unfeasible due to informational constraints, incentive incompatibilities, or there not existing a level of compensation that *each and everyone* of the potential Losers would be willing to accept for their welfare loss. Anti-trust regulations, tax design, trade liberalisation, infrastructure developments, public health measures, and the redistribution of resources in fixed supply, among many other settings, offer further examples of Hard Problems.

How should we make collective choices when faced with Hard Problems? Sometimes the status quo may well be the optimal compromise between the Winners and Losers. But it seems plausible that not every situation is regarded as an optimal compromise once it is a status quo. This is especially the case when there are very many Winners who have a huge welfare gain from moving away from the status quo, and only a single Loser who has to withstand but a tiny loss. At least a small move away from the status quo would seem warranted.

However, it turns out that any such compromise violates a fairness condition that is hard to object to. This condition, which we call *Expansion Solidarity*, requires that if new Pareto-improving possibilities become available under the same status quo, then neither the Winners nor the Losers should become worse off relative to their welfare before these new possibilities became available. For example, a new technology that reduces carbon emissions should not be a reason to reduce even further the welfare of the current generation, nor that of future generations, in fighting climate change. We show that any solution that satisfies Expansion Solidarity must be a Tyranny of the Loser: the status quo must always be chosen.

In a Hard Problem, everyone is *either* a Winner *or* a Loser. However, there may also be settings where the status quo is Pareto optimal, but some people, or even everyone, might be a Winner in some of the feasible alternatives, but a Loser in the other alternatives. For example, different actions to fight climate change might place the burden on different groups of people. We show that the Tyranny of the Loser is robust to extending Hard Problems to these *Extended Hard Problems* where the identity of Winners and Losers is not fixed.

The framework in which we study Hard and Extended Hard Problems is formally the same as the one used to study cooperative bargaining and rationing problems. The only differences between these collective choice problems are that there may only be Winners in a cooperative bargaining problem, only Losers in a cooperative rationing problem, and that the status quo is not in general Pareto optimal in cooperative bargaining. There are many possible non-dictatorship solutions to cooperative bargaining and rationing. Our results show that all these possibilities disappear for Hard and Extended Hard Problems. When there is just one Loser among Winners, just one Winner among Losers, or when the status quo is Pareto optimal, all non-dictatorship solutions violate Expansion Solidarity.

Common ways to escape impossibilities are to allow for greater information about the measurability and interpersonal comparability of welfare, as well as to make further domain restrictions. We show that the Tyranny of the Loser holds regardless of the information allowed about the measurability and interpersonal comparability of welfare, as well as under further domain restrictions that ensure a sufficient 'richness' of the sets of feasible alternatives, and thus the existence of a sufficiently large set of possible compromises.

We extend our core analysis to what we call *contextualised social welfare relations*. Standard social welfare relations compare welfare profiles in isolation from the social context. Specifically, they do not take the status quo into account. However, the essence of a Hard Problem is precisely the presence of a Pareto efficient status quo. We propose contextualised social welfare relations as an extension of social welfare relations that may compare welfare profiles given a status quo. We show that there is no contextualised social welfare ranking[1] that satisfies Expansion Solidarity together with relatively mild convexity, continuity, and efficiency properties.

Finally, in search of possibilities, we turn to a liberal non-interference principle instead of a fairness one. This principle requires that if someone suffers a strict welfare loss in a way in which no one else is affected, then this person should not be punished further. Such a punishment would be unjustified on liberal grounds. The Tyranny of the Loser re-emerges under this principle when only unit-comparable cardinal measurability of welfare is allowed, and the contextualised social welfare ranking satisfies mild continuity, efficiency, and anonymity properties.

[1] That is, a contextualised social welfare relation which is reflexive, s-transitive, and s-complete. See Section 5 for details.

1.1 Related literature

Our framework differs from most welfarist settings in social choice, such as those used to study collective choice problems (e.g. Peters and Wakker 1991; Ok and Zhou 1999; Mariotti 2000), those used in the 'old' and 'new' welfare economics (e.g. Kaldor 1939; Hicks 1939; Bergson 1938; Samuelson 1947; 1950; Ng 2004), and the welfarist social welfare functional framework (Sen 1970; see d'Aspremont and Gevers 2002 for a survey), in that it does not abstract away the context of the collective choice. We explicitly take into account a Pareto-efficient status quo relative to which are groups of Winners and Losers. This is different from cooperative bargaining problems (see Thomson 1994; 2022 for surveys), where there may only be Winners, and also from cooperative rationing problems (e.g. Chun and Thomson 1992; Herrero 1997; Mariotti and Villar 2005), where there may only be Losers.

A paper with a similar flavour is Bogomolnaia et al. (2017), which finds that if the zero vector is Pareto-efficient in a set of feasible alternatives, then it is the competitive allocation. This is reminiscent of our Tyranny of the Loser if the zero vector is interpreted as the status quo. However, Bogomolnaia et al. (2017) focus on a specific environment of homogeneous of degree one, concave, and continuous utility functions over finite commodities.

Our paper also relates to the literature on 'hard choices' (Levi 1986; Chang 1997; 2017). 'Hard choices' are formalised as sets of feasible alternatives over which a given choice function yields an empty choice set (Gerasimou 2018; Hees et al. 2021). This captures the idea that a choice is 'hard' when no alternatives may justifiably be chosen over others. Gerasimou (2018) and Hees et al. (2021) give internal consistency conditions on choice functions to characterise the circumstances under which 'hard choices' arise. Given that making no choice means indirectly choosing the status quo, these results bear a conceptual relation with our Tyranny of the Loser. However, unlike Gerasimou (2018) and Hees et al. (2021), we define Hard Problems directly with respect to the welfare of agents relative to a status quo, instead of indirectly through a choice function. This allows us to study solutions to Hard Problems that are not a dictatorship of the Loser. Moreover, the conditions we use to characterise our Tyranny of the Loser admit interpretations that are external to the choices made, such as fairness, and the measurability and interpersonal comparability of welfare. This is in contrast to the internal consistency conditions used by Gerasimou (2018) and Hees et al. (2021) that do not admit interpretations external to the choices themselves (see also Sen 1993).

The contextualised social welfare relations we propose are formally equivalent to the difference relations that have been used in a decision theoretic setting to characterise cardinally measurable utility functions (e.g. Krantz et al. 1971; Shapley 1975; Wakker 1988; Köbberling 2006; Gerasimou 2021). Specifically, difference relations between extended profiles are used to characterise utility difference representations, such as $(a, b) \succsim (a', b')$ if and only if $u(a) - u(b) \geq u(a') - u(b')$. These characterisations depend on a concatenation condition which requires that if a welfare improvement from a profile a to b and from b to c are respectively just as good as improvements from a' to b' and b' to c', then the overall improvement from a to c should be just as good as one from a' to c'. Our results instead use conditions on these relations that have social welfare interpretations.

Finally, the liberal principle of non-interference we use to characterise the Tyranny of the Loser for contextualised social welfare relations is due to Mariotti and Veneziani (2009). See also the subsequent literature studying liberal principles of distributive justice (e.g. Alcantud 2013; Lombardi et al. 2016; Mariotti and Veneziani 2013; 2017; 2018). Our paper extends this liberal non-interference principle to the context of Hard Problems and contextualised social welfare relations.

This paper is structured as follows. Section 2 outlines the framework used in the paper. Section 3 presents solutions to Hard Problems. Section 4 establishes the Tyranny of the Loser. Section 5 extends the core analysis to contextualised social welfare relations. Section 6 concludes.

2 Preliminaries

Let $\mathcal{N}$ be any countable set of agents. For any set $\mathcal{T}$, let T denote its cardinality. We develop here the notation for the case where N is finite, with the extension to the infinite case being straightforward.

Let $\mathbb{R}$ be the set of real numbers. We interpret each $x \in \mathbb{R}^N$ as an allocation of 'welfare' to the N agents. 'Welfare' may be understood as a share of a divisible good, utility, an index of access to primary goods, among many other things, as long as it may be represented as a real number. Let vector inequalities on $\mathbb{R}^N$ be $x \geq y$ if and only if $x_i \geq y_i$, all $i \in \mathcal{N}$, $x > y$ if and only if $x \geq y$ and $x \neq y$, and $x \gg y$ if and only if $x_i > y_i$, all $i \in \mathcal{N}$. For any set of feasible allocations of welfare, or *set of feasible alternatives*, $A \subseteq \mathbb{R}^N$, let

$$P(A) = \{a \in A \mid \text{there does not exist } b \in A \text{ with } b > a\}$$

be the *Pareto-efficient alternatives*. Let $s \in P(A)$ be a Pareto-efficient *status quo*. Denote by $\mathcal{D}$ the domain of all pairs (A, s) of sets of feasible alternatives and status quos. Let $\mathcal{W}$ and $\mathcal{L}$ form a partition of $\mathcal{N}$ into a set of *Winners* and *Losers* relative to a status quo.

Definition 2.1. A *Hard Problem* is a pair (A, s), with $A \subseteq \mathbb{R}^N$ and $s \in P(A)$, such that for all $a \in A$, $a_w \geq s_w$ for all $w \in \mathcal{W}$ and $a_\ell \leq s_\ell$ for all $\ell \in \mathcal{L}$.

In a Hard Problem, every feasible alternative entails a welfare loss to some people, despite a welfare gain to the others, relative to a Pareto-efficient status quo. See Figure 1 for an example.

We focus on Hard Problems that satisfy the following domain conditions.

H1: There exists an alternative $a \in P(A)$ such that $a \neq s$.

H2: There exists an alternative $a \in A$ such that $a \notin P(A)$.

H3: A is convex.

H1 is a non-triviality condition. If the status quo were the only Pareto-efficient alternative, then it would arguably become the natural collective choice. *H2* ensures a sufficient 'richness' in the set of feasible alternatives. Specifically, it requires that not all alternatives are Pareto efficient. This opens up the possibility to sacrifice Pareto-efficiency for the sake of a compromise. *H3* is a standard condition that admits several interpretations. One is that the set of feasible alternatives are utility profiles of agents with concave utility functions. Another interpretation is that *H3* allows us to study settings with risk. Specifically, assuming that agents form preferences over lotteries that satisfy the von Neumann-Morgenstern axioms would let us interpret each feasible alternative as distributions of von Neumann-Morgenstern expected utilities. *H3* may also be interpreted as allowing for rich enough possibilities for compromises. That is, when compromises are understood as $\lambda a + (1 - \lambda)s$ for some $\lambda \in (0, 1)$ and $a \in A \setminus \{s\}$.

Requiring *H1*, *H2*, and *H3*, while desirable on its own right, shows that our results do not depend on non-standard sets of feasible alternatives. Moreover, given that our results are 'impossibilities' in the sense that we give conditions under which dictatorships must be implemented, extending the domain by relaxing *H1*, *H2*, or *H3* is not a way of escaping these impossibilities.

We shall focus on Hard Problems where there is a single Loser. Specifically, where $\mathcal{L} := \{\ell\}$. Extending all of our results to Hard Problems where there are additional Losers is straightforward. We focus on only one Loser to emphasise that this is enough to recover our results. Having a single Loser also significantly expands the set of real life situations to which our analysis applies. Denote by $\mathcal{H}$ the domain of all Hard Problems that satisfy *H1*, *H2*, *H3*, and only have a single Loser.

We may extend Hard Problems to allow agents to be both a Winner and a Loser. That is, to only require of a Hard Problem that the status quo be Pareto efficient.

Definition 2.2. An *Extended Hard Problem* is a pair (A, s) such that $A \subseteq \mathbb{R}^N$ and $s \in P(A)$.

Extended Hard Problems are 'hard' in that it remains impossible to improve the welfare of all agents over the status quo, however each agent can in principle become a Loser as society moves away from the status quo. Denote the domain of all Extended Hard Problems that satisfy *H1*, *H2*, and *H3* by $\mathcal{E}$.

A *positive affine transformation* is a function $\tau : \mathbb{R}^N \to \mathbb{R}^N$ such that, for some real numbers $\alpha_i > 0$ and β_i, $\tau_i(x) = \alpha_i x + \beta_i$ for all $i \in \mathcal{N}$. A positive affine transformation is *unit comparable* if $\alpha_i = \alpha_j$ for all $i, j \in \mathcal{N}$. For any $A \subseteq \mathbb{R}^N$ and positive affine transformation τ, denote $\tau(A) = \{b \in \mathbb{R}^N \mid b = \tau(a) \text{ for some } a \in A\}$.

Denote $\mathbf{0} \in \mathbb{R}^N$ the zero vector. For any $A \subseteq \mathbb{R}^N$, denote its power set by 2^A, convex hull by $\text{co}\{A\}$, interior by $\text{int}(A)$, and boundary by $\partial A := A \setminus \text{int}(A)$. Let $\|\cdot\|$ be the Euclidean norm, $\overline{B}(x, \varepsilon) := \{y \in \mathbb{R}^N \mid \|x - y\| \leq \varepsilon\}$ a closed ball of radius $\varepsilon > 0$ centred at $x \in \mathbb{R}^N$, and $\text{dist}(x, A) := \inf_{a \in A}\|x - a\|$ the distance between a point $x \in \mathbb{R}^N$ and a set $A \subseteq \mathbb{R}^N$. Let $\langle \cdot, \cdot \rangle$ be the standard inner product, $a^T \in \mathbb{R}^N$ the transpose of $a \in \mathbb{R}^N$, and $Y := \{x \in \mathbb{R}^N \mid \langle a^T, x - x_0 \rangle = 0\}$ the hyperplane orthogonal to $a \in \mathbb{R}^N$ and passing through $x_0 \in \mathbb{R}^N$. Denote the closed half-spaces of a hyperplane Y by $U_Y := \{x \in \mathbb{R}^N \mid \langle a^T, x - x_0 \rangle \geq 0\}$ and $L_Y := \{x \in \mathbb{R}^N \mid \langle a^T, x - x_0 \rangle \leq 0\}$. A permutation is a bijection, $\pi : \mathcal{N} \to \mathcal{N}$, such that for all $a \in \mathbb{R}^N$, $a^\pi := (a_{\pi(i)})_{i \in \mathcal{N}}$.

3 Solutions to Hard Problems

We investigate ways of addressing Hard Problems. This is done by studying solutions understood as a choice function over the feasible alternatives in light of a status quo.

Definition 3.1. A *solution* on $\mathcal{D}$ is a function $f : \mathcal{D} \to \mathbb{R}^N$ such that $f(A, s) \in A$ for all $(A, s) \in \mathcal{D}$.

Many solutions on $\mathcal{H}$ imply some form of compromise between Winners and Losers. Take the classical utilitarian solution, $f(A, s) = \arg\max_{a \in A} \sum_{i \in \mathcal{N}} a_i$, for all $(A, s) \in \mathcal{H}$.[2] A compromise is chosen whenever the sum of utility gains to the Winners is larger than the utility losses to the Losers,

$$\sum_{\ell \in \mathcal{L}} (s_\ell - a_\ell) < \sum_{w \in \mathcal{W}} (a_w - s_w).$$

Further examples are the maximin solution that seeks to formalise the difference principle of Rawls (1971), $f(A, s) = \arg\max_{a \in A} \min_{i \in \mathcal{N}} a_i$ and dictatorships of a single agent, $f(A, s) = \arg\max_{a \in A} a_i$, for some $i \in \mathcal{N}$ the dictator.[3]

We introduce our key property on solutions.

Expansion Solidarity (EXP): For all $(A, s), (B, s) \in \mathcal{D}$, if $A \subset B$ and for all $a \in P(A) \setminus \{s\}$, there exists $b \in B$ such that $b > a$, then $f(B, s) \geq f(A, s)$.

[2] There are several extensions of the classical utilitarian solution to the infinite case. See Asheim (2010) for a survey.

[3] Additional restrictions on these examples may be imposed to ensure that they are always well-defined functions. All of our results hold under analogous conditions when we allow solutions to be correspondences.

Expansion Solidarity requires that if the set of alternatives expands in such a way that new allocations emerge which Pareto-dominate every existing alternative except for the status quo, then no one should become worse off because of this expansion. Expansion Solidarity also admits an interpretation when there is a contraction to the set of feasible alternatives in the sense that every Pareto-efficient alternative except for the status quo becomes unfeasible. The condition requires solidarity in the sense that everyone should weakly bear a welfare loss under such a contraction.

Expansion Solidarity is not simply a monotonicity property requiring that if the feasible set expands, then the new solution point should weakly dominate the old one. Consider, for example, $A := \text{co}\{(0,0),(1,-1),(0,-1)\}$, $s = (0,0)$, $f(A,s) = (1,-1)$, and $A(\delta) := \text{co}\{(0,0),(\delta,-\delta),(0,-\delta)\}$ where $\delta > 1$. It *might* sometimes be reasonable to have $f(A(\delta),s) = (\delta,-\delta)$ for δ large enough. Expansion Solidarity only considers expansions where there are Pareto improvements upon every Pareto-efficient alternative different from the status quo. This makes the condition arguably much weaker.

4 The Tyranny of the Loser

This section finds that any solution which satisfies Expansion Solidarity must be a dictatorship of the Loser where the status quo is always chosen.

Theorem 1. (Tyranny of the Loser). *A solution on $\mathcal{H}$ satisfies EXP if and only if $f(A,s) = s$ for all $(A,s) \in \mathcal{H}$.*

Proof. The 'if' part is immediate. For the 'only if' part, take $(A,s) \in \mathcal{H}$ arbitrary. We have two cases.

Case 1: $D_A := \{d \in A \mid d < s\} = \emptyset$. Suppose by contradiction that $f(A,s) \neq s$. Because (A,s) is a Hard Problem and $D_A = \emptyset$, there exists $w \in \mathcal{W}$ such that,

$$f_w(A,s) > s_w. \tag{4.1}$$

Let $e^\ell \in \mathbb{R}^N$ be such that $e^\ell_\ell = 1$ and $e^\ell_i = 0$ for all $i \neq \ell$. Let $Y^\ell := \{x \in \mathbb{R}^N \mid \langle (e^\ell)^T, x - s\rangle = 0\}$ be the hyperplane orthogonal to e^ℓ and passing through s. A is contained in L_{Y^ℓ}. (To see this, take $a \in A$ arbitrary. We have, $\langle (e^\ell)^T, a-s\rangle = a_\ell - s_\ell \leq 0$ because (A,s) is a Hard Problem.) Observe that, if $y \in Y^\ell$, then $y_\ell = s_\ell$. Denote $B^\ell := \cup_{x\in A}\overline{B}(x, \frac{\text{dist}(x,Y^\ell)}{2})$ and $Q := \{x \in \mathbb{R}^N \mid x_w \geq s_w \text{ for all } w \in \mathcal{W} \text{ and } x_\ell \leq s_\ell\}$. Define $B := B^\ell \cap Q$. See Figure 2 for an example.

(B,s) satisfies *H1* and *H2* by construction. To show that B is convex, because an arbitrary intersection of convex sets is convex, and Q is convex, it is enough to show that

B^ℓ is convex. Take $u, v \in B^\ell$ arbitrary. By construction,

$$\|u - a_u\| \leq \frac{\text{dist}(a_u, Y^\ell)}{2} \tag{4.2}$$

$$\|v - a_v\| \leq \frac{\text{dist}(a_v, Y^\ell)}{2} \tag{4.3}$$

for some $a_u, a_v \in A$. Take $\lambda \in (0, 1)$ arbitrary. We have,

$$\begin{aligned} \|\lambda u + (1-\lambda)v - (\lambda a_u + (1-\lambda)a_v)\| &\leq \|\lambda(u - a_u)\| + \|(1-\lambda)(v - a_v)\| \\ &\leq \frac{1}{2}(\lambda \text{dist}(a_u, Y^\ell) + (1-\lambda)\text{dist}(a_v, Y^\ell)) \\ &\leq \frac{1}{2}\text{dist}(\lambda a_u + (1-\lambda)a_v, Y^\ell). \end{aligned}$$

where the first inequality follows from the triangle inequality, the second inequality follows from Equations 4.2 and 4.3, and the last inequality uses that $\text{dist}(x, \partial X) : X \to \mathbb{R}$ is a concave function for all $X \subseteq \mathbb{R}^N$ convex, that the set L_{Y^ℓ} is convex, and that $Y^\ell = \partial L_{Y^\ell}$. Because A is convex, $\lambda a_u + (1-\lambda)a_v \in A$ so that $\lambda u + (1-\lambda)v \in \overline{B}(\lambda a_u + (1-\lambda)a_v, \frac{\text{dist}(\lambda a_u + (1-\lambda)a_v, Y^\ell)}{2}) \subset B^\ell$. Therefore, we conclude that B^ℓ is convex, and so is B.

Clearly, $A \subset B$. Moreover, for each $a \in P(A) \setminus \{s\}$, there exists $b \in P(B)$ such that $b > a$. Therefore, by *EXP*, $f(B, s) \geq f(A, s)$. By Equation 4.1, this means $f(B, s) \neq s$. Specifically, $f_\ell(B, s) < s_\ell$. Let $c \in \mathbb{R}$ be such that $f_\ell(B, s) < c < s_\ell$. Consider $C := A \cap \{x \in \mathbb{R}^N \mid x_\ell \geq c\}$. $(C, s) \in \mathcal{H}$ by construction. Moreover, because $C \subseteq A$, by *EXP*, $f(C, s) \leq f(B, s)$. Specifically, $f_\ell(C, s) \leq f_\ell(B, s) < c$ which contradicts $f_\ell(C, s) \geq c$. Therefore, $f(A, s) = s$.

<u>Case 2</u>: $D_A \neq \emptyset$. Therefore, there exists $d \in D_A$ and, by *H1*, there exists $a \in P(A) \setminus \{s\}$. Let $a^d \in A$ be such that $a^d = \gamma a + (1-\gamma)d$ for some $\gamma \in (0, 1)$. Consider $G := \text{co}\{a^d, s\}$. By construction, $(G, s) \in \mathcal{H}$ and $G \subset A$. Moreover, for all $g \in P(G) \setminus \{s\}$, there exists $a \in A$ such that $a > g$. To see this, because $a^d < a^s := \gamma a + (1-\gamma)s$, we have $g = \varepsilon a^d + (1-\varepsilon)s < \varepsilon a^s + (1-\varepsilon)s \in A$ for some $\varepsilon \in (0, 1)$. By Case 1, $f(G, s) = s$. It follows from *EXP* that $f(A, s) \geq f(G, s) = s$ which means $f(A, s) = s$ because $s \in P(A)$. □

The proof of Theorem 1 may straightforwardly be extended to allow for additional Losers by defining the set B to instead be given by $B := (\cap_{\ell \in \mathcal{L}} B^\ell) \cap Q$.

Theorem 1 may also be extended to hold for *Extended Hard Problems* where some people, or even everyone, may be both a Winner and a Loser.

Proposition 1. (Tyranny of the Loser for Extended Hard Problems). *A solution on $\mathcal{E}$ satisfies EXP if and only if $f(A, s) = s$ for all $(A, s) \in \mathcal{E}$.*

Proof. The 'if' part is immediate. For the 'only if' part, take $(A, s) \in \mathcal{E}$ arbitrary. By *H1*, there exists $a \in P(A)$ such that $a \neq s$. By *H2*, there exists $b \in A$ such that $b \notin P(A)$.

Because $a \in P(A)$ and $b \notin P(A)$, there exists $\gamma \in (0,1)$ such that $c := \gamma a + (1-\gamma)b \nleq s$. To see this, because $a \nleq s$, there exists $i \in \mathcal{N}$ such that $a_i > s_i$. Let $\gamma \in (0,1)$ be close enough to 1 such that $\gamma a_i + (1-\gamma)b_i > s_i$. Moreover, $c \notin P(A)$ because $b \notin P(A)$ so that there exists $b^* \in A$ such that $c = \gamma a + (1-\gamma)b < \gamma a + (1-\gamma)b^* =: c^*$. $c^* \in A$ by *H3*. Let $\mathcal{L} := \{\ell \in \mathcal{N} \mid c_\ell < s_\ell\}$. Choose $\delta \in (0,1)$ close enough to 1 such that $d := \delta c + (1-\delta)c^*$ is such that $c_\ell \leq d_\ell < s_\ell$ for all $\ell \in \mathcal{L}$. $d \in A$ by *H3*. Denote $U := \{x \in \mathbb{R}^N \mid x_i \geq s_i \text{ for all } i \in \mathcal{N} \text{ such that } c_i \geq s_i\}$ and $L := \{x \in \mathbb{R}^N \mid x_i \leq s_i \text{ for all } i \in \mathcal{N} \text{ such that } c_i < s_i\}$. Define $C := \text{co}\{c, d, s\} \cap U \cap L$. By construction, (C, s) satisfies *H1*, *H2*, and *H3*. Indeed, $d \in P(C)$ is such that $d \neq s$ and $c \in C$ is such that $c \notin P(C)$. By *H3*, $C \subseteq A$. Moreover, take $x \in C \setminus \{s\}$ arbitrary. By construction, $x = \lambda_1 c + \lambda_2 d + \lambda_3 s$ where $\sum_{i=1}^{3} \lambda_i = 1$ and $\lambda_3 < 1$ so that $\lambda_1 \neq 0$ or $\lambda_2 \neq 0$. We have, $x < \lambda_1 d + \lambda_2 c^* + \lambda_3 s =: x^*$ where $x^* \in A$ by *H3*. Using the same argument as in the proof of Theorem 1, but with $B := (\cap_{\ell \in \mathcal{L}} B^\ell) \cap Q$ because there may be additional Losers, we have $f(C, s) = s$. By *EXP*, $f(A, s) \geq f(C, s) = s$ which means $f(A, s) = s$ because $s \in P(A)$. $\square$

Theorem 1 and Proposition 1 show that any non-dictatorial solution for a Hard or Extended Hard Problem must violate the relatively weak fairness condition of Expansion Solidarity. Another way to see Theorem 1 and Proposition 1 is as impossibility results in a framework where there are usually possibilities. Specifically, Theorem 1 and Proposition 1 show that all the non-dictatorship solutions which are possible for cooperative bargaining and rationing problems disappear under Expansion Solidarity when there is just one Loser among Winners, just one Winner among Losers, or when the status quo is Pareto efficient. This is summarised in Table 1.

	$\mathcal{W} \neq \emptyset$	$\mathcal{W} = \emptyset$
$\mathcal{L} \neq \emptyset$	Hard Problems (Impossibility)	Cooperative Rationing (Possibility)
$\mathcal{L} = \emptyset$	Cooperative Bargaining (Possibility)	Extended Hard Problems when $s \in P(A)$ (Impossibility) Cooperative Bargaining when $s \notin P(A)$ (Possibility)

Table 1: Relationship between Hard and Extended Hard Problems, and Cooperative Bargaining and Rationing, with the possibility or impossibility of non-dictatorship solutions that satisfy Expansion Solidarity in brackets.

Allowing for greater information about the measurability and interpersonal comparability of welfare, as well as imposing further domain restrictions, are common ways to escape impossibilities in normative economics and social choice theory. However, Theorem 1 and Proposition 1 hold regardless of the information allowed about the measurability or interpersonal comparability of welfare. Indeed, for any positive monotonic transformation,

$\phi : \mathbb{R}^N \to \mathbb{R}^N$, we have,

$$\begin{aligned}\phi(f(A,s)) &= \phi(s) \\ &= f(\phi(A), \phi(s))\end{aligned}$$

for all $(A, s) \in \mathcal{H} \cup \mathcal{E}$ whenever the solution satisfies Expansion Solidarity. That is, the Tyranny of the Loser satisfies all informational requirements, even full measurability and comparability. It is not possible to say that Theorem 1 and Proposition 1 do not hold when enough information about the measurability and comparability of welfare is allowed.

As for further restricting the domain of Hard Problems, it is not always clear *a priori* what kind of relevant restrictions would allow possibility results to emerge. However, we are able to show that the Tyranny of the Loser does not depend on an insufficient 'richness' of the set of feasible alternatives which makes compromises difficult.

H4: If $a \in P(A)\backslash\{s\}$, then for all $\lambda \in (0,1)$, there exists $b \in P(A)$ such that $\lambda a + (1-\lambda)s < b$.

H5: For all $a \in A$ and $x \in \mathbb{R}^N$, if $x \leq a$, then $x \in A$.

H6: A is closed.

H4, *H5*, and *H6* guarantee sufficient 'richness' in the sets of feasible alternatives. Specifically, *H4* requires the set of feasible alternatives to be strictly convex above. That is, for there to always be Pareto improvements on 'convex' compromises between Winners and Losers of the form $\lambda a + (1 - \lambda)s$ for $a \in P(A) \setminus \{s\}$ and $\lambda \in (0, 1)$. This ensures that the Pareto-efficient compromises are not restricted to just being these 'convex' compromises. *H5* is the comprehensiveness condition from cooperative bargaining problems. It may be interpreted as allowing for free disposal. That is, for it to be feasible for every existing alternative to be made worse in the sense that it leads to a welfare loss for everyone. *H6* is another condition from cooperative bargaining which requires that the boundary is contained in the set of feasible alternatives. This ensures that there exist infinitely many Pareto-efficient alternatives beyond just s and the a guaranteed by *H2*.[4] Denote by $\mathcal{R}$ the domain of all Hard Problems in $\mathcal{H}$ that also satisfy *H4*, *H5*, and *H6*.

The Tyranny of the Loser persists when we restrict the domain from $\mathcal{H}$ to $\mathcal{R}$ when only unit-comparable cardinal measurability of welfare is allowed.

Unit-Comparable Cardinal Measurability (UCAR): For all $(A, s) \in \mathcal{D}$, $f(\tau(A), \tau(s)) = \tau(f(A, s))$ for all τ unit-comparable positive affine transformations.

Unit-Comparable Cardinal Measurability allows for welfare to be measured on a cardinal

[4] Proposition 2 also holds when *H5* and *H6* are replaced by *H7*: A is compact.

scale, as well as for it to be meaningful to interpersonally compare welfare units or differences. This is in a sense a 'strict lower bound' on the information required on welfare to recover possibilities when we restrict the domain from $\mathcal{H}$ to $\mathcal{R}$.

Proposition 2. (Tyranny of the Loser Under Further Domain Restrictions). *A solution on $\mathcal{R}$ satisfies EXP and UCAR if and only if $f(A, s) = s$ for all $(A, s) \in \mathcal{R}$.*

Proof. The 'if' part is immediate. For the 'only if' part, take any $(A, s) \in \mathcal{R}$. Let f be a solution on $\mathcal{R}$ satisfying *EXP* and *UCAR*. Consider a unit-comparable positive affine transformation τ given by $\tau(x) = 2x - s$. We have, $\tau(s) = s$ and $A \subset \tau(A)$, because, for all $a \in A$, $2a - s \in \tau(A)$, and $b = \frac{1}{2}a + \frac{1}{2}s \in A$ by *H3* and $\tau(b) = a$. Moreover, because $a \in P(A)$ implies $(2a - s) \in P(\tau(A))$ and $\tau(A)$ satisfies *H4*, for all $a \in P(A) \setminus \{s\}$, there exists $b \in \tau(A)$ such that $b > a$. By *EXP*, we have, $f(\tau(A), \tau(s)) \geq f(A, s)$. Moreover, by *UCAR*, we have $2f(A, s) - s \geq f(A, s)$, or equivalently, $f(A, s) \geq s$. It follows that $f(A, s) = s$ because $s \in P(A)$. $\square$

The next section extends our core analysis to contextualised social welfare relations.

5 Contextualised Social Welfare Relations

Standard social welfare relations abstract away from the context of the profiles that are evaluated. Specifically, they abstract away from history as incorporated in the status quo. However, a status quo is an essential component of a Hard Problem. We propose contextualised social welfare relations as a way to take status quos explicitly into account.[5]

A *contextualised social welfare relation* $\succsim$ over $\mathbb{R}^{2N}$ is a binary relation such that, for any *extended profile* $(a, s), (b, t) \in \mathbb{R}^{2N}$, $(a, s) \succsim (b, t)$ means that the profile a given the status quo s is at least as socially preferred to b given the status quo t. $\succ$ and $\sim$ are the asymmetric and symmetric components respectively. We focus on contextualised social welfare relations that are reflexive, s-transitive, and s-complete.[6] We call these *contextualised social welfare rankings.* We may also define an analogue to a solution in this setting.

Definition 5.1. A *social welfare solution* on $\mathcal{D}$ with respect to $\succsim$ is a correspondence $F : \mathcal{D} \to 2^{\mathbb{R}^{2N}}$ such that, for all $(A, s) \in \mathcal{D}$,

$$\begin{aligned} F(A, s) &:= \max(A \times \{s\}, \succsim) \\ &= \{(a, s) \in A \times \{s\} \mid \text{there does not exist } (b, s) \in A \times \{s\} \text{ such that } (b, s) \succ (a, s)\}. \end{aligned}$$

[5]Proposition 3 holds for standard social welfare relations under even weaker conditions. Theorem 2 does not have an analogue for standard social welfare relations.

[6]A contextualised social welfare relation is reflexive when for all $(a, s) \in \mathbb{R}^{2N}$, $(a, s) \succsim (a, s)$; s-transitive when for all $(a, s), (b, s), (c, s) \in \mathbb{R}^{2N}$, if $(a, s) \succsim (b, s)$ and $(b, s) \succsim (c, s)$, then $(a, s) \succsim (c, s)$; and s-complete when for all $(a, s), (b, s) \in \mathbb{R}^{2N}$, if $(a, s) \neq (b, s)$, then $(a, s) \succsim (b, s)$ or $(a, s) \precsim (b, s)$.

Social welfare solutions maximise contextualised social welfare rankings, which allows us to explicitly identify the full set of social rankings underlying a solution, as well as the conditions we require of these rankings. Social welfare solutions also allow us to extend solutions from being functions to being correspondences.

Expansion Solidarity may be extended to social welfare solutions.

Expansion Solidarity (EXP*): For all $(A, s), (B, s) \in \mathcal{D}$, if $A \subset B$ and for all $a \in P(A) \setminus \{s\}$, there exists $b \in B$ such that $b > a$, then for all $(x, s) \in F(A, s)$, there exists $(y(x), s) \in F(B, s)$ such that $y(x) \geq x$.

We focus on contextualised social welfare rankings that satisfy a standard efficiency condition for extended profiles with the same status quo.

s-Strong Pareto Efficiency (s-SP): For all $(a, s), (b, s) \in \mathbb{R}^{2N}$, if $a > b$, then $(a, s) \succ (b, s)$.

We also require contextualised social welfare rankings to satisfy standard regularity conditions under the same status quo.

s-Convexity (s-CONV): For all $(a, s) \in \mathbb{R}^{2N}$, the upper contour set $\{(b, s) \in \mathbb{R}^{2N} \mid (b, s) \succsim (a, s)\}$ is convex.

s-Continuity (s-CONT): For all $(a, s) \in \mathbb{R}^{2N}$, the upper and lower contour sets $\{(b, s) \in \mathbb{R}^{2N} \mid (b, s) \succsim (a, s)\}$ and $\{(b, s) \in \mathbb{R}^{2N} \mid (b, s) \precsim (a, s)\}$ are closed.

s-Convexity embodies a minimal notion of fairness in the sense that welfare losses to one agent should be compensated by welfare gains to another. s-Continuity rules out contextualised social welfare rankings based on the leximin ordering, a prominent social welfare ordering with an egalitarian interpretation (see also Roemer 1996, pp. 136–137). This may seem undesirable, but as Example 5.1 shows, a contextualised social welfare relation based on leximin is already ruled out by *EXP**.

Example 5.1. (Leximin Violates *EXP**). Let $s = (1, 0)$. Consider $A := \text{co}\{(\frac{3}{4}, \frac{1}{4} - \varepsilon), (1, 0), (\frac{3}{4}, 0)\}$ for some $\varepsilon \in (0, \frac{1}{4})$ and $B := \text{co}\{(0, 1), (1, 0), (0, 0)\}$. If F maximises the leximin ordering, then $F(A, (1, 0)) = \{((\frac{3}{4}, \frac{1}{4} - \varepsilon), (1, 0))\}$ and $F(B, (1, 0)) = \{((\frac{1}{2}, \frac{1}{2}), (1, 0))\}$, which violates *EXP**.

These conditions on contextualised social welfare rankings lead to an impossibility result.[7]

Proposition 3. *There is no contextualised social welfare ranking which satisfies s-SP, s-CONV, s-CONT, and is maximised by a social welfare solution on $\mathcal{H}$ that satisfies EXP*.*

[7]We again focus on the domain $\mathcal{H}$. Extending all the results to a domain with additional Losers, as well as with the further restrictions *H4*, *H5*, and *H6*, is straightforward.

Proof. Suppose by contradiction that there exists such a contextualised social welfare ranking $\succsim$. We show that $(s,s) \in F(A,s)$ for all $(A,s) \in \mathcal{H}$, then show that this is impossible. Take $(A,s) \in \mathcal{H}$ arbitrary. Suppose by contradiction that $(s,s) \notin F(A,s)$. Because $\succsim$ is an ordering for any fixed status quo, and satisfies *s-SP*, *s-CONV*, and *s-CONT*, the indifference sets $I(a,s) := \{(b,s) \in \mathbb{R}^{2N} \mid (a,s) \sim (b,s)\}$ are well-defined and unique for all $(a,s) \in A \times \{s\}$. Because $\{s\}$ is convex, $A \times \{s\}$ is convex by *H3*. Moreover, because $(A,s) \in \mathcal{H}$, $s \in P(A)$ so that $(s,s) \in \partial(A \times \{s\})$. By the supporting hyperplane theorem and *s-CONV*, there exists a hyperplane H passing through (s,s) such that $\{(b,s) \in \mathbb{R}^{2N} \mid (s,s) \precsim (b,s)\}$ is contained in one of the half-spaces generated by H. Denote the other half-space by L and $A_L := A \cap L$. By construction, $(s,s) \in F(A_L,s)$.

Starting from A, consider B as constructed in the proof of Theorem 1. By construction, $(B,s) \in \mathcal{H}$, $A_L \subseteq A \subset B$, and for all $a_L \in P(A_L) \setminus \{s\}$, there exists $a \in P(A)$ and $b \in B$ such that $a_L \leq a < b$. By *EXP**, there exists $(x,s) \in F(B,s)$ such that $x \geq s$. Because $s \in P(B)$, we have $(x,s) = (s,s)$. That is, $(s,s) \in F(B,s)$. Because $(s,s) \notin F(A,s)$, there exists some $(a,s) \in A \times \{s\}$ such that $(a,s) \succ (s,s)$. Because $A \subset B$, $(a,s) \in B \times \{s\}$ which contradicts $(s,s) \in F(B,s)$. Therefore, $(s,s) \in F(A,s)$ for all $(A,s) \in \mathcal{H}$.

Take any $(C,s) \in \mathcal{H}$. Consider any $y \in \mathbb{R}^N$ such that $y > s$ and $y_\ell = s_\ell$. By *s-SP*, $(y,s) \succ (s,s)$. By *s-CONT*, $\{(b,s) \in \mathbb{R}^{2N} \mid (b,s) \succ (s,s)\}$ is open so that there exists $\varepsilon > 0$ small enough such that $B((y,s),\varepsilon) \subseteq \{(b,s) \in \mathbb{R}^{2N} \mid (b,s) \succ (s,s)\}$. Define $y^\varepsilon \in B((y,s),\varepsilon)$ to be such that $y^\varepsilon_\ell := y_\ell - \frac{\varepsilon}{2}$ and $y^\varepsilon_w = y_w$ for all $w \in \mathcal{W}$. By construction $(y^\varepsilon,s) \succ (s,s)$. Define $C_\varepsilon := \text{co}(C,\{y^\varepsilon\})$. By construction, $(C_\varepsilon,s) \in \mathcal{H}$ so that $(s,s) \in F(C_\varepsilon,s)$ which contradicts $(y^\varepsilon,s) \succ (s,s)$. □

Proposition 3 shows that when we require a contextualised social welfare ranking to satisfy standard efficiency, convexity, and continuity properties, it becomes impossible to even find a social welfare solution that satisfies *EXP**. Many contextualised social welfare rankings based on prominent social welfare orderings, such as the utilitarian, prioritarian, and Nash social welfare orderings, satisfy s-Strong Pareto Efficiency, s-Convexity, and s-Continuity. Proposition 3 shows that they are all ruled out by *EXP**.

We are able to recover a possibility result for contextualised social welfare rankings by turning to a liberal non-interference principle instead of a fairness one.

s-Individual Damage Principle (s-IDP): For all $(a,s),(b,s),(a',s),(b',s) \in \mathbb{R}^{2N}$, if $(a,s) \succ (b,s)$, and

$$a'_i < a_i \text{ and } b'_i < b_i \text{ for some agent } i \in \mathcal{N} \text{ and,}$$
$$a'_j = a_j \text{ and } b'_j = b_j \text{ for every other agent } j \neq i,$$

then $(b',s) \not\succ (a',s)$ whenever $a'_i > b'_i$.

The s-Individual Damage Principle applies in settings where one agent $i \in \mathcal{N}$ suffers strict welfare losses from (a, s) and (b, s) to (a', s) and (b', s), respectively, in a way in which no one else is affected. An example is a bad individual choice that is harmless to others. The principle requires that if for whatever reason $(a, s) \succ (b, s)$, and i has higher welfare under (a', s) than (b', s), then i should not be punished any further by having $(b', s) \succ (a', s)$. Given that nobody else is affected, such a punishment is unjustified on liberal grounds. This condition is due to Mariotti and Veneziani (2009).[8]

We also maintain a minimal form of fairness. Specifically, for contextualised social welfare rankings to be such that the identity of the agents does not matter.

s-Anonymity (s-ANO): For all $(a, s), (b, s) \in \mathbb{R}^{2N}$, if there exist two agents $i, j \in \mathcal{N}$ such that $a_i = b_j$, $a_j = b_i$, $s_i = s_j$, and, for all other agents $k \notin \{i, j\}$, $a_k = b_k$ and $s_k = s_k$, then $(a, s) \sim (b, s)$.

s-Anonymity, together with s-transitivity, implies that for any two $(a, s), (b, s) \in \mathbb{R}^{2N}$ such that $b = a^\pi$, $s = s^\pi$, for any permutation π, we have that $(a, s) \sim (b, s)$.

s-Strong Pareto Efficiency may be relaxed to s-Weak Pareto Efficiency.

s-Weak Pareto Efficiency (s-WP): For all $(a, s), (b, s) \in \mathbb{R}^{2N}$, if $a \gg b$, then $(a, s) \succ (b, s)$.

Finally, here we reformulate Unit-Comparable Cardinal Measurability to hold for contextualised social welfare relations instead of focusing on solutions.

s-Unit-Comparable Cardinal Measurability (s-UCAR): For all $(a, s), (b, s) \in \mathbb{R}^{2N}$, $(a, s) \succsim (b, s)$ if and only if $(\tau(a), \tau(s)) \succsim (\tau(b), \tau(s))$ for all τ unit-comparable positive affine transformations.

s-Unit-Comparable Cardinal Measurability allows for welfare to be cardinally measurable and unit comparable over extended profiles that have the same status quo.

We find that any contextualised social welfare ranking that satisfies the s-Individual Damage Principle, s-Anonymity, s-Weak Pareto Efficiency, and s-Continuity must be a *maximin difference relation*, $\succsim_{MD}$, when only s-Unit-Comparable Cardinal Measurability is allowed. The maximin difference relation is defined as follows. For any $(a, s), (b, s) \in \mathbb{R}^{2N}$, denoting $a^\Delta := a - s$ and $b^\Delta := b - s$, let $(a, s) \succsim_{MD} (b, s)$ if and only if $\min_{i \in \mathcal{N}} a_i^\Delta \geq \min_{i \in \mathcal{N}} b_i^\Delta$.[9] Maximising the maximin difference relation recovers the Tyranny of the Loser.

Theorem 2. (Tyranny of the Loser for Contextualised Social Welfare Rankings). *A contextualised social welfare ranking satisfies s-IDP, s-WP, s-ANO, s-CONT, and s-UCAR if and only if it is the maximin difference relation. Hence, $F(A, s) = \{(s, s)\}$ for all $(A, s) \in \mathcal{H}$.*

[8]See also Mariotti and Veneziani (2013; 2017) for detailed discussions.

[9]Note that the maximin difference relation, though s-complete, is otherwise incomplete because it does not rank extended profiles that have different status quos.

Proof. For the 'if' part, it is immediate to see that $\succsim_{MD}$ is a contextualised social welfare ranking that satisfies *s-WP*, *s-ANO*, and *s-UCAR*. To see that it satisfies *s-IDP*, consider any $(a,s),(b,s),(a',s),(b',s) \in \mathbb{R}^{2N}$ such that for some $j \in \mathcal{N}$,

$$a'_j < a_j \text{ and } b'_j < b_j$$

and $a'_i = a_i$ and $b'_i = b_i$ for all $i \neq j$. If $(a,s) \succ_{MD} (b,s)$, then $\min_{i\in\mathcal{N}} a_i^{\Delta} > \min_{i\in\mathcal{N}} b_i^{\Delta}$. Because $a_j > a'_j$, $b_j > b'_j$, and $a'_j > b'_j$, we have $\min_{i\in\mathcal{N}} a_i'^{\Delta} > \min_{i\in\mathcal{N}} b_i'^{\Delta}$. That is, $(b',s) \not\succ (a',s)$.

To see that $\succsim_{MD}$ satisfies *s-CONT*, take any $(a,s) \in \mathbb{R}^{2N}$. Take any $(c,s) \in \{(x,s) \in \mathbb{R}^{2N} \mid (x,s) \prec_{MD} (a,s)\}$. We have, $\min_{i\in\mathcal{N}} a_i^{\Delta} < \min_{i\in\mathcal{N}} c_i^{\Delta}$. Let $\delta := \min_{i\in\mathcal{N}} c_i^{\Delta} - \min_{i\in\mathcal{N}} a_i^{\Delta} > 0$. Take any $(d,s) \in \mathbb{R}^{2N}$ such that $\|(c,s)-(d,s)\| < \delta$. We have, $\min_{i\in\mathcal{N}} d_i^{\Delta} > \min_{i\in\mathcal{N}} a_i^{\Delta}$ so that $B((d,s),\delta) \subseteq \{(x,s) \in \mathbb{R}^{2N} \mid (x,s) \prec_{MD} (a,s)\}$. A similar argument shows that $\{(x,s) \in \mathbb{R}^{2N} \mid (x,s) \succ_{MD} (a,s)\}$ is open.

For the 'only if' part, consider any $\succsim$ that is contextualised social welfare ranking which satisfies *s-IDP*, *s-WP*, *s-ANO*, *s-CONT*, and *s-UCAR*. By s-completeness, it is enough to show that for any $(a,s),(b,s) \in \mathbb{R}^{2N}$, $(a,s) \succ (b,s)$ implies $(a,s) \succ_{MD} (b,s)$, and $(a,s) \sim (b,s)$ implies $(a,s) \sim_{MD} (b,s)$. Take $(a,s),(b,s) \in \mathbb{R}^{2N}$ arbitrary. It is without loss of generality to consider s such that $s = s^{\pi}$ for all permutations π. To see this, by *s-UCAR*, $(a,s) \succsim (b,s)$ if and only if $(\tau(a),\tau(s)) \succsim (\tau(b),\tau(s))$ for all unit-comparable positive affine transformations. This means we may focus on the specific τ given by $\tau_i(x) = x_i + \beta_i$ where $\beta_i = \max_{k\in\mathcal{N}} s_k - s_i$ for all $i \in \mathcal{N}$ and $x \in \mathbb{R}^{2N}$. By construction, $\tau(s) = \tau^{\pi}(s)$ for all permutations π. Moreover, by *s-ANO*, s-transitivity, and noting that s is invariant to permutations, it is without loss of generality to reorder the components in a and b in ascending order so that $a_k \leq a_{k+1}$ and $b_k \leq b_{k+1}$ for all $k = 1,\dots,N-1$.

First, we show that if $(a,s) \succ_{MD} (b,s)$, then $(a,s) \succ (b,s)$. Suppose by contradiction that $(a,s) \succ_{MD} (b,s)$, but $(b,s) \succ (a,s)$. Because $(a,s) \succ_{MD} (b,s)$, we have,

$$\begin{aligned} a_k^{\Delta} &\geq \min_{m\in\mathcal{N}} a_m^{\Delta} \\ &> \min_{m\in\mathcal{N}} b_m^{\Delta} \\ &=: b_1^{\Delta} \end{aligned}$$

for all $k \in \mathcal{N}$. In particular, $a_1^{\Delta} > b_1^{\Delta}$ so that $a_1 > b_1$.

Step 1: Because $(b,s) \succ (a,s)$, by *s-WP*, $b_j \geq a_j$ for some $j \in \mathcal{N}$. Therefore, $b_1 < a_1 \leq a_j \leq b_j$. Define $(a^*,s),(b^1,s) \in \mathbb{R}^{2N}$ to be such that, for some $\gamma,\delta > 0$,

$$a_1^* := a_1 - \gamma \in (b_1,\infty) \quad \text{and} \quad b_j^1 := b_j^1 - \delta \in (a_1^*, a_j)$$

with $a_k^* = a_k$ for all $k \neq 1$ and $b_k^1 = b_k$ for all $k \neq j$. Consider π_{1j} a permutation that swaps the components 1 and j. By *s-ANO*, we have $(b^{\pi_{1j}}, s^{\pi_{1j}}) \sim (b, s) \succ (a, s)$, where,

$$a_1^* < a_1 \quad \text{and} \quad (b^1)_1^{\pi_{1j}} < b_1^{\pi_{1j}} \quad \text{and} \quad (b^1)_1^{\pi_{1j}} > a_1^*.$$

By *s-IDP*, *s*-completeness, *s-ANO*, and *s*-transitivity, $(b^1, s) \sim ((b^1)^{\pi_{1j}}, s^{\pi_{1j}}) \succsim (a^*, s)$ where $a_1^* > b_1^1$ and $a_j^* > b_j^1$.

Step 2: Let $\eta > 0$ be such that $\eta < \min_{m \in \mathcal{N}}\{a_m^* - b_m^1 \mid a_m^* > b_m^1\}$. Denote $a^1 \in \mathbb{R}^N$ to be such that $a_k^1 = a_k^* - \eta$ for all $k \in \mathcal{N}$. By *s-WP*, $(a^*, s) \succ (a^1, s)$, and by *s*-transitivity, $(b^1, s) \succ (a^1, s)$. If $a_k^1 > b_k^1$ for all $k \in \mathcal{N}$, then this contradicts *s-WP*. So $b_\ell^1 \geq a_\ell^1$ for some $\ell \notin \{1, j\}$ and $(b^1, s) \succ (a^1, s)$.

We may repeat Steps 1 and 2 finitely many times until we reach some $n \in \mathbb{N}$ such that $a_k^n > b_k^n$ for all $k \in \mathcal{N}$ which contradicts *s-WP*. Therefore, $(a, s) \succsim (b, s)$ whenever $(a, s) \succ_{MD} (b, s)$. It remains to show that $(a, s) \sim (b, s)$ is impossible.

Suppose by contradiction that $(a, s) \sim (b, s)$. Let $\varepsilon > 0$ be such that $a_1^\Delta - \varepsilon > b_1^\Delta$. Denote $a^\varepsilon \in \mathbb{R}^N$ to be such that $a_k^\varepsilon = a_k - \varepsilon$ for all $k \in \mathcal{N}$. By construction, $(a^\varepsilon, s) \succ_{MD} (b, s)$. Using the same argument as above, we have, $(a^\varepsilon, s) \succsim (b, s)$. However, by *s-WP* and *s*-transitivity, $(b, s) \sim (a, s) \succ (a^\varepsilon, s)$, a contradiction. Therefore, $(a, s) \succ (b, s)$ whenever $(a, s) \succ_{MD} (b, s)$.

Second, we show that if $(a, s) \sim_{MD} (b, s)$, then $(a, s) \sim (b, s)$. By construction, $a_1^\Delta = b_1^\Delta$. Suppose by contradiction that $(a, s) \succ (b, s)$. By *s-CONT*, we have that $(a^\zeta, s) \succ (b, s)$ where $a_k^\zeta := a_k^* - \zeta$ for all $k \in \mathcal{N}$ for some $\zeta > 0$ small enough. However, by construction, $(a^\zeta, s) \prec_{MD} (b, s)$, a contradiction. An analogous argument rules out $(a, s) \prec (b, s)$. □

Theorem 2 shows that, when only *s*-Unit-Comparable Cardinal Measurability is allowed, finding any compromise between Winners and Losers must be in tension with a liberal principle of non-interference. Either this principle or standard fairness, efficiency, and continuity properties must be violated.

6 Conclusion

So how should we make collective choices when faced with Hard Problems? This paper shows that we must sacrifice a relatively weak form of fairness when making such choices, or else face a Tyranny of the Loser that rules out any compromise between Winners and Losers. This is *even when there are a large number of Winners and just a single Loser*. An important question must therefore be answered in order to find fair compromises to Hard Problems: who should bear the welfare losses that may arise when new Pareto-improving alternatives emerge under the same status quo? This is a normative question that does not admit obvious answers. However, the only alternative to grappling with it is a complete choice paralysis. The status quo would never be abandoned.

This paper also shows that all the possible compromises to cooperative bargaining and rationing problems violate Expansion Solidarity under Hard and Extended Hard Problems. Whenever there is just one Loser among Winners, one Winner among Losers, or a Pareto-efficient status quo, all these possibilities disappear under Expansion Solidarity. There remains only the Tyranny of the Loser.

There are a number of avenues for further work. First, some Hard Problems may involve large jumps away from the status quo. Consider, for example, the establishment of universal healthcare in many countries. Studying these Hard Problems where there do not exist feasible alternatives that may be made arbitrarily close to the status quo would require an extension of the framework presented in this paper. Second, our framework is completely welfarist in that it ignores any information about the well-being of the population that cannot be represented as a real vector. Extending our framework to a non-welfarist setting would allow us to study Hard Problems in a way that takes into account things like the functionings and capabilities of people in the economy (see also Sen 1985). Third, the non-cooperative implementation of solutions to Hard Problems is an interesting question that is exogenous to our framework. Such an implementation may be through a competitive allocation as in Bogomolnaia et al. (2017), but may also be studied in a non-cooperative game setting, as has been done in the Nash rationing context in Mariotti and Wen (2021).

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A Independence of Axioms

A.1 Axioms in Proposition 2

The utilitarian solution satisfies *UCAR* but not *EXP*. An example of a social welfare relation that satisfies *EXP* but not *UCAR* is as follows.

Example A.1. Let $\boldsymbol{\gamma} \in \mathbb{R}^N$ be such that $\boldsymbol{\gamma}_\ell = \gamma > 0$ and $\boldsymbol{\gamma}_w = 0$ for all $w \in \mathcal{W}$. For all $(A, s) \in \mathcal{H}$ that also satisfies *H4*, *H5*, and *H6*, $f(A, s) = s - \boldsymbol{\gamma}$

A.2 Axioms in Proposition 3

Contextualised social welfare rankings based on the utilitarian, prioritiarian, and Nash social welfare orderings, among others, satisfy *s-SP*, *s-CONV*, and *s-CONT*, but not *EXP**. An example of a social welfare relation that satisfies *EXP**, but not *s-SP*, *s-CONV*, and *s-CONT* is as follows.

Example A.2. For all $(a, s), (b, s) \in \mathbb{R}^N$, $(a, s) \succsim (b, s)$ if and only if $a_w \leq b_w$ for some $w \in \mathcal{W}$ if $a, b \in \mathbb{Q}^N$, and $a_\ell \geq b_\ell$ otherwise.

To see that Example A.2 satisfies *EXP**, it is enough to show that $(s, s) \in F(A, s)$ and $F(A, s) \subseteq \{(d, s) \in A \times \{s\} \mid d \leq s\}$ for all $(A, s) \in \mathcal{H}$. Indeed, this would mean that for any $(B, s) \in \mathcal{H}$ such that $A \subset B$ and for all $a \in P(A) \setminus \{s\}$, there exists $b \in P(B)$ such that

$b > a$, we have that for all $(a, s) \in F(A, s)$, there exists $(s, s) \in F(B, s)$ such that $s \geq a$ as required by *EXP**.

Take $(A, s) \in \mathcal{H}$ arbitrary. If $s \in \mathbb{Q}^N$, then for all $a \in A$ such that $a \in \mathbb{Q}^N$, $s_w \leq a_w$ for all $w \in \mathcal{W}$ so that $(s, s) \succsim (a, s)$. Moreover, if $s \notin \mathbb{Q}^N$ or $a \in A$ is such that $a \notin \mathbb{Q}^N$, $s_\ell \geq a_\ell$ so that $(s, s) \succsim (a, s)$. This means $(s, s) \in F(A, s)$. Moreover, we show that if $(a, s) \succsim (s, s)$, then $a \leq s$. Suppose there exists $w \in \mathcal{W}$ such that $a_w > s_w$. Then, if $a \in \mathbb{Q}^N$, $(s, s) \succ (a, s)$. And, if $a \notin \mathbb{Q}^N$, because $a \neq s$, $a_\ell < s_\ell$ so that $(s, s) \succ (a, s)$.

$\succsim$ does not satisfy *s-SP* because take any $a, b \in \mathbb{Q}^N$ such that $a > b$. We have, $(a, s) \prec (b, s)$. $\succsim$ does not satisfy *s-CONV* because take any $a, b, c \in \mathbb{Q}^N$ such that $a, b < c$. We have that $(a, s) \succsim (c, s)$ and $(b, s) \succsim (c, s)$, but $\lambda a + (1-\lambda)b < c$ for any $\lambda \in (0, 1) \cap \mathbb{R} \setminus \mathbb{Q}$ so that $(c, s) \succ \lambda(a, s) + (1-\lambda)(b, s)$. $\succsim$ does not satisfy *s-CONT* because take any $a, b \in \mathbb{Q}^N$ such that $a > b$ so that $(b, s) \succ (a, s)$. Suppose by contradiction that $\succsim$ satisfies *s-CONT*. There exists $\varepsilon > 0$ such that $(b + \varepsilon, s) \succ (a, s)$ for all $\varepsilon \in [0, r)$ for some $r > 0$. Moreover, there exists $t \in \mathbb{R} \setminus \mathbb{Q}$ such that $r > t > 0$ and $b_i + t < a_i$ for all $i \in \mathcal{N}$. Therefore, $a_i > b_i + r$ for all $i \in \mathcal{N}$ so that $(a, s) \succ (b, s)$, a contradiction.

A.3 Axioms in Theorem 2

A contextualised social welfare ranking based on a dictatorship only violates *s-ANO*. $\succsim$ such that $(a, s) \sim (b, t)$ for all $(a, s), (b, t) \in \mathbb{R}^{2N}$ only violates *s-WP*. An example that only violates *s-IDP* and *s-WP* is as follows. For any $a \in \mathbb{R}^N$, denote $\overline{a} \in \mathbb{R}^N$ to be a permutation of a such that the components are ordered in increasing order.

Example A.3. For any $(a, s), (b, t) \in \mathbb{R}^{2N}$, set $(a, s) \succsim (b, t)$ if and only if $\overline{a}_1 \leq \overline{b}_1$.

An example that satisfies *s-WP* but violates *s-IDP* is as follows.

Example A.4. For any $(a, s), (b, t) \in \mathbb{R}^{2N}$, we have

Case 1: $s \neq t$. $\succsim$ is not defined.

Case 2: $s = t$ and $a \gg b$. Set $(a, s) \succ (b, t)$.

Case 3: $s = t$, not $a \gg b$, and $a_i \in \mathbb{R} \setminus \mathbb{Q}$ and $b_i \in \mathbb{Q}$ for all $i \in \mathcal{N}$. $(a, s) \succ (b, t)$.

Case 4: $s = t$, not $a \gg b$, and $a_i \in \mathbb{Q}$ for all $i \in \mathcal{I}$ for some non-empty $\mathcal{I} \subset \mathcal{N}$, $a_j \in \mathbb{R} \setminus \mathbb{Q}$ for all $j \in \mathcal{J} := \mathcal{N} \setminus \mathcal{I}$, and $b_i \in \mathbb{Q}$ or $b_i \in \mathbb{R} \setminus \mathbb{Q}$ for all $i \in \mathcal{N}$. $(a, s) \succ (b, t)$.

Case 5: Not Cases 1 to 4. $(a, s) \sim (b, t)$.

A contextualised social welfare ranking based on leximin only violates *s-CONT* and *s-UCAR*. $\succsim$ such that $(a, s) \succsim (b, s)$ if and only if $\min_{i \in \mathcal{N}} a_i \geq \min_{i \in \mathcal{N}} b_i$ violates *s-UCAR* but not *s-CONT*. $\succsim$ such that $(a, s) \succsim (b, s)$ if and only if $g(a, s) \geq g(b, s)$ where

$$g(a, s) = \begin{cases} \frac{\prod_{i \in \mathcal{N} \setminus \{j\}}(a_i - s_i)}{a_j - s_j} & \text{if } a_j - s_i \neq 0 \\ \gamma > 1 & \text{otherwise} \end{cases}$$

for some $j \in \mathcal{N}$ violates s-*CONT* but not s-*UCAR*. To see that $\succsim$ violates s-*CONT* consider, for example, $a := (1,1)$, $b_n := (1, \frac{1}{n})$, and $s = \mathbf{0}$. We have, $(a, \mathbf{0}) \succ (b_n, \mathbf{0})$ for all $n \geq 1$, but $b_n \to b := (1,0)$ is such that $(b, \mathbf{0}) \succ (a, \mathbf{0})$.

B Figures

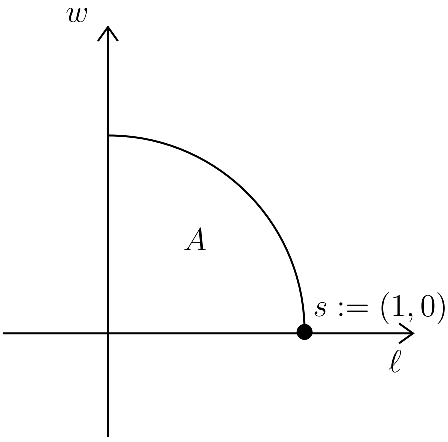

Figure 1: An example of a Hard Problem where ℓ is the Loser, w is the Winner, A is the fully positive quadrant of $\{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$, and $s := (1,0)$.

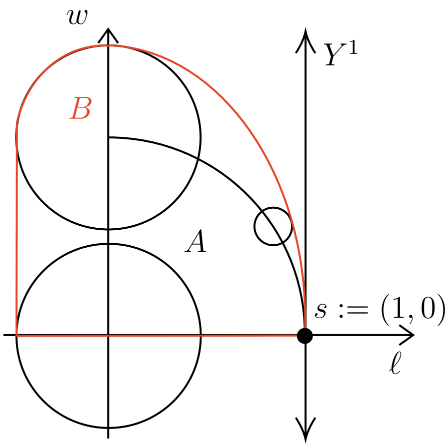

Figure 2: An example of B where ℓ is the Loser, w is the Winner, A is the fully positive quadrant of $\{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$, and $s := (1,0)$.

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