Quantum Field theories, Quivers and Word combinatorics

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"Quivers as Calculators : Counting, correlators and Riemann surfaces," arxiv:1301.1980, J. Pasukonis, S. Ramgoolam

"Quivers, Words and Fundamentals," arxiv:1412.5991, P. Mattioli and S.Ramgoolam

+ REFS therein

INTRODUCTION

Quantum Field theory (QFT) is the mathematical structure underlying the standard model of particle physics.

In any quantum system the dynamical variable is subject to quantum (Heisenberg) uncertainty relations. In quantum mechanics of particle in one dimension the position $x(t)$ is subject to such uncertainty.

In field theory, the dynamical variable is a field $\phi(x_1, x_2, x_3, t) = \phi(x_\mu, t)$.

Observable quantities we compute in QFT are probabilities of out-particles given some in-particles (such as at the LHC) with specified momenta.

Getting at these probabilistic observables, we have to compute expectation values

$$\langle \phi(x_\mu^{(1)}, t^{(1)}) \phi(x_\mu^{(2)}, t^{(2)}) \cdots \phi(x_\mu^{(n)}, t^{(n)}) \rangle$$

These expectation values are obtained from path integrals

$$\langle \mathcal{O}_1 \mathcal{O}_2 \cdots \mathcal{O}_n \rangle = \int d\phi \; \mathcal{O}_1 \mathcal{O}_2 \cdots \mathcal{O}_n e^{-S(\phi)}$$

These are computed in perturbation theory

$$S = S_0 + \delta S$$

The $\mathcal{O}_i$ can be polynomials in the elementary fields (one field for each particle type) - composite fields.

String theory is a quantum theory for particles, including gravitons. Naturally lives in 10 dimensions: reduces to quantum field theory at low energies.
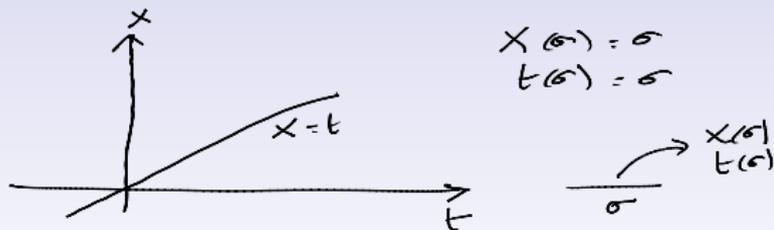
Particles in space-time can be described in terms of coordinate functions on one-dimensional word-lines.

For strings : world-line becomes two-dimensional world-surface.

particles -> strings -> branes.

For p-branes, we have $p + 1$ dimensional world-volumes.

For p-branes in D dimensions : we have D-coordinate fields which are functions of $p + 1$-dimensional world-volume.

$$X(\sigma) = \sigma$$
$$t(\sigma) = \sigma$$

$$\xrightarrow{\hspace{1cm}} \quad X(\sigma)$$
$$\qquad\qquad t(\sigma)$$
$$\sigma$$

- Map from worldline to space-time
- in D-spacetime dims:
$$X^0, X^1 \cdots X^{D-1}$$
- particle $\longrightarrow$ string
$$\Rightarrow \quad \sigma \longrightarrow \{\sigma_1, \sigma_2\}$$

$$X^\mu(\sigma_1, \sigma_1) \quad : \quad D\text{-spacetime}$$
$$\underbrace{\hspace{2cm}} \qquad \text{fields in 2 dims}$$

- p-brane:
$$\Rightarrow X^r(\sigma_1, \sigma_2, \cdots \sigma_{p+1})$$
$$\underbrace{\hspace{4cm}}$$

String theory contains extended objects called branes. One way to realize (3+1)-dimensional QFTs in string theory is to consider (3+1)-dimensional branes.

These (3+1)-dimensional branes have quantum fields corresponding to the transverse geometry E.g. one field for each transverse dimension.
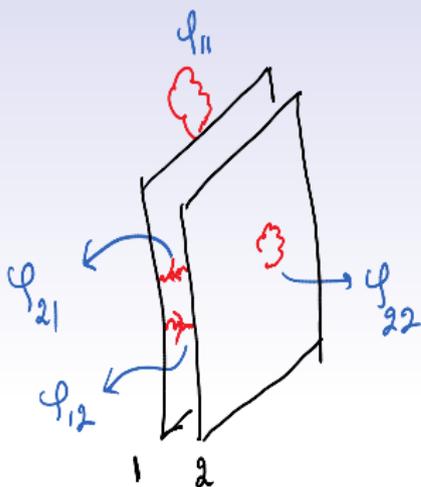
Brane fields and transverse geometry

2-brane in 3D :

$$X_3 (x_1, x_2, t)$$
$$\shortparallel$$
$$\varphi_3 (x_1, x_2, t)$$

$$\xrightarrow{\quad} x_3$$

3-brane in 10 D :

$$\begin{cases} \varphi_4 (x_1, x_2, x_3, t) \\ \varphi_5 (x_1, x_2, x_3, t) \\ \vdots \\ \varphi_9 (x_1, x_2, x_3, t) \end{cases} \quad \begin{matrix} \text{six} \\ \text{transverse} \\ \text{fields} \end{matrix}$$

Matrix quantum fields from strings.

φ_{11}

$\varphi(x,t)$

φ_{21}

φ_{22}

φ_{12}

1   2

$$\begin{pmatrix} \varphi_{11} & \varphi_{12} \\ \varphi_{21} & \varphi_{22} \end{pmatrix}$$

2-copies of brane ⟹ 2×2 Matrix transverse field

N-copies of brane ⟹ N×N Matrix transverse field

When we have multiple branches, say N of them, the transverse scalars become matrix-scalars.

$$\phi(x_\mu, t) \to \phi^i_j(x_\mu, t)$$

where i, j range in $\{1, \cdots, N\}$.

The theories are symmetric under unitary group ($U(N)$) transformations of the matrix fields. The transformations (called gauge transformations) are

$$\phi \to U\phi U^\dagger$$

The unitary group symmetries U are $N \times N$ matrices obeying $UU^\dagger = 1$

For a large class of transverse geometries, the gauge theory of branes is a has a product group $U(N_1) \times U(N_2) \times \cdots U(N_n)$ as symmetry, along with a collection of matrix fields.

The symmetry-group and matter content is parametrised by a directed graph called a Quiver.

A Quiver is a directed graph : a collection of nodes, with edges joining these nodes. Each edge has an orientation.

Each node labelled by $a \in \{1, \cdots, n\}$ corresponds to a $U(N_a)$ factor in the product gauge group.

Each edge corresponds to a matter field $(\Phi_{ab})^{i_a}_{i_b}$ which transforms as

$$\Phi_{ab} \rightarrow U_a \Phi_b U_b^\dagger$$

These fields- each transforming under a pair of $U(N_a)$ group factors are called bi-fundamental fields.
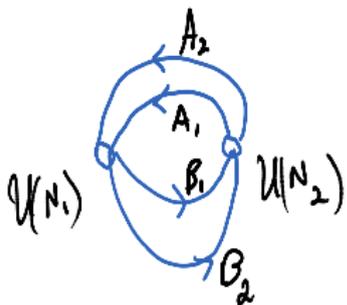
$$: \quad Z \to U Z U^\dagger$$

$$Y \to U Y U^\dagger$$
$$Z \to U Z U^\dagger$$

$$A_i \to U_2 A_i U_1^\dagger$$

$$B_i \to U_1 B_i U_2^\dagger$$

The Quiver gauge theory has a product group as gauge symmetry, and matter fields corresponding to the edges.

In a gauge theory, the observables of interest are invariant under the action of the group, i.e gauge invariant observables.

This leads to the problem of enumerating polynomials in the matter fields which are invariant under the action of the gauge group.

IN the limit where the N_a are larger than the degree of the polynomials, there is an elegant solution to the enumeration problem.

The solution is found by using the mathematical connections between unitary groups and permutation groups.

$$U(N) \longleftrightarrow \bigoplus_{k=0}^{\infty} \mathbb{C}(S_k)$$

I will describe this solution.

When we inspected the result of the counting problem, we found a connection to word counting.
Suppose we have a set of letters a_1, a_2 and another set b_1, b_2,
Consider finite sequences of a's and b's. Impose

$$a_1 a_2 = a_2 a_1$$
$$b_1 b_2 = b_2 b_1$$

But a, b do not commute.
Subject to these rules, specify number of a_1, a_2, b_1, b_2 and count the sequences (or words). E.g. If we specify these to be $(1, 1, 0, 0)$ we have one word :

$$a_1 a_2 = a_2 a_1$$

If we specify $(1, 0, 1, 0)$:

$$a_1 b_1, b_1 a_1$$

we have two words.

These words, defined by partial commutation relations, were studied by mathematicians Cartier and Foatai in the sixties.

There is a product on the set of words. Product in the monoid is concatenation. It is associative. Identity is empty word.

In the context of counting gauge invariant observables in Quiver gauge theories, we will encounter monoids where the letters correspond to SIMPLE loops in the graph.

OUTLINE

- The Quiver gauge theory counting problem : Examples and generating function for a general Quiver.
- Relation to Underlying graph-based word problems.
- The counting formulae at finite N : Littlewood Richardson coefficients.
- Conclusions/Discussions.

Simplest case is one-node, One Edge Quiver.
The Gauge theory problem is to count $U(N)$ polynomial
invariants in one-matrix Z which transforms as

$$Z \to UZU^\dagger$$

Traces are invariant :

$$\mathrm{tr}Z^k$$

And products of traces.

For degree 3, for example :

$$\mathrm{tr}Z^3 \to 3 = 3$$
$$\mathrm{tr}Z^2\mathrm{tr}Z \to 3 = 2 + 1$$
$$(\mathrm{tr}Z)^3 \to 3 = 1 + 1 + 1$$

for general n, we have the number of partitions of n - denoted $p(n)$.

For $n < N$ this always works.

At $n > N$, there are relations between these traces and multi-traces. The trZ^{N+1} can be written in terms of products of lower traces by the Cayley-Hamilton theorem.

$$\mathcal{Z}(n, N) = p(n) \equiv \mathcal{Z}(n) \text{ for } n < N$$

A generating function for $p(n)$ is

$$\mathcal{Z}(z) = \sum_{n=0}^{\infty} p(n)z^n$$

It is known that

$$\mathcal{Z}(z) = \prod_{i=1}^{\infty} \frac{1}{(1 - z^i)}$$

Example : Consider $n = 3$.

$$\frac{1}{(1 - z)(1 - z^2)(1 - z^3)\cdots}$$
$$= (1 + z + z^2 + z^3 + \cdots)(1 + z^2 + z^4 \cdots)(1 + z^3 + z^6 + \cdots)$$
$$= (1 + z + z^2 + z^3)(1 + z^2)(1 + z^3) + \cdots$$
$$= 1 + z + 2z^2 + 3z^3 + \cdots$$

For the quiver with one node and 2 edges, we have

$$Z \to UZU^\dagger$$
$$Y \to UYU^\dagger$$

The invariants are traces, e.g. for $n_x = 2, n_y = 2$

$$(\mathrm{tr}Y)^2(\mathrm{tr}Z)^2 \ , \ \mathrm{tr}(Y^2)(\mathrm{tr}Z)^2 \ , \ (\mathrm{tr}Y)^2\mathrm{tr}(Z^2) \ , \ \mathrm{tr}Z^2\mathrm{tr}Y^2$$
$$\mathrm{tr}(YZ)\mathrm{tr}YZ) \ , \ \mathrm{tr}YZ\mathrm{tr}Y\mathrm{tr}Z$$
$$\mathrm{tr}(Y^2Z)\mathrm{tr}Z \ , \ \mathrm{tr}Y\mathrm{tr}YZ^2$$
$$\mathrm{tr}Y^2Z^2 \ , \ \mathrm{tr}YZYZ$$

The counting result is $\mathcal{Z}(2,2) = 10$. In general
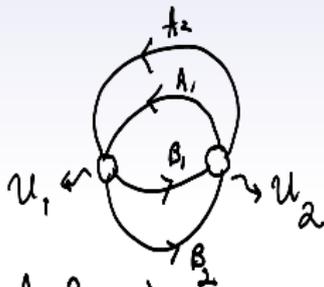
$$\mathcal{Z}(y,z) = \sum_{n,m} \mathcal{Z}(n,m)y^n z^m$$

The generating function is found to be

$$\mathcal{Z}(y,z) = \prod_{i=1}^{\infty} \frac{1}{(1 - z^i - y^i)}$$

$$A_i \rightarrow \mathcal{U}_2 A_i \mathcal{U}_i^\dagger$$

$$B_i \rightarrow \mathcal{U}_1 B_i \mathcal{U}_2^\dagger$$

Examples:

$$\mathrm{tr}\left(A_{i_1} B_{j_1} A_{i_2} B_{j_2}^{-1}\right)$$

Now we specify the numbers m_1, m_2 of A_1, A_2 and n_1, n_2 of B_1, B_2 and count $\mathcal{Z}(m_1, m_2, n_1, n_2)$.

We ask what is the generating function

$$\mathcal{Z}(a_1, a_2, b_1, b_2) = \sum_{m_1, m_2, n_1, n_2} \mathcal{Z}(m_1, m_2, n_1, n_2) a_1^{m_1} a_2^{m_2} b_1^{n_1} b_2^{n_2}$$
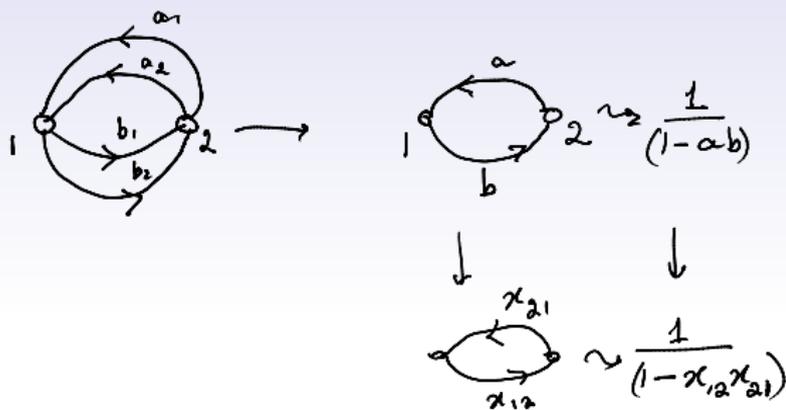
The answer is

$$\mathcal{Z}(a_1, a_2, b_1, b_2) = \prod_{i=1}^{\infty} \frac{1}{(1 - a_1^i b_1^i - a_1^i b_2^i - a_2^i b_1^i - a_2^i b_2^i)}$$
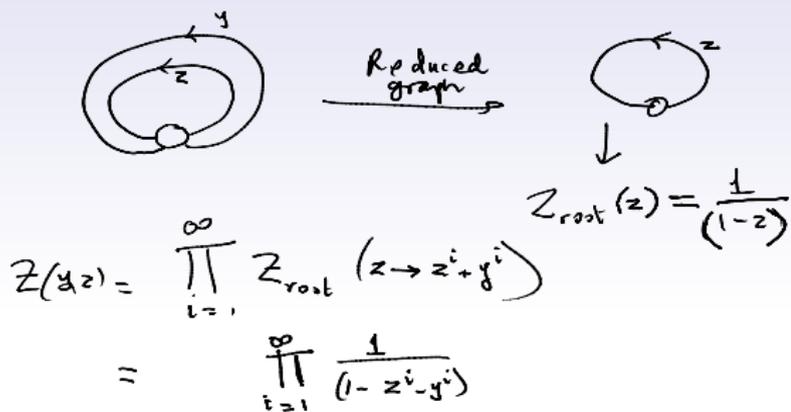
Let us re-write this as

$$
\begin{aligned}
\mathcal{Z} &= \prod_i \frac{1}{(1 - a_1^i b_1^i - a_1^i b_2^i - a_2^i b_1^i - a_2^i b_2^i)} \\
&= \prod_i \frac{1}{(1 - (a_1^i + a_2^i)(b_1^i + b_2^i))}
\end{aligned}
$$

$$
\mathcal{Z}_{root}(a, b) = \frac{1}{(1 - ab)}
$$

$$
\mathcal{Z}(a_1, a_2, b_1, b_2) = \prod_i \mathcal{Z}_{root}(a \to (a_1^i + a_2^i), b \to (b_1^i + b_2^i))
$$

The root function can be naturally associated with a reduced graph.

The 2-edge-1-node case : Reduction, root function, infinite product.

For a general quiver, define a reduced graph by collapsing multiple edges with fixed start and end-points, to a single edge.

So we have a single variable for each pair of nodes, labelled by integers (a, b), which range from 1 to n - the number of nodes in the graph.

The reduced graph can be associated to a matrix X (of size $n \times n$) whose entries are x_{ab} if there is an edge from a to b and zero otherwise.
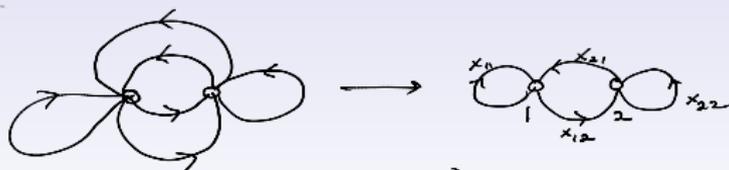
This is called the **weighted adjacency matrix** of the reduced graph.

The root function is

$$\mathcal{Z}_{root}(x_{ab}) = \frac{1}{det(1-X)}$$

The quiver gauge theory generating function, depends on variables $x_{ab;\alpha_{ab}}$ where α_{ab} runs over the number of edges going from a to b.

$$\mathcal{Z}(x_{ab;\alpha_{ab}}) = \prod_{i=1}^{\infty} \mathcal{Z}(x_{ab} \to \sum_{\alpha_{ab}} x^i_{ab;\alpha_{ab}})$$

$$\det \left(\mathbb{1} - \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \right) = (1 - x_{11})(1 - x_{22}) - x_{12} x_{21}$$

$$Z\left(x_{11}, x_{22}, x_{12,1}, x_{12,2}, x_{21,1}, x_{21,2} \right)$$

$$= \prod_{i=1}^{\infty} \frac{1}{(1 - x_{11}^i)(1 - x_{22}^i) - (x_{12,1}^i + x_{12,2}^i)(x_{21,1}^i + x_{21,2}^i)}$$

Application of the determinant formula to a more complicated 2-node quiver.

PART II : The hidden word problem

It turns out : The expansion of the inverse determinant itself has only positive coefficients. Suggests the root function itself is counting something. What can it be ?

Consider the 1-node 2-edge quiver.

$$\mathcal{Z}_{root} = \frac{1}{(1 - x - y)}$$

$$\mathcal{Z}_{root} \quad = 1 + \sum_{n=1}^{\infty}(x + y)^n$$
$$= 1 + \sum_{n=1}^{\infty}\sum_{n_1=0}^{n} x^{n_1} y^{n-n_1} \binom{n}{n_1}$$

This has a word-counting interpretation. Take a language with two letters $\hat{x}, \hat{y}$ which do not commute. Consider the number of words with n_1 copies of $\hat{x}$ and $(n - n_1)$ copies of $\hat{y}$. The number of these words is the binomial coefficient.
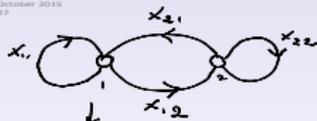
There are letters $\hat{y}_1, \hat{y}_2, \hat{y}_{12}$ for each **simple** loop in this reduced quiver.

Two letters commute if they do not share a node. These words form a monoid. Product is concatenation.

This story is very general.

For any *n*-node reduced quiver, there is a monoid (called a trace monoid). The generators correspond to simple loops.

Simple loops are loops that visit any node no more than once, and traverse every edge no more than once.

The trace monoid has a letter for each simple loop. They commute if they do not share a node.

The root function has an expansion in positive powers of the simple-loop variables y_s, with positive coefficients.

These coefficients are the counting functions for the words, subject to the partial commutation relations.

This monoid features in the work of Cartier and Foata.

The formulation above is in terms of simple closed loops. There is also a word formulation in terms of open-edges.

In the 2-node case,

$$y_1^{n_1} y_{12}^{n_{12}} y_2^{n_2} = x_{11}^{n_{11}} (x_{12} x_{21})^{n_{12}} x_{22}^{n_{22}}$$

In the open-edge word formulation, we have letters :

$$x_{11} \rightarrow \hat{x}_{11}$$
$$x_{12} \rightarrow \hat{x}_{12}$$
$$x_{21} \rightarrow \hat{x}_{21}$$
$$x_{22} \rightarrow \hat{x}_{22}$$

Consider words made from these letters, subject to partial commutation relations. $\hat{x}_{1a}$ and $\hat{x}_{2b}$ always commute for any a, b.

Letters corresponding to edges with distinct starting points commute.

Letters for distinct edges with the same starting point do not commute

$$\hat{x}_{11}\hat{x}_{12} \neq \hat{x}_{12}\hat{x}_{11}$$
$$\hat{x}_{11}\hat{x}_{21} = \hat{x}_{21}\hat{x}_{11}$$
$$\hat{x}_{11}\hat{x}_{22} = \hat{x}_{22}\hat{x}_{11}$$
$$\hat{x}_{12}\hat{x}_{21} = \hat{x}_{21}\hat{x}_{12}$$
$$\hat{x}_{21}\hat{x}_{22} \neq \hat{x}_{22}\hat{x}_{21}$$

Subject to these rules, count the words for specified numbers of $\hat{x}_{ab}$. The resulting numbers are the coefficients from the expansion of

$$\frac{1}{\det(1-X)}$$

The equivalence between the open-edge word counting and the closed-edge word counting was proved by Cartier and Foata.

Summary of the talk : The root of the quiver gauge theory counting problem is the quiver word counting problem.

The central pun of the talk I gather - mostly from Wikiedia - that these trace monoids are used to model "computational processes" in computer science. The letters are events(computational steps) and the words are processes. The length of a word is "parallel execution time"

Commuting letters are computations that can be performed simultaneously or in any relative order.

IN this usage of trace, the word trace means a history of events which is the process.

Nothing to do with matrix traces.
This talk - Count the words in the trace monoid with a generating function. Take infinite products with appropriate substitutions. The result counts matrix traces in quiver gauge theories !!

Finite N counting : Young Diagrams and LR coefficients.

Questions :

1. The asymptotics of $p(n)$ were worked out by Hardy and Ramanujan. Have lots of applications in string theory.

2. The 2-matrix case

$$\prod_i \frac{1}{(1 - x^i - y^i)}$$

Asymptotics are of interest in string theory. Are they also of interest in computer science ?

3. Finite N counting with LR coefficients are related to computational complexity problems .... finite N quiver gauge theory counting could lead to further common mathematical structures.

4. Counting is the first step - Next is correlators ... role of trace monoids there ?