4D Quiver Gauge theory combinatorics and 2D TFTs

Sanjaye Ramgoolam

Queen Mary, University of London

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New trends in Quantum integrability

“Quivers as Calculators : Counting, correlators and Riemann surfaces,” arxiv:1301.1980, J. Pasukonis, S. Ramgoolam
“Quivers, fundamentals, ... ‘’ P. Mattioli, S. Ramgoolam (to appear)

Introduction and Summary

4D gauge theory ($U(N)$ and $\prod_a U(N_a)$ groups) problems – counting and correlators of local operators in the free field limit – theories associated with Quivers (directed graphs) -

2D gauge theory (with S_n gauge groups) - topological lattice gauge theory, with defect observables associated with subgroups $\prod_i S_{n_i}$ - on Riemann surface obtained by thickening the quiver. n is related to the dimension of the local operators. For a given 4D theory, we need all n.

1D Quiver diagrammatics - quiver decorated with S_n data - is by itself a powerful tool. Finite N information.

OUTLINE

Part 1 : 4D theories - examples and motivations

Introduce some examples of the 4D gauge theories and motivate the study of these local operators.

- AdS/CFT and branes in dual AdS background.

- SUSY gauge theories, chiral ring

- Will work in the free limit - e.g. $g^2_{YM} = 0$ in $N = 4$ SYM. More generally, chiral ring with superpotential switched off.

Motivations for studying the free fixed point :

- non-renormalization theorems for some correlators

- a stringy regime of AdS/CFT - supergravity is not valid. Dual geometry should be constructed from the combinatoric data of the gauge theory.

- A point of enhanced symmetry and enhanced chiral ring.

- Contains information about the weakly coupled chiral ring - which is obtained by imposing super-potential relations on the space of gauge invariants ; or for more detailed information, solving a Hamiltonian acting on the ring of gauge invariants.

OUTLINE

Part 2 : 2d lattice TFT - Symmetric groups, subgroups, defects.

- Introduce the 2d lattice gauge theories and defect observables.
- 2d TFTs : counting and correlators of the 4d CFTs at large N.
- Generating functions for the counting at large N.

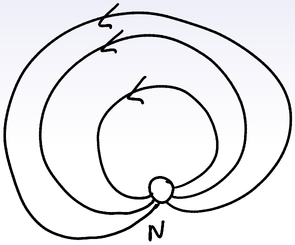
OUTLINE

Part 3 : Quiver - as 1D calculator

- Finite N counting with decorated Quiver.
- Orthogonal basis of operators and Quiver characters.
- Chiral ring structure constants.

Part 1 : Examples

Simplest theory of interest is $U(N)$ gauge theory, with $\mathcal{N} = 4$ supersymmetry. As an $\mathcal{N} = 1$ theory, it has 3 chiral multiplets in the adjoint representation ($\rightarrow$ 3 complex matrix scalars)



Dual to string theory on $AdS_5 \times S^5$ by AdS/CFT. Half-BPS (maximally super-symmetric sector) reduces to a single arrow – Contains dynamics of gravitons and super-symmetric branes (giant gravitons).

$ADS_5 \times S^5 \leftrightarrow$ CFT : $N = 4$ SYM $U(N)$ gauge group on $R^{3,1}$

Radial quantization in (euclidean) CFT side :

$ADS_5 \times S^5 \leftrightarrow$ CFT : $N = 4$ SYM $U(N)$ gauge group on $R^{3,1}$

Radial quantization in (euclidean) CFT side :

Time is radius
Energy is scaling dimension Δ.

Local operators e.g. $tr(F^2)$, TrX_a^n
correspond to quantum states.

Part 1 : 4D theories

Half-BPS states are built from matrix $Z = X_1 + iX_2$. Has $\Delta = 1$. Generate short representations of supersymmetry, which respect powerful non-renormalization theorems.

Holomorphic gauge invariant states :

$$\Delta = 1 \quad : \quad \mathrm{tr}\, Z$$

$$\Delta = 2 \quad : \quad \mathrm{tr}\, Z^2, \mathrm{tr}\, Z \mathrm{tr}\, Z$$

$$\Delta = 3 \quad : \quad \mathrm{tr}\, Z^3, \mathrm{tr}\, Z^2 \mathrm{tr}\, Z, (\mathrm{tr}\, Z)^3$$

For $\Delta = n$, number of states is

$p(n)$ = number of partitions of n

The number $p(n)$ is also the number of irreps of S_n and the number of conjugacy lasses.

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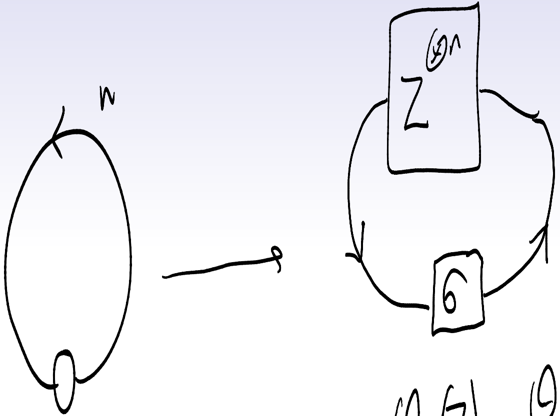
To see S_n – Any observable built from n copies of Z can be constructed by using a permutation.

$$\mathcal{O}_\sigma = Z^{i_1}_{i_{\sigma(1)}} Z^{i_2}_{i_{\sigma(2)}} \cdots Z^{i_n}_{i_{\sigma(n)}}$$

All indices contracted, but lower can be a permutation of upper indices.

$$\mathcal{O}_\sigma = Z^{i_1}_{j_1} Z^{i_2}_{j_2} \cdots Z^{i_n}_{j_n} \quad \delta^{j_1}_{i_{\sigma(1)}} \cdots \delta^{j_n}_{i_{\sigma(n)}}$$

Part 1 : 4D theories



$$\mathcal{O}_{\sigma}(z) = \mathcal{O}_{\gamma\sigma\gamma^{-1}}(z)$$

Conjugacy classes are Cycle structures

For $n = 3$, permutations have 3 possible cycle structures.

$$(123), (132)$$

$$(12)(3), (13)(2), (23)(1)$$

$$(1)(2)(3)$$

Hence 3 operators we saw.

Part 1 : 4D theories

More generally - in the eighth-BPS sector - we are interested in classification/correlators of the local operators made from X, Y, Z.

Viewed as an $\mathcal{N} = 1$ theory, this sector forms the chiral ring.

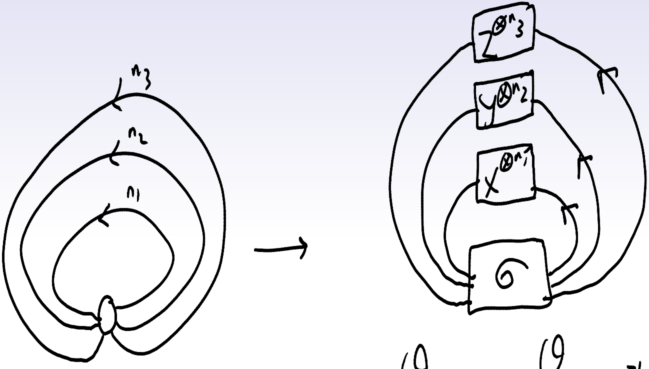
Away from the free limit, we can treat the X, Y, Z as commuting matrices, and get a spectrum of local operators in correspondence with functions on $S^N(\mathbb{C}^3)$ - the symmetric product.

Part 1 : 4D theories

This is expected since $\mathcal{N} = 4$ SYM arises from coincident 3-branes with a transverse $\mathbb{C}^3$.

At zero coupling, we cannot treat the X, Y, Z as commuting, and the chiral ring - or spectrum of eight-BPS operators - is enhanced compared to nonzero coupling.

Part 1 : 4D theories

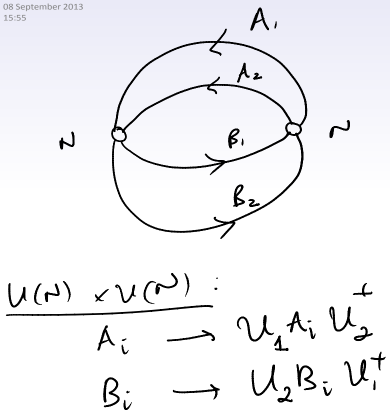


$$\mathcal{O}_{\sigma} = \mathcal{O}_{\gamma \sigma \gamma^{-1}}$$

$$\gamma \in S_{n_1} \times S_{n_2} \times S_{n_3}$$

$$\sigma \in S_{n_1+n_2+n_3}$$

Conifold Theory :



Specify n_1, n_2, m_1, m_2, numbers of A_1, A_2, B_1, B_2, and want to count holomorphic gauge invariants.

Part 1 : 4D theories

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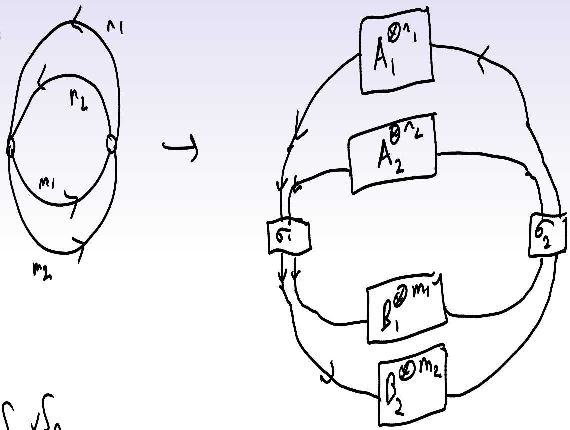


$\gamma_1 \in S_{n_1} \times S_{n_2}$

$\gamma_2 \in S_{m_1} \times S_{m_2}$

$$= \mathcal{O}_{(\sigma_1, \sigma_2)} \sim \mathcal{O}_{(\gamma_1 \sigma_1 \gamma_2^{-1}, \gamma_2 \sigma_2 \gamma_1^{-1})}$$

Part 1 : 4D theories

Having specified (m_1, m_2, n_1, n_2) we want to know the number of invariants under the $U(N) \times U(N)$ action $N(m_1, m_2, n_1, n_2)$

Counting is simpler when $m_1 + m_2 = n_1 + n_2 \leq N$. In that case, we can get a nice generating function - via 2d TFT.

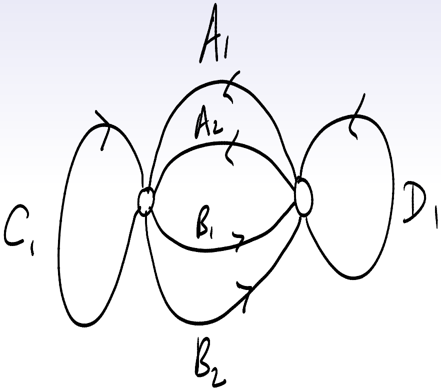
Also want to know about the matrix of 2-point functions :

$$< \mathcal{O}_\alpha(A_1, A_2, B_1, B_2) \mathcal{O}^\dagger_\beta(A_1, A_2, B_1, B_2) >$$
$$\sim \frac{M_{\alpha\beta}}{|x_1 - x_2|^{2(n_1+n_2+m_1+m_2)}}$$

The quiver diagrammatic methods produce a diagonal basis for this matrix.

Part 1 : 4D theories

$\mathbb{C}^3/Z_2$



Part 2 : 2D TFT from lattice gauge theory, 4D large N, generating functions

Edges $\rightarrow$ group elements $\sigma_{ij} \in G = S_n$

σ_P : product of group elements around plaquette.

Partition function Z :

$$Z = \sum_{\{\sigma_{ij}\}} \prod_P Z(\sigma_P)$$

Plaquette weight invariant under conjugation e.g trace in some representation.

Take the group $G = S_n$ for some integer n.

Symmetric Group of $n!$ rearrangements of $\{1, 2, \cdots, n\}$.

Plaquette action :

$$\begin{aligned} Z_P(\sigma_P) &= \delta(\sigma_P) \\ \delta(\sigma) &= 1 \text{ if } \sigma = 1 \\ &= 0 \quad \text{otherwise} \end{aligned}$$

Partition function :

$$Z = \frac{1}{n!^V} \sum_{\{\sigma_{ij}\}} \prod_P Z_P(\sigma_P)$$

This simple action is topological. Partition function is invariant under refinement of the lattice.

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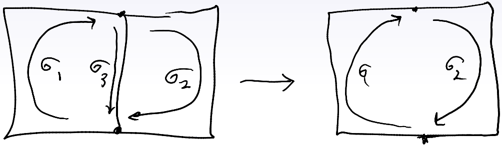


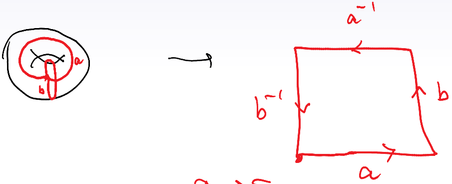
$$\sum_{\sigma_3} \delta(\sigma_1 \sigma_3)\, \delta(\sigma_3^{-1} \sigma_2) \longrightarrow \delta(\sigma_1 \sigma_2)$$

integrating out an edge $\longrightarrow$ Plaquette weight of new Plaquette

The partition function – for a genus G surface– is

$$Z_G = \frac{1}{n!} \sum_{s_1, t_2, \cdots, s_G, t_G \in S_n} \delta(s_1 t_1 s_1^{-1} t_1^{-1} s_2 t_2 s_2^{-1} t_2^{-1} \cdots s_G t_G s_G^{-1} t_G^{-1})$$

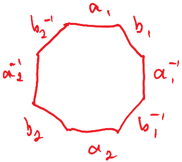
$G = 1$



$a \to s_1$
$b \to t_1$

$\sigma_p = s_1 t_1 s_1^{-1} t_1^{-1}$

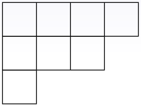
$G = 2$



$\longrightarrow \sigma_p = (s_1 t_1 s_1^{-1} t_1^{-1} s_2 t_2 s_2^{-1} t_2^{-1})$

Part 2 : 2d TFTs ... gen. functions

The delta-function can also be expanded in terms of characters of S_n in irreps. There is one irreducible rep for each Young diagram with n boxes. e.g for S_8 we can have



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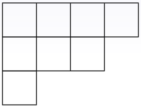

Label these R. For each partition of n

$$n = p_1 + 2p_2 + \cdots + np_n$$

there is a Young diagram.

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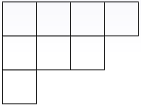

Label these R. For each partition of n

$$n = p_1 + 2p_2 + \cdots + np_n$$

there is a Young diagram.

The delta function is a class function :

$$\delta(\sigma) = \sum_{R \vdash n} \frac{d_R \chi_R(\sigma)}{n!}$$

The partition function

$$Z_G = \sum_{R \vdash n} (\frac{d_R}{n!})^{2-2G}$$

Fix a circle on the surface, and constrain the permutation associated with it to live in a subgroup.

$$Z(T^2, S_{n_1} \times S_{n_2}; S_{n_1+n_2}) = \frac{1}{n_1!n_2!} \sum_{\gamma \in S_{n_1} \times S_{n_2}} \sum_{\sigma \in S_n} \delta(\gamma\sigma\gamma^{-1}\sigma^{-1})$$

This kind of Fourier transformation on the group,in refined form, will play a role in subsequent developments.

Part 2 : 2d TFTs .... gen functions

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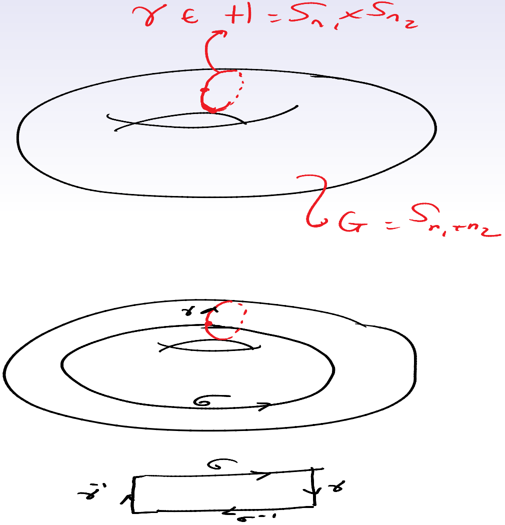

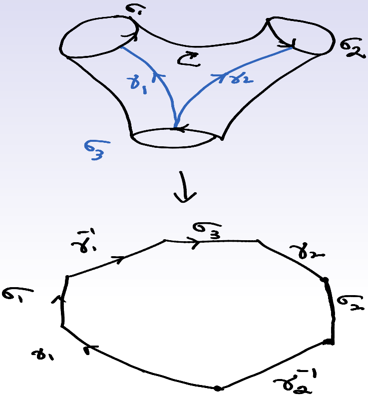

Conjugacy class product

$\uparrow$

$$Z(\sigma_1,\sigma_2,\sigma_3) \sim \sum_{\gamma_1,\gamma_2} \delta\left(\gamma_1\sigma_1\gamma_1^{-1}\,\gamma_2\sigma_2\gamma_2^{-1}\sigma_3\right)$$

if constrain $\gamma_1, \gamma_2 = 1$

group product

$$Z(\sigma_1,\sigma_2,\sigma_3) = \delta(\sigma_1\sigma_2\sigma_3)$$

Back to 4D

Start with simplest quiver. One-node, One edge. Gauge invariant operators $\mathcal{O}_\sigma$ with equivalence

$$\mathcal{O}_\sigma = \mathcal{O}_{\gamma\sigma\gamma^{-1}}$$

The set of $\mathcal{O}_\sigma$'s is acted on by γ. Burnside Lemma gives number of orbits as the average of the number of fixed points of the action.

number of orbits $= \frac{1}{n!}$ number of fixed points of the γ action on the set of σ

Hence number of distinct operators

$$\begin{aligned} p(n) &= \frac{1}{n!} \sum_{\sigma,\gamma \in S_n} \delta(\gamma\sigma\gamma^{-1}\sigma^{-1}) \\ &= Z_{TFT2}(T^2, S_n) \end{aligned}$$

In the case of $\mathbb{C}^3$, we specify n_1, n_2, n_3, the numbers of X, Y, Z and we can construct any observable $\mathcal{O}_\sigma(X, Y, Z)$ by using a permutation $\sigma \in S_n$, where $n = n_1 + n_2 + n_3$.

There are equivalences

$$\sigma \sim \gamma\sigma\gamma^{-1}$$

where $\gamma \in H \equiv S_{n_1} \times S_{n_2} \times S_{n_3} \subset S_n$.

Again using Burnside Lemma

$$\begin{aligned} N(n_1, n_2, n_3) &= \frac{1}{n_1!n_2!n_3!} \sum_{\gamma \in H} \sum_{\sigma \in S_n} \delta(\gamma\sigma\gamma^{-1}\sigma^{-1}) \\ &= Z_{TFT2}(T^2, H, S_n) \end{aligned}$$

Part 2 : 2d TFTs ....4D ... gen functions

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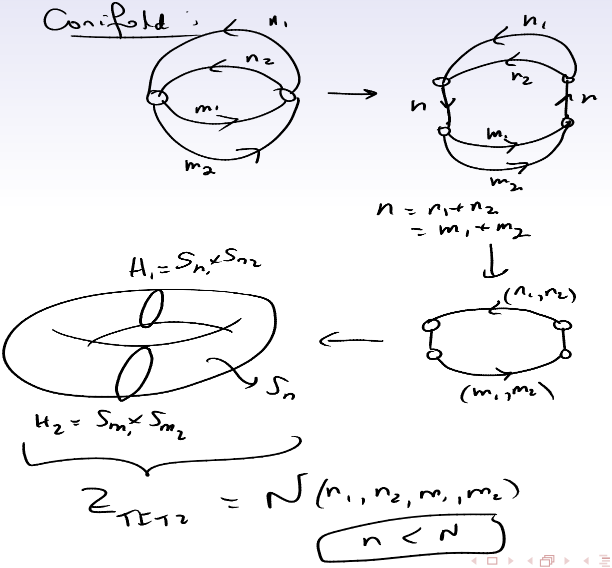

In terms of delta functions

$$N_{conifold}(n_1, n_2, m_1, m_2) = \sum_{\sigma_1 \in S_n} \sum_{\sigma_2 \in S_n} \sum_{\gamma_1 \in S_{n_1} \times S_{n_2}} \sum_{\gamma_2 \in S_{m_1} \times m_2} \delta(\gamma_1 \sigma_1 \gamma_2^{-1} \sigma_1^{-1}) \delta(\gamma_2 \sigma_2 \gamma_1^{-1} \sigma_2^{-1})$$

One delta function for each gauge group.
One permutation σ_a contracting the upper with lower indices for each $U(N_a)$. Equivalences

$$(\prod_b \gamma_{ba}) \sigma_a \prod_b \gamma_{ab}^{-1} \sim \sigma_a$$

γ_{ab} is in $\prod_\alpha S_{n^\alpha_{ab}}$.

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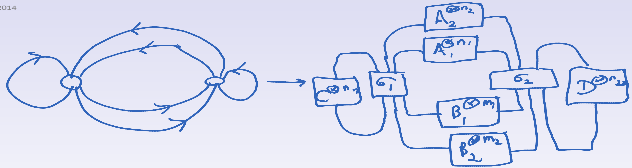

$$\sigma_1 \sim (\gamma_{11} \circ \gamma_{21})\, \sigma_1\, (\gamma_{11}^{-1} \circ \gamma_{12}^{-1})$$

$$\sigma_2 \sim (\gamma_{22} \circ \gamma_{12})\, \sigma_2\, (\gamma_{22}^{-1} \circ \gamma_{21}^{-1})$$

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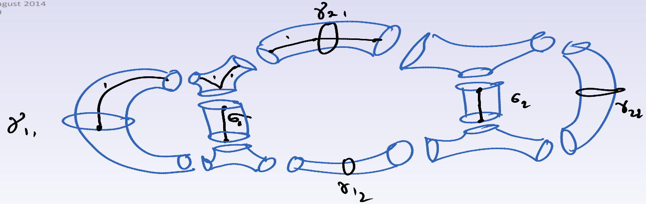

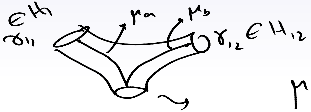

$\mu_a \in G$

$\mu_a \gamma_{11} \mu_a^{-1} \cdot \mu_b \gamma_{12} \mu_b^{-1}$

Need constraint /defect :

$\mu_a = 1$
$\mu_b = 1$

These large N formulae in terms of delta functions can be used to derive simple generating functions - in the form of infinite products. The form of the denominators are simply related to the structure of the quiver - will illustrate by examples (general formula in 1301.1980).

1-node, 1-edge (Half-BPS)

$$\prod_{i=1}^{\infty} \frac{1}{(1-t^i)}$$

1-node, 3-edges (eighth-BPS)

$$\prod_{i=1}^{\infty} \frac{1}{(1-t_1^i-t_2^i-t_3^i)}$$

This formula was first written in F. Dolan 2005

Conifold case

$$\mathcal{N}(a_1, a_2, b_1, b_2) = \sum_{n_1, n_2, m_1, m_2} N(n_1, n_2, m_1, m_2) a_1^{n_1} a_2^{n_2} b_1^{m_1} b_2^{m_2}$$

$$= \prod_{i=1}^{\infty} \frac{1}{(1 - a_1^i b_1^i - a_1^i b_2^i - a_2^i b_1^i - a_2^i b_2^i)}$$

This is a remarkably simple formula - obtained by converting permutation sums, into sums over conjugacy classes, labelled by cycles lengths i.

Even simpler - as obtained by substitution :

$$F(y_{12}, y_{21}) = \frac{1}{(1 - y_{12} y_{21})}$$

$$\mathcal{N}(a_1, a_2, b_1, b_2) = \prod_i F(y_{21} \to a_1^i + a_2^i ; y_{12} \to b_1^i + b_2^i)$$

$\mathbb{C}^3/Z_2$ case

$$\mathcal{N}_{\mathbb{C}^3/Z_2}(a_1, a_2, b_1, b_2, c, d)$$
$$= \prod_{i=1}^{\infty} \frac{1}{1 - a_1^i b_1^i - a_1^i b_2^i - a_2^i b_1^i - a_2^i b_2^i - c^i - d^i + c^i d^i}$$

Again there is a basic F function, $F(y_{11}, y_{12}, y_{21}, y_{22})$ which gives the above after substitution

$$\mathcal{N}_{\mathbb{C}^3/Z_2}(a_1, a_2, b_1, b_2, c, d) =$$
$$\prod_i F(y_{11} \to c^i, y_{21} \to a_1^i + a_2^i, y_{12} \to b_1^i + b_2^i, y_{22} \to d^i)$$

where

$$F(y_{ab}) = \frac{1}{(1 - y_{11} - y_{22} - y_{12}y_{21} + y_{11}y_{22})}$$

In general the F function is

$$F = \frac{1}{Det(1 - Y)}$$

and there is a simple substitution to get the desired trace generatin gfunction

$$\mathcal{N}(x_{ab;\alpha}) = \prod_i F(y_{ab} \to \sum_\alpha x^i_{ab;\alpha})$$

Part 3 : Quiver as Calculators - Finite N counting and orthogonal bases

The above formulae are valid when N is sufficiently large. The finite N counting formulae can be written in terms of Littlewood Richardson coefficients - the form of the expression can be read off from the quiver diagram.

for the 1-node, 1-edge quiver

$$N(n,N) = p_N(n) = \sum_{\substack{R \vdash n \\ l(R) \leq N}} 1$$

giant graviton physics in AdS/CFT - stringy exclusion principle

For the 1-node, 3-edge quiver

$$N(n_1,n_2,n_3,N) = \sum_{r_1 \vdash n_1} \sum_{r_2 \vdash n_2} \sum_{r_3 \vdash n_3} \sum_{\substack{R \vdash n \\ l(R) \leq N}} g(r_1,r_2,r_3;R)^2$$

Part : Quivers as calculators, finite N, orthogonality

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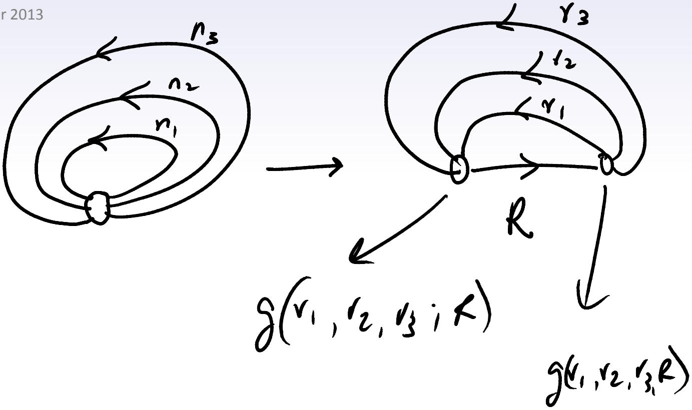


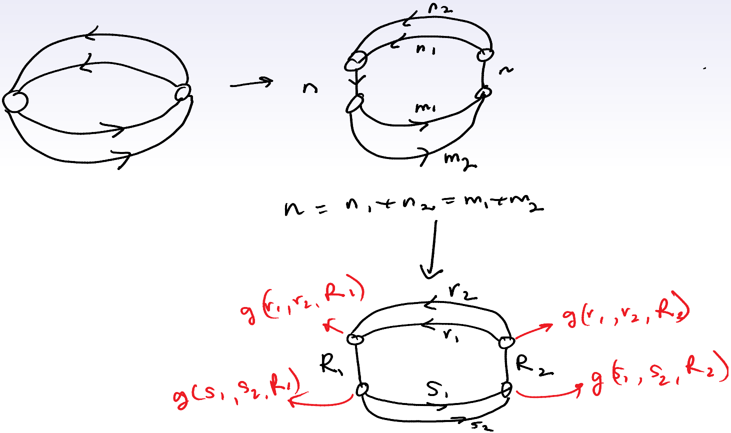
For conifold :

$$N(n_1, n_2, m_1, m_2) = \sum_{\substack{R_1 \vdash n \\ l(R_1) \leq N}} \sum_{\substack{R_2 \vdash n \\ l(R_2) \leq N}} \sum_{r_1 \vdash n_1} \sum_{r_2 \vdash n_2} \sum_{s_1 \vdash m_1} \sum_{s_2 \vdash m_2}$$

$$g(r_1, r_2, R_1) g(r_1, r_2, R_2) g(s_1, s_2, R_1) g(s_1, s_2, R_2)$$

$n = n_1 + n_2 = m_1 + m_2$.

For the conifold



For the $\mathbb{C}^3/Z_2$ case

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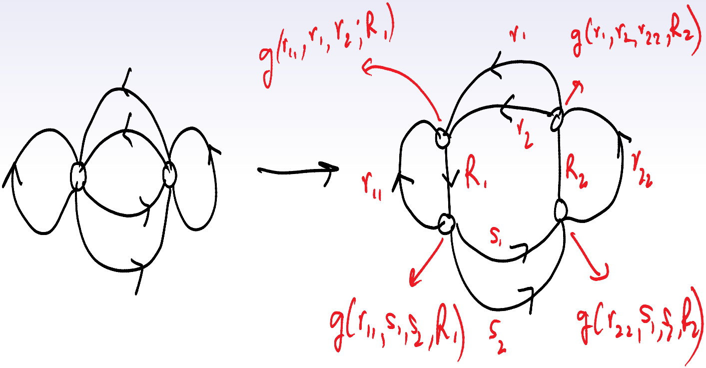


Part 3 b : Orthogonal bases

Back to 1-node, 1-edge quiver :
Using Wick's theorem and the basic 2-point function

$$< Z^i_j (Z^\dagger)^k_l >= \delta^k_j \delta^i_l$$

we can calculate the correlators

$$< \mathcal{O}_{\sigma_1} \mathcal{O}^\dagger_{\sigma_2} >$$

which give an inner product on the space of local operators.

This inner product is diagonalized by

$$\mathcal{O}_R = \sum_\sigma \chi_R(\sigma) \mathcal{O}_\sigma$$

$$< \mathcal{O}_R \mathcal{O}_S^\dagger >= f_R \delta_{RS}$$

Proof uses orthogonality properties of characters e.g.

$$\frac{1}{n!} \sum_\sigma \chi_R(\sigma) \chi_S(\sigma) = \delta_{RS}$$

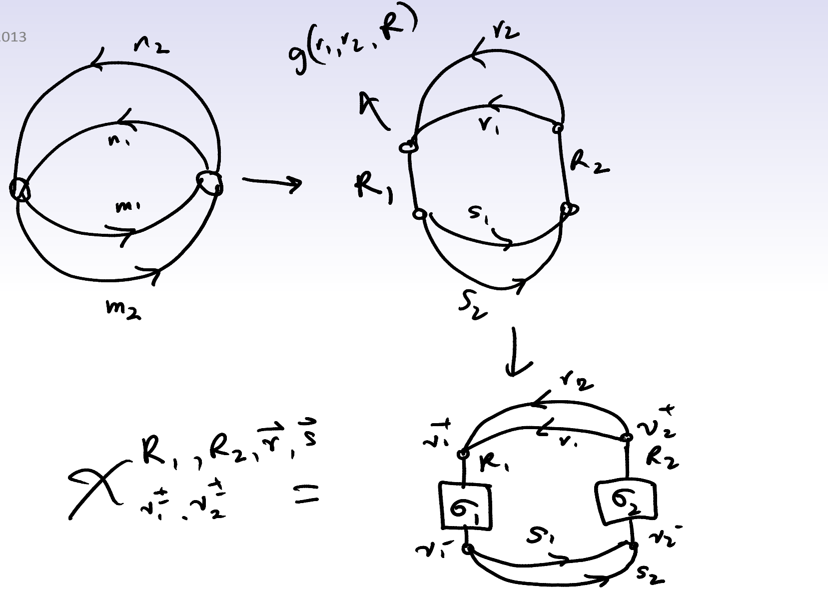
This diagonalization was done and used to propose a map between Young diagram operators and giant gravitons in AdS/CFT
Corley, Jevicki, Ramgoolam 2001
extended to half-BPS sugra backgrounds Lin, Lunin, Maldacena 2004
Recent tests (2011-2012) using DBI in $AdS \times S$ - Bissi ,Kristkjanssen, Young, Zoubos ; Caputa, de Mello Koch, Zoubos ; Hai Lin

Part 3 : Quivers as calculators, finite N, orthogonality

For general quivers, the $\chi_R(\sigma)$ are replaced by what we called Quiver characters, which are obtained by inserting permutations in the quiver diagram, interpreting the resulting in terms of $D^R_{ij}(\sigma)$ and branching coefficients $B^{R \to r_1, r_2 \cdots ; \nu}_{i, i_1, i_2 \cdots}$

The quiver characters have analogous orthogonality properties to ordinary S_n characters. And lead to orthogonal multi-matrix operators for quiver theories.

For the multi-edge single node quiver, this was understood in 2007/2008,
Kimura, Ramgoolam
Brown,Heslop,Ramgoolam
Collins, De Mello Koch, Bhattacharyya, Stephanou



n2
n1
m1
m2
g(r1, r2, R)
r2
r1
R1
R2
s1
S2
r2
r1
R1
R2
σ1
σ2
s1
s2

The LR coefficients $g(R_1, R_2, R_3)$, with $R_1 \vdash n_1, R_2 \vdash n_2, R_3 \vdash n_1 + n_2$, give the multiplicity of $R_1 \otimes R_2$ of the subgroup $S_{n_1} \times S_{n_2}$ in the reduction of the irrep R_3 of $S_{n_1+n_2}$.

$$V_{R_3}^{(S_{n_1+n_2})} = \bigoplus_{R_1,R_2} V_{R_1} \otimes V_{R_2} \otimes V_{R_3}^{R_1,R_2}$$

The multiplicity space can be given an orthogonal basis, labelled by an index ν which takes values $1 \leq \nu \leq g(R_1, R_2, R_3)$ Correspondingly there are branching coefficients

$$|R, i >= |R_1, R_2, \nu; i_1, i_2 >< R_1, R_2, \nu; i_1, i_2|R, i >$$

These branching coefficients are associated with vertices of the diagram, and $D_{ij}^{R}(\sigma)$ to the lines. This gives a quantity labelled by σ_1, σ_2 and the $\vec{R}$ and $\vec{\nu}$ labels. (no state labels - all contracted).

These quiver characters have the invariances we saw before

$$\chi^Q_{\mathbf{L}}(\sigma_a) = \chi^Q_{\mathbf{L}}\left(\prod_b \gamma_{ba} \;\; \sigma_a \;\; \prod_b \gamma^{-1}_{ba}\right)$$

and obey orthogonality relations e.g

$$\sum_{\sigma_a} \chi^Q_{\mathbf{L}_1}(\sigma_a)\chi^Q_{\mathbf{L}_2}(\sigma_a) \sim \delta_{\mathbf{L}_1,\mathbf{L}_2}$$

The operators

$$\mathcal{O}_{\mathbf{L}}(X_{ab;\alpha}) = \sum_{\sigma_a} \chi^Q_{\mathbf{L}}(\sigma_a)\mathcal{O}_{\sigma_a}(X_{ab;\alpha})$$

are orthogonal in the free field inner product - obtained by Wick contraction rule from

$$\langle X_{a_1 b_1;\alpha_1} X^\dagger_{a_2,b_2;\alpha_2} \rangle = \delta_{a_1 a_2}\delta_{b_1 b_2}\delta_{\alpha_1,\alpha_2}$$

Part 4 : Comments and future directions.

- The 2D TFT also gives a description of the correlators at large N.
- Chiral ring structure constants - selection rules - all Young diagrams combine according to LR rule. Multiplicity indices more complicated - but the structure can be captured by a diagram - obtained by cutting and gluing the diagrams for the 3 operators.
- There are equivalences - in some cases the same 4D observable can be given in different ways in the TFT2. A complete characterization of the equivalences would be good - categorical description of the TFT2 + defects.

19 August 2014
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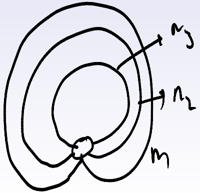
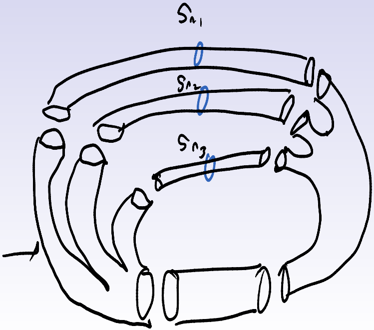
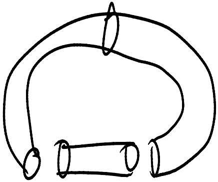




$S_{n_1} \times S_{n_2} \times S_{n_3} \subset S_n$



- Hamiltonians on the gauge invariants e.g 2-matrix problem

$$H_2 = tr[X, Y][\check{X}, \check{Y}]$$

In planar limit- Heisenberg spin chain. At finite N brane arguments imply that the BPS states of this Hamiltonian connect with $S^N(\mathbb{C}^2)$, i.e N bosons on $\mathbb{C}^2$.

- There should be analogous statements for general quivers. The Hamiltonians are not known - from first principle. But - conceivably, could be determined by requiring the correct space of ground states - $S^N(X)$; integrability at large N + knowledge of the space of marginal operators (comment of Alessandro) .....
- BPS states (null eigenstates of H_2) can be described as symm-traces + $1/N$ corrections using permutation groups. (papers of Vaman, Verlinde (2002) ; Brown, Heslop, Ramgoolam (2007) , Brown (2010) Pasukonis, Ramgoolam (2010)

$$\Omega^{-1} P$$

- A complete orthogonal finite N description is missing - although there are partial results using Brauer algebras (

  Kimmura, Ramgoolam - Branes, anti-Branes and Brauer algebras (2007) ; Kimura - Quarter BPS from Brauer algebra (2010)).

- "Permutation TFT2" formulations of 4D QFT combinatorics away from zero coupling also relevant to integrability in giant graviton dynamics - where we expand around a large Yong diagram $\chi_R(X)$ and study the Y-imputities using the $\chi^R_{r_1,r_2;\nu_1,\nu_2}$ restricted Schur basis for 2-matrix system.

  Giant graviton oscillators - Giatanagas, de Mello Koch , Dessein, Mathwin (2011)

  A double coset ansatz for integrability in AdS/CFT - de Mello Koch, Ramgoolam (2012)

- Permutation TFT2 - a unifying description of a vareity of QFT combinatorics of gauge invariant operators ...interesting to explore further ...