Matrix models, Hurwitz spaces and the absolute Galois group

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with Robert de Mello Koch.

Introduction : Physics Background

String theory :

Point particles moving in spacetime are replaced by one-dimensional objects.

Worldlines - curves describing trajectories of points in space-time - are replaced by Worldsheets (two-dimensional surfaces).

A string theory is a theory of maps from worldsheets to space-time.

Worldsheets have a genus h. A string theory has a parameter g_{st}, called the *string coupling* .

Contributions to physical observables, from genus h world-sheets, are weighted by g_{st}^{2-2h}

Traditionally string theories are described by writing a worldsheet action

$$S = \int_{\Sigma_h} \sqrt{g} g^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu}$$

X^μ are space-time coordinates. They are dynamical variables $X^\mu(\sigma, \tau)$ depending on world-sheet coordinates (σ, τ).

This traditional approach leads to integrals over $\mathcal{M}_{h,n}$, the moduli space of conformal structures of the world-sheet metric g

A recent theme in last 20 years :

Emergent string theories.

Simplest string theories emerge from Matrix integrals.

$$\mathcal{Z} = \int dX \;\; e^{-\frac{1}{2} trX^2}$$

X is an $N \times N$ matrix, can be restricted to be hermitian today.

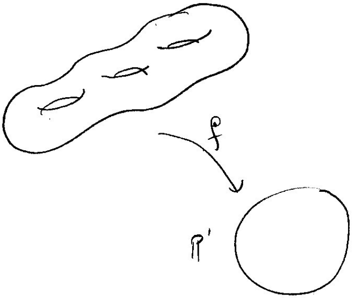
The string theory thus emerging is simplest at large N. The contributions from genus h are weighted by N^{2-2h}.

In Quantum field theories the exponent is replaced by an action e.g $e^{\frac{1}{2}\int d^4 x tr \partial_\mu X \partial^\mu X - trX^2}$ and the integral dX becomes a path integral.

In this talk : I will show how the Gaussian Matrix integral leads to a very simple string theory

The spacetime is $\mathbb{P}^1$

The maps are holomorphic.



$$\bar{\partial} f = 0$$

What kind of holomorphic maps ?

The curve and map are defined by equations involving coefficients which are algebraic numbers i.e they are in $\bar{\mathbb{Q}}$.

Any curve defined over $\bar{\mathbb{Q}}$ can appear among the worldsheets for appropriate observables.

OUTLINE

- The Matrix model : From correlators to counting triples of permutations.
- Physical Interpretation : Target space of $\mathbb{P}^1$ and strings over algebraic numbers.
- Multi-Matrix Model : New invariants of the absolute Galois group.
-
    - Open problems
    - Emergent string theories.

PART 1 : One Matrix Model

$$\mathcal{Z} = \int dX \;\; e^{-\frac{1}{2}trX^2}$$

X : $N \times N$ Hermitian matrix

$$dX \equiv \prod_{i<j} dRe(X_{ij}) dIm(X_{ij}) \prod_i dRe(X_{ii})$$

$$\mathcal{Z}(g) = \int dX e^{-\frac{1}{2}trX^2 + V(X,g)} = \int dX e^{-\frac{1}{2}trX^2 + g_3 trX^3 + g_4 trX^4 + \cdots}$$

One Matrix Model : Obserbvables

The Observables of interest : Trace moments of the matrix variables.

$$\langle \mathcal{O}(X) \rangle = \frac{1}{\mathcal{Z}} \int dX e^{-\frac{1}{2} trX^2} \mathcal{O}(X) \cdots$$

The $\mathcal{O}(X)$ is a function of traces, e.g $\mathcal{O}(X) = (trX)^{p_1} (trX^2)^{p_2} \cdots$.

Fixing the total number of X to be n, the number of these observables is $p(n)$. The number of partitions of n.

$$n = p_1 + 2p_2 + 3p_3 + \cdots$$

Partitions of n correspond to conjugacy classes of the symmetric group S_n, of all permutations of n objects.

It is possible to associate observables to permutations

$$\mathcal{O}_\sigma(X)$$

which only depend of the conjugacy class.

$$\mathcal{O}_{\alpha\sigma\alpha^{-1}}(X) = \mathcal{O}_\sigma(X)$$

$$X : V \to V$$

$$X \otimes X : V \otimes V \to V \otimes V.$$

$$\mathrm{tr} X \, \mathrm{tr} X = \mathrm{tr}_{V \otimes V}(X \otimes X)$$

$$\mathrm{tr} X^2 \; = \; \mathrm{tr}_{V \otimes V}((X \otimes X)\sigma).$$

where $\sigma(v_{i_1} \otimes v_{i_2}) = v_{i_2} \otimes v_{i_1}$.

$$\mathcal{O}_\sigma(X) = X^{i_1}_{i_{\sigma(1)}} X^{i_2}_{i_{\sigma(2)}} \cdots X^{i_n}_{i_{\sigma(n)}}$$

Defining $\mathbf{X} = X \otimes X \otimes \cdots X$ we can write the above as

$$\mathcal{O}_\sigma(X) = \mathrm{tr}_{V^{\otimes n}}(\mathbf{X}\,\sigma)$$

Note :

- invariance under conjugation by σ, and
-

$$\mathcal{O}_\sigma(X) = (\mathrm{tr}X)^{p_1(\sigma)}(\mathrm{tr}X^2)^{p_2(\sigma)} \cdots (\mathrm{tr}X^n)^{p_n(\sigma)}$$

The numbers $p_1(\sigma), p_2(\sigma)$.. give the numbers of 1-cycles, 2-cycles etc. of the permutation – this cycle structure determines the conjugacy class of the permutation in the symmetric group S_n.

Example : A permutation (123)(45) in S_5 cyclically permutes $1, 2, 3$ and swops $4, 5$.

In this case

$$\mathcal{O}_\sigma(X) \quad \sim \quad trX^3 trX^2$$

We will choose a normalization of observables as

$$\begin{aligned} \mathcal{O}_\sigma(X) &= N^{-n+p_1(\sigma)+p_2(\sigma)+\cdots+p_n(\sigma)}(trX)^{p_1}(trX^2)^{p_2}\cdots(trX^n)^{p_n} \\ &= N^{C_\sigma - n}(trX)^{p_1}(trX^2)^{p_2}\cdots(trX^n)^{p_n} \end{aligned}$$

We will define a delta function over the symmetric group

$$\begin{aligned} \delta(\sigma) &= 1 \text{ if } \sigma = 1 \\ &= 0 \text{ otherwise} \end{aligned}$$

Theorem 1 :

$$\langle \mathcal{O}_\sigma \rangle = \frac{1}{(2n)!} \sum_{\sigma \in [\sigma]} \sum_{\gamma \in [2^n]} \sum_{\tau \in S_{2n}} \delta(\sigma\gamma\tau) N^{C_\sigma + C_\tau - n}$$

The sum γ is over the conjugacy class $[2^n]$ – of permutations with n cycles of length 2.

This is the sum over Feynman diagrams of the Gaussian matrix model.

Outline of Proof

Basic results of Gaussian integration :

$$\langle X^{i_1}_{j_1} X^{i_2}_{j_2} \rangle = \delta^{i_1}_{j_2} \delta^{i_2}_{j_1}$$

$$\langle X^{i_1}_{j_1} X^{i_2}_{j_2} X^{i_3}_{j_3} \rangle = 0$$

or for any odd number.

$$\langle X^{i_1}_{j_1} X^{i_2}_{j_2} X^{i_3}_{j_3} X^{i_4}_{j_4} \rangle = \langle X^{i_1}_{j_1} X^{i_2}_{j_2} \rangle \langle X^{i_3}_{j_3} X^{i_4}_{j_4} \rangle + \langle X^{i_1}_{j_1} X^{i_3}_{j_3} \rangle \langle X^{i_2}_{j_2} X^{i_4}_{j_4} \rangle + \langle X^{i_1}_{j_1} X^{i_4}_{j_4} \rangle \langle X^{i_2}_{j_2} X^{i_3}_{j_3} \rangle$$

A general correlator factors into a product of quadratic correlators. One sums over all ways of pairing the variables. This is WICK's theorem and basic result in quantum field theory.

The basic 2-point function

$$\langle X^{i_1}_{j_1} X^{i_2}_{j_2} \rangle = \delta^{i_1}_{j_2} \delta^{i_2}_{j_1}$$

The RHS is the matrix elements of a permutation (12) acting on $V \otimes V$.

$$\langle e^{i_1} \otimes e^{i_2} | (12) | e_{j_1} \otimes e_{j_2} \rangle$$

Each elementary Wick contraction is a permutation. The sum over Wick contractions is a sum over permutations which products of cycles of length 2.

These two permutations σ, γ appear as follows :

$$\langle tr_{V^{\otimes 2n}}(\mathbf{X}\sigma)\rangle = \sum_{\gamma \in [2^n]} tr_{V^{\otimes 2n}}(\gamma\sigma)$$

The trace of the permutation, viewed as an operator in $V^{\otimes 2n}$, is $N^{C_{\gamma\sigma}}$.

A minor re-writing leads to the result stated

$$\langle \mathcal{O}_\sigma \rangle = \frac{1}{(2n)!} \sum_{\sigma \in [\sigma]} \sum_{\gamma \in [2^n]} \sum_{\tau \in S_{2n}} \delta(\sigma\gamma\tau) N^{C_\sigma + C_\tau - n}$$

Use a classic theorem : The Riemann Existence theorem, which relates the counting of such strings of permutations to the counting of equivalence classes of holomorphic maps $f : \Sigma_h \to \mathbb{P}^1$, from Riemann surface Σ_h of genus h to target $\mathbb{P}^1$.

Equivalence of maps :

$$\begin{array}{ccccc} \Sigma_h & & \stackrel{\phi}{\longrightarrow} & & \Sigma_h \\ & f_1 \searrow & & \swarrow f_2 & \\ & & \mathbb{P}^1 & & \end{array}$$

$$f_1 = f_2 \circ \phi.$$

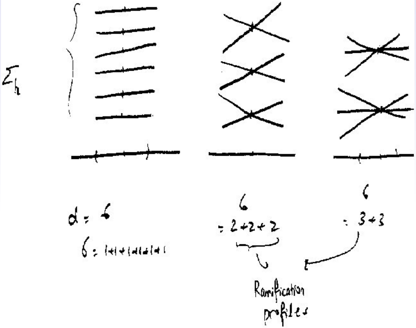
Automorphisms : Bi-holomorphic $\phi : X \rightarrow X$

$$\begin{array}{ccccc} \Sigma_h & & \stackrel{\phi}{\longrightarrow} & & \Sigma_h \\ & f \searrow & & \swarrow f & \\ & & \mathbb{P}^1 & & \end{array}$$

$$f = f \circ \phi.$$

Holomorphic maps between Riemann surfaces are branched covers.

An interval through a generic point on the target Riemann surface : inverse image has d intervals, where d is the degree of the map. A branch point has fewer inverse images.



Σh
d = 6
6 = 1+1+1+1+1+1
6 = 2+2+2
6 = 3+3
Ramification profiles

Each branch point has a ramification profile which is a partition of the degree d. The ramification data determines the genus h of Σ_h by the Riemann Hurwitz formula.

$$\langle \mathcal{O}_\sigma \rangle = \sum_{f:\Sigma_h \to \mathbb{P}^1} \frac{1}{|Aut\ f|} N^{2-2h}$$

The Gaussian Matrix model correlator is a sum over equivalence classes of holomorphic maps to $\mathbb{P}^1$, branched at 3 points, with ramification profiles $[\sigma]$, $[\gamma] = [2^n]$ and $[\tau]$ which is general.

Weighted by g_{st}^{2h-2} where $g_{st} = \frac{1}{N}$

For a Gaussian model perturbed by the potential $V(g_i, X) = g_3 trX^3 + g_4 trX^4 + \cdots$, the exponential of the potential can be viewed as an observable.

Hence the above observation about correlators and their relation to counting triples of permutations continues to hold for the perturbed Gaussian model, and for observables inserted in that model.

PART 2 : Physics Interpretation

The Gaussian Matrix model is equivalent to a topological string theory, with target space $\mathbb{P}^1$ which localizes to holomorphic maps with three branch points.

A perturbed Gaussian model also has such an interpretation with e^V treated as an observable.

Implication : Hurwitz counting results from Saddle point

By considering a potential $V = trX^m$ we can get explicit Hurwitz space counting results for maps where $[\sigma] = [m^n]$. The cases $m = 3, 4, 6$ were done in the paper - for genus genus zero worldsheets using saddle point methods.

For the case $m = 6$, $[\sigma] = [6^n]$, $[\gamma] = [2^{3n}]$ and $[\tau]$ is summed over all possible. For such maps,

$$\sum_f \frac{1}{|Autf|} = \frac{1}{2} \frac{(10)^n (3n-1)!}{(2n+1)!(n+1)!}$$

MEANING OF THREE ?

Belyi theorem : A Riemann surface is defined over algebraic numbers iff it admits a map to $\mathbb{P}^1$ with three branch points.

Riemann surface can be described by algebraic equations, e.g an elliptic curve

$$y^2 = x^3 + ax^2 + bx$$

If a, b are algebraic numbers, i.e. solutions to polynomial equations with rational coefficients $\mathbb{Q}$, then the Riemann surface is defined over $\bar{\mathbb{Q}}$, i.e for $x \in \bar{\mathbb{Q}}$, $y \in \bar{\mathbb{Q}}$

The field $\bar{\mathbb{Q}}$ is the algebraic closure of $\mathbb{Q}$ and contains all solutions of polynomial equations with rational coefficients.

It contains finite extensions of $\mathbb{Q}$ such as $\mathbb{Q}(\sqrt{2})$.

This is numbers of the form $a + b\sqrt{2}$, where a, b are rational. They form a field, closed under addition, multiplication, division.

An important group associated to this extension is the group of automorphisms which preserves the rationals. In this case, the only non-trivial element of the group is $\sqrt{2} \to -\sqrt{2}$. We say $Gal(\mathbb{Q}(\sqrt{2})/\mathbb{Q}) = \mathbb{Z}_2$

The absolute Galois group $Gal(\bar{\mathbb{Q}}/\mathbb{Q})$ contains as subgroups, all the finite Galois groups of finite dimensional extensions.

It acts on the algebraic numbers coefficients of the defining equations of the curve Σ_h and of the map f.

Hence the Galois group acts on the (equivalence classes of) permutation triples, equivalently the Feynman graphs of the 1-matrix model.

Calculations in Quantum Field theory – and Matrix integration – are often done diagrammatically. For the computation of $< trX^4 >$, we could draw on a plane, a black dot with four vertices coming out of it.

The sum over Wick contractions is a sum over ways of joining the lines. If the lines cross when drawn on a plane, then we can add a handle that the line rides over. So we have a graph embedded on a higher genus surface.

These are the Feynman graphs – in the simple case of 1-Matrix model. 't Hooft (70's) observed that the contributions to physical quantities of higher genus graphs are sub-leading in the large N limit.

In more complicated physics problems, the lines are associated with space-time functions (propagators). They can be of different types. Electron propagators or photon propagators which are different functions of space-time.

Grothendieck associated Dessins to the permutation triples, which are essentially the Feynman graphs of the 1-matrix model.

More precisely, for each propagator, we add a white vertex. In Grothendiek Dessins, with all edges labelled, we have a permutation σ describing the cyclic permutations around the black vertices. And permutation γ describing permutations around the white vertices.

The multiplicity of Feynman graphs can be organised into orbits of the Galois group action.

Elements in the same orbit contribute with equal weight, since *Autf* is a Galois invariant.

$$\langle \mathcal{O}_{[\sigma]} \rangle = F_1 + F_2 + F_3 + F_4 + F_5$$

$F_1 + F_2 + F_3$: Galois orbit

$F_4 + F_5$: Galois orbit

F_1: Equivalence class of σ, γ, τ ;

$$\frac{1}{\mathrm{Aut}\, f} = \frac{1}{\mathrm{Aut}(\sigma, \gamma, \tau)}$$

This is a Galois invariant

$\Downarrow$

Feynman diagrams in a Galois orbit contribute with same weight

PART 3 : Multi-matrix models

An obvious generalization to consider is multi-matrix models, where we have integrals over multiple matrix variables, e.g X, Y.

The edges of the Feynman graphs, which are propagators are now colored, i.e they can be X or Y propagators. So they correspond to colored-edge versions of Grothendieck Dessins.

Computations of observables can be done using sums of triples of permutations.

Again can use permutations to construct observables

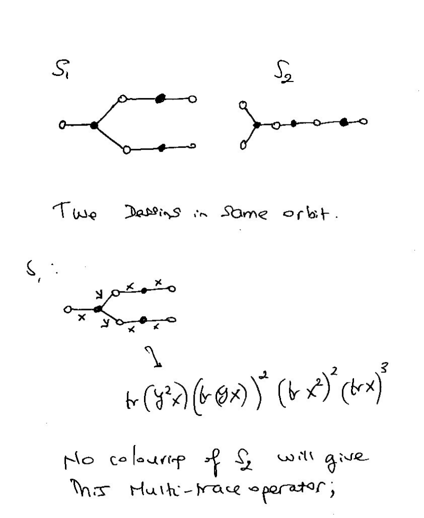
$$\mathrm{tr}_{V^{\otimes m+n}}(X^{\otimes m} \otimes Y^{\otimes n} \sigma)$$

But σ and $\alpha\sigma\alpha^{-1}$ only give the same observable if $\alpha \in S_n \times S_m$ subgroup of S_{n+m}.

Wick contractions are sums over conjugacy classes of subgroups.

Holomorphic maps, with coloring of the ramification points above 1.

A given multi-matrix observable, e.g $trX^2Y^2trXtrY^3$ can receive zero contribution from one Dessin in a Galois orbit and non-zero from another.



- A generic correlator in the multi-matrix model is not a Galois invariant.
- But we can build invariants from multi-matrix model correlators with a few steps.

GAUSSIAN = CLEAN !!

General Grothendieck Dessins have arbitrary $[\sigma], [\gamma], [\tau]$. The branch points can be chosen at $0, 1, \infty$.

The Dessins are bi-partitite graphs drawn the sphere, with black and white vertices. Black vertices have numbers of edges corresponding to the cycle structure of $[\sigma]$. White vertices correspond to $[\gamma]$.

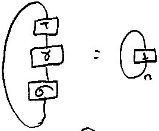
Dessins with $[\gamma] = [2^n]$ are **Clean Dessins**

Any dessin can be mapped to a clean Dessin. Any map with three branch points at $0, 1, \infty$ can be mapped to a new map with only simple branchings using $f \to 4f(1-f)$.

Cleaning is a process used to define Galois invariants in the Dessins literature.

The restriction $[\gamma] = [2^n]$ is not significant from the point of view of Galois invariants.

(12)

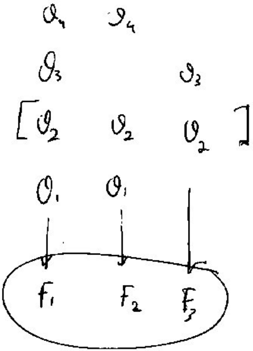


$\Sigma = (\sigma \circ \gamma)$

$\Gamma =$  $= t$

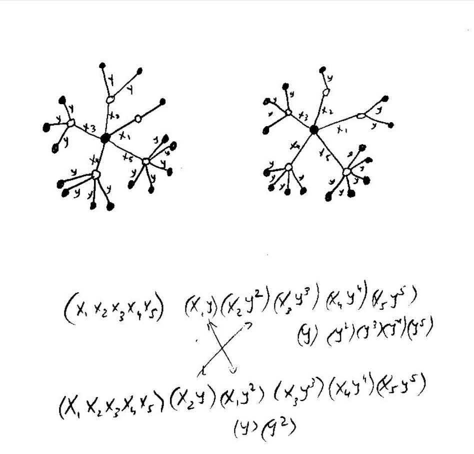
$\tau = t(\gamma^{-1} \cdot \sigma^{-1})$

- Colorings of the Dessins allow the definition of new invariants of the action of $Gal(\bar{\mathbb{Q}}/\mathbb{Q})$ on the Dessins : constructed from lists of multi-matrix observables which receive contributions from the Dessin.
- Take certain unions and intersections.



Intersection of lists of Multi-Matrix Observables is a Galois Invariant.

Known invariants can be described in terms of these lists, e.g Flower-shaped trees.



Two of Many questions

How are the Physics-inspired invariants constructed from coloured Dessins related to number theoretic invariants ?

Belyi theorem suggests that the string theory of 1-matrix model can be defined over $\bar{\mathbb{Q}}$. Is there an explicit construction of string amplitudes and path integrals over $\bar{\mathbb{Q}}$?

This model was studied, with the possibility of an additional general potential $V(g_i, X)$, around early nineties.

By tuning g_i to different critical points, one had a series of dual string theories $c < 1$ models coupled to 2D gravity (Liouville theory).