Thesis submitted for the degree of
Doctor of Philosophy

# Argyres-Douglas Theories and Chiral Algebras

Zoltan Laczko

Supervisors

Dr Matthew Buican
Dr Pau Figueras

July 9, 2020

School of Mathematical Sciences
Queen Mary University of London
*and*
Centre for Research in String Theory
School of Physics and Astronomy
Queen Mary University of London

*This thesis is dedicated to my family*
*who inspired my love for science.*

# Abstract

In this thesis we present various non-perturbative results about Argyres-Douglas (AD) theories and their chiral algebras. We also study renormalization group flows emanting from AD theories with accidental supersymmetry enhancement.

In the first introductory chapter we motivate the use of extended symmetries as a means to understanding strongly coupled quantum field theories. Various candidates for these symmetries are discussed, with the most attention given to supersymmetry. We close with a short summary of the coming chapters and their relations to each other.

After an introduction in the second chapter to $\mathcal{N} = 2$ supersymmetric quantum field theories in four dimensions, we present a summary of the key techniques used throughout the thesis. Topics include the superconformal index, the 4D/2D chiral algebra correspondence and the main subject of our thesis, Argyres-Douglas theories.

Our discussion in the third chapter explores the simplest Argyres-Douglas generalization of Argyres-Seiberg S-duality. In particular, we study interesting properties of the exotic AD theory emerging from this duality by bootstrapping its chiral algebra.

In the fourth chapter we use a relation between characters of affine Kac-Moody algebras to give a free field realization to certain observables in a subsector of an infinite family of AD theories. Interestingly, the free fields turn out to be non-unitary.

In chapter five we consider another family of AD theories and relate their logarithmic chiral algebras to a set of rational conformal field theories. We then study renormalization group (RG) flows emanting from our AD theories and show how certain pieces of the topological data associated with the infrared (IR) theory can be calculated from that of the ultraviolet (UV).

In the sixth chapter we introduce another infinite set of RG flows and argue that they lead to supersymmetry enhancement from $\mathcal{N} = 2$ to $\mathcal{N} = 4$ in four dimensions.

We conclude in chapter seven with a summary of our main results and some questions they raise.

# Acknowledgements

First and foremost, I want to thank my supervisor, Dr Matthew Buican for his guidance throughout my PhD and for sharing his great ideas with me. I also wish to thank Dr Pau Figueras whose help made this thesis possible.

I am grateful to Prof. Christopher Beem and Prof. Yang-Hui He for their comments and suggestions regarding my thesis and for serving on my viva committee. I also wish to thank Prof. David Arrowsmith for organizing and chairing my viva.

During my time at CRST, the atmosphere has been both stimulating and thoroughly enjoyable. This is due in no small part to the wonderful people who work there, both students and academics. They deserve all the praise for their kindness and dedication.

I am grateful to my previous supervisor, Zoltán Bajnok for his mentoring during my undergraduate years, it has been vital to my progress as a student.

I thank my friends from ELTE and EJC for sharing many laughs and adventures with me over the years, I hope there will be plenty more to enjoy together in the future.

I cannot thank my family enough for the love and support they gave me throughout my life, I am incredibly lucky to have them. I wish to thank my grandfather, Zoltán Laczkó for helping me as soon as I started showing interest in physics.

Last but not least, I thank Lilla for sharing the ups and downs of this journey with me. I could not have asked for a better companion.

# Declaration

I, Zoltan Balazs Laczko, confirm that the research included within this thesis is my own work or that where it has been carried out in collaboration with, or supported by others, that this is duly acknowledged below and my contribution indicated. Previously published material is also acknowledged below.

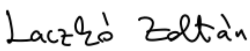
I attest that I have exercised reasonable care to ensure that the work is original, and does not to the best of my knowledge break any UK law, infringe any third party's copyright or other Intellectual Property Right, or contain any confidential material.

I accept that the College has the right to use plagiarism detection software to check the electronic version of the thesis.

I confirm that this thesis has not been previously submitted for the award of a degree by this or any other university.



Signature:




Date: July 9, 2020

Details of collaboration and publications:

This thesis describes research carried out with my supervisor Matthew Buican, which was published in [1–4]. We collaborated with Takahiro Nishinaka in [1,4]. Where other sources have been used, they are cited in the bibliography.

# Contents


**1 Introduction** **7**

**2 Review** **11**

2.1 $\mathcal{N} = 2$ Quantum Field Theories . . . . . . 11

2.1.1 $\mathcal{N} = 2$ SUSY algebra . . . . . . 12

2.1.2 Lagrangian Theories . . . . . . 12

2.1.3 Moduli space . . . . . . 13

2.2 $\mathcal{N} = 2$ Superconformal Field Theories . . . . . . 17

2.2.1 Lagrangian $\mathcal{N} = 2$ SCFTs . . . . . . 18

2.2.2 $\mathcal{N} = 2$ superconformal representations . . . . . . 19

2.2.3 Superconformal Index . . . . . . 20

2.2.4 $\mathcal{N} = 2$ index . . . . . . 21

2.2.5 Limits of the index . . . . . . 24

2.3 Chiral algebra correspondence . . . . . . 26

2.3.1 Chiral algebra from 4D $\mathcal{N} = 2$ SCFT . . . . . . 26

2.3.2 Notable elements of the chiral algebra correspondence . . . . . . 27

2.4 Argyres-Douglas theories . . . . . . 30

2.4.1 Review of Class $\mathcal{S}$ theories . . . . . . 31

2.4.2 Class $\mathcal{S}$ construction . . . . . . 31

**3 $\mathcal{N} = 2$ S-duality Revisited** **33**

3.1 Introduction . . . . . . 34

3.2 A chiral algebra argument for $\mathcal{T}_{3,\frac{3}{2}} = \mathcal{T}_X \oplus$ hyper . . . . . . 40

3.3 The Schur index of $\mathcal{T}_X$ from $S$-duality and inversion . . . . . . 41

3.4 A chiral algebra conjecture . . . . . . 44

3.5 Bootstrapping the Chiral Algebra of $\mathcal{T}_X$ . . . . . . 45

3.6 Re-writing the index in terms of AKM characters . . . . . . 48

3.7 A remark on the Hall-Littlewood chiral ring of $\mathcal{T}_X$ and the Schur sector 52

3.8 Witten's anomaly and the chiral algebra . . . . . . 53

**4 A Non-Unitary Surprise** **55**

4.1 Introduction . . . . . . 55

4.2 Modular $S$-transformations and an AKM relation . . . . . . . . . . . . . 57
4.3 A 4D interpretation . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 58
4.4 Infinitely many generalizations . . . . . . . . . . . . . . . . . . . . . . . 61

**5 Rationalizing CFTs and Anyonic Imprints on Higgs Branches 63**
5.1 Introduction . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 63
5.2 The 4D theories and their associated non-unitary chiral algebras . . . . 69
5.3 From logarithmic theories to RCFT . . . . . . . . . . . . . . . . . . . . 70
5.3.1 Flavoring the correspondence . . . . . . . . . . . . . . . . . . . . 72
5.4 MTCs and the RG flow . . . . . . . . . . . . . . . . . . . . . . . . . . . 74
5.4.1 MTC / TQFT basics . . . . . . . . . . . . . . . . . . . . . . . . . 75
5.4.2 Galois action, RG flows, and quantum dimensions . . . . . . . . 77
5.5 Connections with other theories . . . . . . . . . . . . . . . . . . . . . . 81

**6 Flowing from 16 to 32 Supercharges 83**
6.1 Introduction . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 83
6.2 The UV starting points . . . . . . . . . . . . . . . . . . . . . . . . . . . 85
6.3 The $(n,k)=(2,3)$ case . . . . . . . . . . . . . . . . . . . . . . . . . . . 87
6.3.1 The $r\to\infty$ limit and (exotic?) 4D $\mathcal{N}=4$ . . . . . . . . . . . . . 91
6.4 Generalizations . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 95

**7 Conclusions and Outlook 97**

**A Proof of the XYY formula 102**

**B Details of the Inversion Formula 104**
B.1 Inversion Theorem . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 106

**C $q\to 1$ and $S^3$ partition function 110**

**D The Hall-Littlewood index of $\mathcal{T}_X$ 113**

**E Proof of (5.50) 115**

**Bibliography 117**

# Chapter 1

# Introduction

The second half of the last century saw an incredible advance in our understanding of the subatomic world. This was made possible by a series of particle discoveries in accelerators, hinting at an inner structure within protons and neutrons. This was explained theoretically by quantum chromodynamics (QCD), the theory of the strong interaction between quarks and gluons. QCD was later combined with theories behind the electromagnetic and weak interactions into a framework called the Standard Model (SM).

The Standard Model turned out to be one of the most successful theories in all of science. Its numerical predictions have been verified extensively and with unparalleled accuracy. Notably, the discovery of the Higgs boson in 2012 filled the last remaining hole in the Standard Model's particle predictions.

Despite its long list of successes, the Standard Model does have its limitations as it fails to incorporate neutrino masses or a dark matter candidatate and requires unexplained input parameters to work. It also makes no mention of gravity and therefore has to break down eventually at short distances, before gravity becomes relevant.

Mathematically, the Standard Model is described by a special type of quantum field theory (QFT) called gauge theory, in which gauge bosons are responsible for mediating interactions betweeen matter particles. The strengths of interactions are determined by values of gauge coupling constants. It was one of the greatest insights gained into the behaviour of quantum field theories that coupling constants depend on the energy scale we are observing the theory at. This phenomena of running couplings is called renormalization group flow. In quantum electrodynamics, the theory behind the electromagnetic interactions, the coupling constant decreases as we go to low energies where particle accelerators do their measurements. This enables us to calculate physical quantities perturbatively in the coupling constants. In practice this is done via the pictorial formalism of Feynman diagrams that encode contributions of certain physical processes.

In QCD however the opposite happens and the coupling constant gets stronger at low energies, gaining it the name of strong interaction. This is known as asymptotic freedom and it produces remarkable phenomena such as color confinement, wherein quarks get locked inside color neutral particles called hadrons, and thus cannot be observed directly. This however also presents a huge challenge since at low energies perturbation theory is no longer valid and we lack effective tools to study the non-linear dynamics of the strong interactions. We can use numerical techniques in the form of lattice simulations but these do not explain the underlying phenomena behind their predictions.

This challenging situation calls for the study of non-perturbative, analytical techniques. This is unfortunately not an easy problem and a sensible way to proceed is to temporarily abandon QCD for theories with more symmetry. More symmetry means more constraints which make it easier to obtain non-perturbative results. The hope is that sometime in the future, we can use the experience gathered and our tools sharpened on these toy models to come back and solve QCD at strong coupling.

One way to introduce more symmetry is to leave behind four spacetime dimensions and study quantum field theories in two dimensions. QFTs that sit at endpoints of renormalization group flows enjoy conformal symmetry which is an infinite dimensional symmetry group in two dimensions. These conformal field theories (CFTs) are relatively well studied and large amounts of non-perturbative information have been gathered about them over the years. Another interesting class is that of integrable quantum field theories where are an infinite number of conservation laws exist and as a result, scattering factorizes. In both cases the bootstrap approach proved very useful which solves some well defined problem simply by enforcing consistency conditions.

We can also stay in four dimensions and introduce various symmetries there. Conformal symmetry can be one of those and even though it is less powerful in 4D than in 2D, the (numerical) bootstrap technique can still be used effectively, as demonstrated first in the seminal paper [5]. Another approach is to study QFTs with various amounts of supersymmetry. Ours will be somewhat of a blend of these, as we will be studying QFTs that are both conformal and supersymmetric, using a relation they have with 2D CFTs.

Back to SUSY, the minimal amount of supersymmetry is called $\mathcal{N} = 1$ and offers the most realistic models. It predicts that all particles come with superpartners that have opposite spin statistics. Although somewhat discouragingly superpartners still have not been observed in the Large Hadron Collider, $\mathcal{N} = 1$ supersymmetric models provide the most promising candidates to beyond the standard model physics. For example one of the unresolved puzzles of particle physics is why the Higgs mass is so incredibly tiny compared to the Planck scale, even though quantum corrections are expected to drive the former closer to the latter. This can be explained naturally with the help of supersymmetry, as in SUSY theories these quantum corrections cancel between superpartners and therefore leave the Higgs mass sufficiently small. Supersymmetry also provides dark matter candidates in the form of the lightest superpartners.

Besides their clear phenomenological appeal, SUSY theories are also important for another reason, namely in supersymmetric theories quantum corrections are kept much more under control, which lets us derive exact relations about quantities protected by supersymmetry. We can therefore gain invaluable insights into the inner workings of QFTs even at strong coupling. Confinement can be studied and many new and interesting phenomena have been discovered such as duality, which relates low-energy limits of two different QFTs. Surprisingly, this is often a strong-weak duality, meaning that the strong coupling region of one theory can be described by the weakly coupled region of the other!

We can do even more in theories with extended supersymmetry, at the cost of abandoning direct phenomenological relevance. The maximum allowed supersymmetry in four dimensions is called $\mathcal{N} = 4$ SUSY, because it has four times as much supersymmetry as the minimal $\mathcal{N} = 1$. These theories have interesting connections with String Theory and the AdS/CFT correspondence, and they even compute certain pieces of scattering amplitudes of QCD. On the other hand, they are believed to be determined uniquely by their gauge group[1] and therefore display much less diversity than $\mathcal{N} = 1$ theories.

Between the two extremes of $\mathcal{N} = 1$ and $\mathcal{N} = 4$ lie $\mathcal{N} = 2, 3$. No purely $\mathcal{N} = 3$ Lagrangian theory can exist and non-Lagrangian $\mathcal{N} = 3$ QFTs [6] are also quite severily restricted [7]. $\mathcal{N} = 2$ theories on the other hand strike a good balance between the variety displayed by $\mathcal{N} = 1$ supersymmetric theories and the manageability of $\mathcal{N} = 4$ ones. They will be the main subjects of this thesis.

More precisely we will study a certain class of $\mathcal{N} = 2$ superconformal field theories (SCFTs), called Argyres-Douglas theories and their RG flows. AD theories are inherently strongly coupled, meaning that they do not admit a weak coupling limit. What is more, they lack the usual Lagrangian construction we are accustomed to in the study of ordinary QFTs. AD theories then provide a useful arena to develop and test non-perturbative techniques to their limits.

In Chapter 2 we begin our review of $\mathcal{N} = 2$ theories with their Lagrangian construction and spaces of vacua. $\mathcal{N} = 2$ SUSY can be combined with conformal symmetry to get the powerful $\mathcal{N} = 2$ superconformal symmetry. The corresponding superconformal algebra and its representations are introduced next, along with an index that counts multiplets protected by the superconformal symmetry. We discuss the crucial invariance of the index under continuous deformations, and a simple recipe to calculate it in Lagrangian theories. Sometimes the full index is not available and we have to con-

[1] We will come back to this statement in the sixth chapter.

tend with its special limits discussed in Sec. 2.2.5. The so-called Schur limit takes an important role in Sec. 2.3, where we consider a non-perturbative correspondence connecting our superconformal field theories with two dimensional chiral algebras. Finally, in Sec. 2.4 we introduce the main protagonists of our story, Argyres-Douglas theories. We discuss their key properties that set them apart from other SCFTs, and a useful way of constructing a large class of them coming from 6D.

In Chapter 3 we study the AD counterpart of a famous duality. Despite the fact that not much is known about $\mathcal{T}_{3,\frac{3}{2}}$, an AD theory involved in the duality, we uncover many of its interesting properties including the Schur limit of its superconformal index and its chiral algebra, the latter with the help of a particular form of the bootstrap principle. We also discuss various consistency checks of our proposals.

In Chapter 4 we explore the connection between free fields and $(A_1, D_4)$, another AD theory that appears in the previous duality. Through the study of its chiral algebra we discover its relation to another chiral algebra, whose 4D ancestor is a free but non-unitary SCFT. Surprisingly, this non-unitary SCFT then describes certain observables in a subsector of the strongly coupled $(A_1, D_4)$ theory. We end the chapter by extending this result to an infinite family of AD theories.

We turn in Chapter 5 to the analysis of a particular set of RG flows between AD theories that is very natural from the 6D point of view. Motivated by the study of the chiral algebras of the AD theories sitting at the starting points of the RG flows, we find that some basic quantities underlying the chiral algebras of the SCFTs at the two endpoints of the flows are related via Galois conjugation.

In Chapter 6 we study an interesting RG flow starting from the $\mathcal{T}_{3,\frac{3}{2}}$ theory we first encountered in Chapter 3, along with its generalizations. The importance of this set of RG flows is emphasized by the fact that they lead to supersymmetry enhancement from $\mathcal{N} = 2$ to $\mathcal{N} = 4$. We analyse in particular the RG flow starting from $\mathcal{T}_{3,\frac{3}{2}}$, to gain a better understanding of the SCFT at the endpoint of the flow.

In Chapter 7 we conclude the thesis with a summary of our main results along with some open questions and recent developments.

# Chapter 2

# Review

In this chapter we conduct a lightning review of the most important concepts used later in the thesis. We first introduce $\mathcal{N}=2$ quantum field theories in the Lagrangian setting and examine their moduli spaces. We then turn to $\mathcal{N}=2$ superconformal field theories where the additional superconformal symmetry enables us to discuss theories without reference to a Lagrangian construction. The next topic is the superconformal index and its various limits, one of which plays an important role in the 4D/2D chiral algebra correspondence presented in the following section. These two will be the main tools in our exploration of SCFTs. Finally we introduce our main object of study, Argyres-Douglas theories. This is where our "Lagrangian free" language pays off as these theories lack such a description, and we instead present a construction coming from 6D. Our aim in this chapter is to present an overview and we refer the reader for details to the original literature.

## 2.1 $\mathcal{N}=2$ Quantum Field Theories

In the first section of this review chapter we collect some basic facts about 4D $\mathcal{N}=2$ QFTs that will be useful later. A large part of this section is dedicated to Lagrangian theories, despite the fact that we will be concerned mainly with non-Lagrangian theories later on. The reason for this is twofold, first, we will also encounter Lagrangian components and second, some of the properties of Lagrangian theories carry over to non-Lagrangian theories as well and are much easier to introduce in the Lagrangian framework.

### 2.1.1 $\mathcal{N}=2$ SUSY algebra

In $\mathcal{N}=2$ supersymmetry, the supercharges transform as doublets of the $SU(2)_R$ part[2] of the $U(2)$ automorphism group of the supersymmetry algebra

$$\{Q^i{}_\alpha, \bar{Q}_{j\dot{\alpha}}\} = 2\delta^i{}_j P_{\alpha\dot{\alpha}} \ , \qquad \{Q^i{}_\alpha, Q^j{}_\beta\} = \left\{\bar{Q}_{i\dot{\alpha}}, \bar{Q}_{j\dot{\beta}}\right\} = 0 \ , \tag{2.1}$$

where undotted and dotted greek indices are chiral and anti-chiral Lorentz spinor indices respectively, while $i, j$ are spin half $SU(2)_R$ indices. In the rest of the section we will restrict to theories with genuine global $SU(2)_R$ symmetry.

### 2.1.2 Lagrangian Theories

There are two fundamental building blocks in Lagrangian $\mathcal{N}=2$ gauge theories, the hypermultiplet and vector multiplet. We will use $\mathcal{N}=1$ SUSY multiplets to describe them.

The $\mathcal{N}=2$ vector multiplet contains an $\mathcal{N}=1$ vector multiplet with components $(\lambda_\alpha, A_\mu)$ and an $\mathcal{N}=1$ chiral multiplet $(\Phi, \tilde{\lambda}_\alpha)$

$$\begin{array}{ccccl} \lambda_\alpha & \longleftrightarrow & A_\mu & & \mathcal{N}=1 \text{ vector multiplet} \\ \updownarrow & & \updownarrow & & \\ \Phi & \longleftrightarrow & \tilde{\lambda}_\alpha & & \mathcal{N}=1 \text{ chiral multiplet,} \end{array} \tag{2.2}$$

vertical arrows indicate the action of the other $\mathcal{N}=1$ supersymmetry that is not manifest in the $\mathcal{N}=1$ description above. The $SU(2)_R$ $R$-symmetry rotates the Weyl fermions $\lambda_\alpha, \tilde{\lambda}_\alpha$ into each other while leaving the bosons unaffected. All component fields in the vector multiplet transform in the adjoint representation of the gauge group.

The hypermultiplet contains an $\mathcal{N}=1$ chiral multiplet $(Q, \psi)$ and an $\mathcal{N}=1$ antichiral multiplet $(\tilde{Q}^\dagger, \tilde{\psi}^\dagger)$

$$\begin{array}{ccccl} Q & \longleftrightarrow & \psi & & \mathcal{N}=1 \text{ chiral multiplet} \\ \updownarrow & & \updownarrow & & \\ \tilde{\psi}^\dagger & \longleftrightarrow & \tilde{Q}^\dagger & & \mathcal{N}=1 \text{ antichiral multiplet.} \end{array} \tag{2.3}$$

Here the scalars $Q, \tilde{Q}^\dagger$ are $SU(2)_R$ doublets, while the fermions are singlets. The $\mathcal{N}=1$ chiral and antichiral multiplets transform in the same representation $\mathcal{R}$ of the gauge group and therefore the two chiral multiplets with bottom components $Q$ and $\tilde{Q}$ transform in conjugate representations $\mathcal{R}$ and $\bar{\mathcal{R}}$.

When $\mathcal{R}$ is pseudo-real, we can obtain a special case of the hypermultiplet by

[2] This is referred to as $R$-symmetry when it descends to a genuine global symmetry of the theory.

imposing the following constraint on the $\mathcal{N}=1$ chiral multiplets

$$Q_a = \epsilon_{ab}\tilde{Q}^b \ . \tag{2.4}$$

The resulting multiplet is called the half-hypermultiplet in representation $\mathcal{R}$, since it effectively contains half the degrees of freedom of a full hypermultiplet.

Now that we have our building blocks at hand we can construct $\mathcal{N}=2$ Lagrangian gauge theories out of them. The conventional way of obtaining Lagrangians for such theories is to start with Lagrangians for their $\mathcal{N}=1$ component superfields and combine them in an $SU(2)_R$ invariant way. This requirement fixes the relative coefficients of terms related by $SU(2)_R$. The resulting Lagrangian for an $SU(N)$ gauge theory with $N_f$ hypermultiplets $Q^a_i, \tilde{Q}^i_a$ in the fundamental representation ($a$ is the gauge and $i$ is the flavor index) takes the following form [8]

$$\begin{aligned}\frac{\operatorname{Im}\tau}{4\pi}\int \mathrm{d}^4\theta\, \operatorname{tr}\Phi^\dagger e^{[V,\cdot]}\Phi + \left(\int \mathrm{d}^2\theta \frac{-i}{8\pi}\tau \operatorname{tr} W_\alpha W^\alpha + \text{cc.}\right) + \\ \int \mathrm{d}^4\theta\Big(Q^{\dagger i}e^V Q_i + \tilde{Q}^i e^{-V}\tilde{Q}^\dagger_i\Big) + \left(\int \mathrm{d}^2\theta\, \tilde{Q}^i\Phi Q_i + \text{cc.}\right) + \left(\int \mathrm{d}^2\theta\, \mu_i \tilde{Q}^i Q_i + \text{cc.}\right) \ ,\end{aligned} \tag{2.5}$$

where gauge indices are suppressed, $\tau$ is the complexified gauge coupling combining the gauge coupling and the theta angle

$$\tau = \frac{4\pi i}{g^2} + \frac{\theta}{2\pi} \ , \tag{2.6}$$

and finally $V$ and $W_\alpha$ are the $\mathcal{N}=1$ vector and field strength superfields respectively.

Armed with a Lagrangian, in the next section we turn to analyzing the space of supersymmetric vacua, known as the moduli space. Due to Lorentz invariance, these moduli spaces are parametrized by the vacuum expectation values of scalar fields in the theory.

### 2.1.3 Moduli space

Classically the moduli space can be found by looking at zeros of the scalar potential corresponding to zero energy field configurations. In non-supersymmetric theories the classical moduli space is generally lifted by quantum corrections unless protected by some global symmetry and the quantum moduli space consists of only a set of isolated points. Not so in supersymmetric theories where large parts of the classical moduli space remain even in the quantum theory due (at least perturbatively) to supersymmetric cancellations in loop diagrams.

To find the classical moduli space of an $\mathcal{N}=2$ gauge theory we must find zeros of the scalar potential coming from D-terms and F-terms in the Lagrangian. The equations we need to solve can be found in [8], with mass terms set to zero they take the following form

$$\begin{gathered}\left[\Phi^\dagger,\Phi\right]=0\\ \left(Q_iQ^{\dagger i}-\tilde{Q}_i^\dagger\tilde{Q}^i\right)\Big|_{\text{traceless}}=0\ ,\qquad Q_i\tilde{Q}^i\Big|_{\text{traceless}}=0\\ \Phi Q_i=\tilde{Q}^i\Phi=\Phi^\dagger Q_i=\tilde{Q}^i\Phi^\dagger=0\ ,\end{gathered}\tag{2.7}$$

where the traceless part of an $N\times N$ matrix is defined as

$$X|_{\text{traceless}}=X-\frac{1}{N}\text{tr}X\ .\tag{2.8}$$

In general it is hard to solve the above equations simultaneously, but the problem simplifies considerably if we concentrate on parts of the moduli space where either hypermultiplet or vector multiplet scalars get VEVs.

The subspace where only hypermultiplet scalars get nonzero VEVs is called the Higgs branch. Since these VEVs generally break the gauge group completely, we end up with just a bunch of massless neutral hypermultiplets in the low-energy theory, which is said to be in the Higgs phase. Quantum mechanically the Higgs branch turns out to be very simple as due to a non-renormalisation theorem in [9] it is classically exact.

The physics is richer on the Coulomb branch where we give VEVs only to $\mathcal{N}=2$ vector multiplet scalars. In this case the classical constraints in (2.7) reduce to just a single equation

$$\left[\Phi^\dagger,\Phi\right]=0\ ,\tag{2.9}$$

which means $\Phi$ is diagonalisable and can be conjugated to an element of the Cartan subalgebra of the gauge group $G$. At generic points on the Coulomb branch the gauge group is broken to the Cartan subalgebra $U(1)^{\text{rank}\,G}$ and the corresponding $\text{rank}\,G$ vector multiplets remain massless. The other $\dim G-\text{rank}\,G$ vector multiplets aquire masses through the super-Higgs mechanism and can be integrated out from the low-energy theory which is therefore in an IR-free abelian Coulomb phase.

Unlike the Higgs branch, the Coulomb branch does receive quantum corrections and as a result singular points can appear where additional massive degrees of freedom become massless. These are not extra gauge bosons however, and we will have much to say about the physics at these singularities. The exact low-energy theory on the Coulomb branch was found by Seiberg and Witten [10,11]. We will give a quick review of this beautiful topic based on [10–12] in the next section.

For completeness we mention that the mixed branch where both hypermultiplet and vector multiplet scalars have nonzero VEVs is locally a product of the Coulomb and Higgs branches [9].

### Seiberg-Witten theory

As a concrete example in this section we will give a taste of the Seiberg-Witten solution of $\mathcal{N}=2$ $SU(2)$ pure Yang-Mills theory [10].

The classical moduli space can be parametrized by the following gauge invariant combination of the expectation value of the $SU(2)$ vector multiplet scalar $\Phi$ satisfying (2.9)

$$u = \operatorname{Tr}\Phi^2 = \frac{1}{2}a^2\ , \qquad \Phi = \frac{1}{2}a\sigma^3\ , \tag{2.10}$$

with $\sigma^3 = \operatorname{diag}(1,-1)$. We take the variable $u$ to parametrize the quantum moduli space but then the relation between the modulus $u$ and the VEV $a$ receives quantum corrections and $a$ becomes some function $a(u)$.

The low-energy effective theory of massless degrees of freedom on the moduli space of any $\mathcal{N}=2$ gauge theory is described by an $\mathcal{N}=2$ nonlinear sigma model whose Lagrangian is fully determined by a single holomorphic function called the prepotential $\mathcal{F}(\Phi)$. In particular the complexified gauge coupling is related to the prepotential by the equation

$$\tau = \frac{\partial^2\mathcal{F}}{\partial\Phi^2}\ , \tag{2.11}$$

since we can integrate this relation to obtain the prepotential we will concentrate on determining the complexified gauge coupling as a function of the moduli space coordinate $u$.

The existence of singular points on the Coulomb branch follows from considering the positiveness of $\operatorname{Im}\tau = 4\pi/g^2$. If the prepotential was holomorphic everywhere on the Coulomb branch, $\operatorname{Im}\tau$ would have to be a harmonic function and unless it was constant it would necessarily be negative somewhere. This would result in the existence of negative norm states leading to violation of unitarity[3]. Consequently the prepotential can only be holomorphic in patches around singular points on the Coulomb branch.

Seiberg and Witten [10,11] found a way to give geometric meaning to $\tau$ guaranteeing positivity of its imaginary part in the process. They introduced the auxiliary object called the Seiberg-Witten curve which takes the following form for our example of $\mathcal{N}=2$ $SU(2)$ pure Yang-Mills [10]

$$y^2 = (x-\Lambda^2)(x+\Lambda^2)(x-u)\ , \tag{2.12}$$

where $\Lambda$ is the dynamically generated mass scale.

[3]This condition can be relaxed in non-unitary theories such as the ones considered in [13]. Note however, that the authors of that paper still find Seiberg-Witten curves for their theories. We will also encounter non-unitary theories in Chapter 4.

This family of elliptic curves describes the double cover of the complex $x$ plane with branch points at $\{-\Lambda^2, \Lambda^2, u, \infty\}$. The two sheets are connected by the branch cuts between the pairs of branch points $(-\Lambda^2, \Lambda^2)$ and $(u, \infty)$. Topologically this complex one dimensional space can be thought of as two spheres connected by tubes along the branch cuts. This is equivalent to a torus as illustrated in Figure 1. To establish connection between the complexified gauge coupling and the Seiberg-Witten curve, first we need to introduce some quantities of the torus associated to the latter.

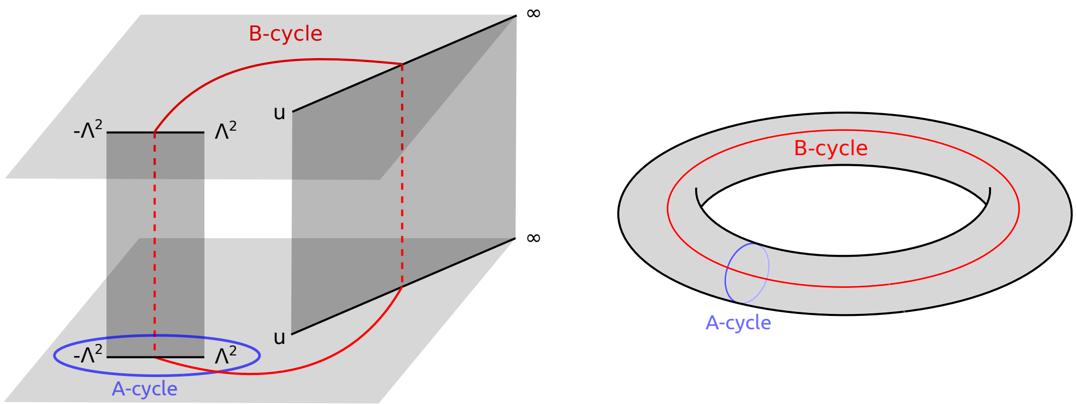



**Figure 1:** The two-sheeted $x$-plane representing the Seiberg-Witten curve (2.12) and the corresponding torus [12].

The modulus $\tau$ of a torus is defined as the ratio of its periods

$$\tau = \frac{\omega}{\omega_D} \, , \tag{2.13}$$

in turn the periods are computed by the following integrals around nontrivial cycles of the torus

$$\omega = \oint_A \frac{\mathrm{d}x}{y} \qquad\qquad \omega_D = \oint_B \frac{\mathrm{d}x}{y} \, . \tag{2.14}$$

Since the imaginary part of the modulus of a torus is guaranteed to be positive definite, it is natural to identify it with the complexified gauge coupling. Starting with the Seiberg-Witten curve then, the complexified gauge coupling can be obtained as a function of $u$ by performing the period integrals in (2.14).

The singular behaviour occurs on the Coulomb branch at the points $u = -\Lambda^2, \Lambda^2, \infty$ where two branch points collide and the corresponding torus degenerates with one or both of its nontrivial cycles shrinking to zero size.

Another important class of quantities we can compute directly from the curve are the masses of a stable BPS particles with electric and magnetic charges $(n_e, n_m)$ determined by the following formula

$$M^2 = 2|Z|^2 = 2|n_e a(u) + n_m a_D(u)|^2 \, , \tag{2.15}$$

where $a_D = \partial\mathcal{F}/\partial a$ is dual to $a$, and classically it is given by the relation $a_D = \tau a$. Similarly to the periods, the dual quantities $a$ and $a_D$ can be computed from the curve by integrating the Seiberg-Witten differential

$$\mathrm{d}\lambda = \frac{\sqrt{2}}{2\pi}\frac{(x-u)}{y}\mathrm{d}x \;, \tag{2.16}$$

along the nontrivial cycles of the torus

$$a = \oint_A \mathrm{d}\lambda \;, \qquad\qquad a_D = \oint_B \mathrm{d}\lambda \;. \tag{2.17}$$

We can see the emergence of massless dyons at points on the Coulomb branch where the torus associated to the Seiberg-Witten curve degenerates.

#### Generalizations

The above construction can easily be generalized for $\mathcal{N}=2$ gauge theories with larger gauge groups or hypermultiplets.

For a general gauge group $G$, the Coulomb branch is an $r = \operatorname{rank} G$ complex dimensional space with $r$ coordinates $u_I$. The Seiberg-Witten curve describes a genus $r$ Riemann surface that is the double cover of the $x$ plane with $r+1$ branch cuts. The complexified gauge coupling becomes a matrix $\tau_{IJ}$ that can be computed by integrals around the $2r$ nontrivial cycles of the Riemann surface analogous to (2.14).

The introduction of hypermultiplets lead to further singularities on the Coulomb branch at points where some of the hypermultiplets turn massless.

It is clear that the crucial ingredient to obtaining the low-energy effective theory on the Coulomb branch of an $\mathcal{N}=2$ gauge theory is its Seiberg-Witten curve. The question is then how can we find such a curve? Unfortunately there is no known prescription in general, although in simple cases it can often be guessed. For 4D $\mathcal{N}=2$ theories with M-theory/brane origin the curve can be derived systematically [14, 15]. We will encounter such a construction a bit later when we introduce theories of class S.

## 2.2 $\mathcal{N}=2$ Superconformal Field Theories

In this section we introduce $\mathcal{N}=2$ superconformal field theories that will be the main focus of the thesis. Conformal field theories are important because they represent endpoints of renormalization group flows of UV complete QFTs.[4] As such, they enjoy an enlarged symmetry group compared to ordinary QFTs which enables us to obtain more detailed information about them. When we combine this conformal symmetry with $\mathcal{N}=2$ SUSY, the resulting $\mathcal{N}=2$ superconformal symmetry allows us to depart

[4]See [16] for a detailed review on the topic of scale versus conformal invariance.

from the conventional Lagrangian setup and consider more general theories that arise quite naturally in the study of $\mathcal{N}=2$ SCFTs. First though, we introduce Lagrangian SCFTs.

### 2.2.1 Lagrangian $\mathcal{N}=2$ SCFTs

Conformal field theories are necessarily scale invariant which translates into the vanishing of their gauge coupling's beta function. In supersymmetric theories the beta function is one loop exact and is determined by the Novikov-Shifman-Vainshtein-Zakharov formula. In the case of $\mathcal{N}=2$ Lagrangian gauge theories it takes the following simple form [8]

$$\Lambda\frac{d}{d\Lambda}\frac{8\pi^2}{g^2} = 2C(adj) - \sum_i C(\mathcal{R}_i) \ , \tag{2.18}$$

where the first term on the right hand side is the contribution of the $\mathcal{N}=2$ vector multiplet in the adjoint of the gauge group while the second term comes from the $\mathcal{N}=1$ chiral multiplets (indexed by $i$) inside hypermultiplets, transforming in representation $\mathcal{R}_i$ of the gauge group[5]. $C$ is the Dynkin index of the representation defined by the equation

$$\operatorname{tr}\rho(T^a)\rho(T^b) = C(\rho)\delta^{ab} \ , \tag{2.19}$$

where $\rho(T^a)$ are the gauge group generators $T^a$ in representation $\mathcal{R}$.

In an $SU(N)$ gauge theory with fundamental hypermultiplets, the $\mathcal{N}=1$ chiral multiplets transform in the the adjoint, fundamental and anti-fundamental representations. The corresponding Dynkin indices are the following

$$C(adj) = N, \qquad C(\square) = C(\bar{\square}) = \frac{1}{2} \ . \tag{2.20}$$

With these in hand we can calculate the beta function for an $\mathcal{N}=2$ $SU(N)$ gauge theory with $N_f$ hypermultiplets

$$\Lambda\frac{d}{d\Lambda}\frac{8\pi^2}{g^2} = 2N - \frac{1}{2}N_f - \frac{1}{2}N_f = 2N - N_f \ . \tag{2.21}$$

In the case of $N_f = 2N$ hypermultiplets, the beta function vanishes giving us our first example of an $\mathcal{N}=2$ SCFT.

The second important example of an SCFT is the $\mathcal{N}=2$ gauge theory with a single hypermultiplet transforming in the adjoint representation, which is just the $\mathcal{N}=2$ formulation of $\mathcal{N}=4$ super Yang-Mills theory.

In both of these cases the gauge coupling is exactly marginal, meaning that its beta

[5] If the gauge theory also involves non-Lagrangian matter components in the form of strongly coupled SCFTs then their contribution to the beta function can be computed from the central charge of their flavor current algebra, part of which is being gauged [17].

function vanishes independently of the value of the coupling. Different values of the exactly marginal coupling then all define distinct CFTs. Spaces of such CFTs are called conformal manifolds and have many interesting properties [18,19]. In particular it can be shown for SCFTs, that a marginal coupling can only fail to be exactly marginal if it pairs up with a conserved current of a global flavor symmetry (from the point of view of $\mathcal{N} = 1$ SUSY) and in this case it must be marginally irrelevant [18]. As a direct consequence, dimension two $\mathcal{N} = 2$ chiral operators (also known as Coulomb branch operators) must be exactly marginal, since the corresponding deformations in the Lagrangians are uncharged under flavor symmetries and therefore cannot pair up with flavor currents [20,21]. We will encounter conformal manifolds in the next chapter, but for now we turn to analyzing the superconformal algebra (SCA) and its representations.

### 2.2.2 $\mathcal{N} = 2$ superconformal representations

The $\mathcal{N} = 2$ superconformal algebra contains the conformal algebra extended by supercharges ${Q^i}_\alpha, \bar{Q}_{i\dot{\alpha}}$ and conformal supercharges ${S_i}^\alpha, \bar{S}^{i\dot{\alpha}}$. Together these form the superalgebra $SU(2,2|2)$.[6] The spectrum of SCFTs organise into representations of the SCA indexed by quantum numbers of its maximal bosonic subalgebra

$$SO(4,2) \times U(2) \subset SU(2,2|2) \; , \tag{2.22}$$

where $SO(4,2)$ is the conformal algebra in 3+1 dimensions and $U(2)$ is the $R$-symmetry group. The conformal algebra contains generators of the Lorentz group along with those of scale and special conformal transformations. The quantum numbers under the Lorentz group $SU(2)_1 \times SU(2)_2$ will be denoted by $j_1, j_2$ and $E$ stands for the conformal dimension. A $U(1)$ factor is now truly part of the $R$-symmetry group of the SCA along with the usual $SU(2)$ we have encountered before for non-conformal $\mathcal{N} = 2$ theories. We denote the quantum numbers under these as $r$ and $R$ respectively.

Superconformal representations are built upon states called superconformal primaries annihilated by the conformal supercharges $S$ and $\bar{S}$, by the action of the 8 supercharges $Q, \bar{Q}$ and $SU(2)_R$ generators. In general this construction leads to a multiplet that is $2^8 = 256$ times as large as the Lorentz times $R$-symmetry representation of the superconformal primary state, this is called a long multiplet. When quantum numbers of the SC primary satisfy some specific relation, a subset of the supercharges annihilate it and we end up with a short multiplet. A complete classification of short multiplets and their shortening conditions in 4D $\mathcal{N} = 2$ SCFTs can be found in [22]. Instead of discussing these in detail, here we present only some of the short multiplets that will be useful for us later.

First, a few words on notation, long multiplets are denoted by $\mathcal{A}^E_{R,r(j_1,j_2)}$ where the

[6] See [22] for more details, e.g. generators and commutation relations of $SU(2,2|2)$.

indices are the quantum numbers of the SC primary. Short multiplets can be identified by the type of shortening condition and a subset of the quantum numbers, since the remaining can be determined from the shortening condition itself.

For example $\mathcal{E}_{r(0,0)} \equiv \mathcal{E}_r$ multiplets[7] are annihilated by all of the unbarred $\mathcal{Q}$ supercharges, and as a consequence satisfy the conditions $E = r$ and $R = 0$. In Lagrangian theories superconformal primaries of $\mathcal{E}_r$ multiplets are gauge invariant composites of vector multiplet scalars parametrizing the Coulomb branch. Similarly in Lagrangian gauge theories the Higgs branch is parametrized by SC primaries in $\hat{\mathcal{B}}_R$ multiplets that are the gauge invariant composites of hypermultiplet scalars. Although in non-Lagrangian theories these primaries lack the interpretation in terms of vector multiplet or hypermultiplet scalars, their vacuum expectation values still parametrize the Coulomb and Higgs branches, whose definition can be extended to non-Lagrangian theories as the subspaces of the moduli space where $U(1)_r$ and $SU(2)_R$ are broken respectively.

Temporarily going back to the realm of Lagrangian theories, free vector multiplets and hypermultiplets sit in $\mathcal{D}_{0(0,0)} \oplus \bar{\mathcal{D}}_{0(0,0)}$ and $\hat{\mathcal{B}}_{\frac{1}{2}}$ multiplets respectively. A cousin of the former is the $\mathcal{D}_{\frac{1}{2}(0,0)} \oplus \bar{\mathcal{D}}_{\frac{1}{2}(0,0)}$ multiplet that houses extra supersymmetry currents on top of the $\mathcal{N} = 2$.

Superconformal primaries of $\hat{\mathcal{C}}$-type multiplets satisfy $r = j_2 - j_1$. The most important multiplet of this type is $\hat{\mathcal{C}}_{0(0,0)}$, also known as the stress tensor multiplet because it contains the stress tensor along with the conserved $R$-symmmetry and supersymmetry currents. Generalizations of this multiplet of the form $\hat{\mathcal{C}}_{0(j_1,j_2)}$ contain higher-spin currents corresponding to decoupled free sectors of the SCFT.

Finally, we mention the $\hat{\mathcal{B}}_1$ multiplet which includes the conserved flavor symmetry current of the theory and is therefore called the flavor-current multiplet. We will learn more about short multiplets in the next section where we introdce the superconformal index.

### 2.2.3 Superconformal Index

The superconformal index [23] is a useful quantity as it contains information about the protected spectrum of an SCFT. It is insensitive to continuous deformations of the theory which renders it computable in many cases of interest.

We define the index with respect to a chosen Poincaré supercharge $\mathcal{Q}$, as a trace over the Hilbert space $\mathcal{H}$ of the theory in radial quantization on $\mathcal{S}^3$ [24]

$$\mathcal{I}(\mu_i) = \mathrm{Tr}_{\mathcal{H}} \, (-1)^F e^{-\mu_i T_i} e^{-\beta \delta} \;, \tag{2.23}$$

where $F$ is the fermion number, $\delta = 2\{\mathcal{Q}, \mathcal{Q}^\dagger\}$ and the $T_i$ are a complete set of com-

[7] The letter $\mathcal{E}$ refers to the particular shortening condition in [22].

muting operators that also commute with $\mathcal{Q}$.[8] This index is a generalization of the Witten index [25] with the chemical potentials $\mu_i$ introduced to refine the counting and regulate contributions from an infinite number of states.

The superconformal index is actually independent of $\beta$ since any state $\phi$ with $\delta \neq 0$ pairs up with state $\mathcal{Q}\phi$ whose quantum numbers under the $T_i$ are the same but their fermion numbers differ by one and therefore their contributions to the index cancel pairwise. This Witten index argument shows that only states with $\delta = 0$ contribute to the index making it independent of $\beta$. Since states with $\delta = 0$ are annihilated by $\mathcal{Q}$ and $\mathcal{Q}^\dagger$, the index counts states in certain short multiplets of the superconformal algebra. This also means that long multiplets do not contribute to the index and neither do sets of short multiplets that are allowed to combine into long multiplets by the rules of superconformal representation theory, see e.g. [26]. This property makes it difficult to extract precise information from the index about the spectrum of short multiplets that can recombine into long ones.[9] At the same time it is also responsible for the usefulness of the index as we explain next.

At the level of the spectrum, exactly marginal deformations of an SCFT can at most lead to splitting of a long multiplet into several short multiplets or recombination in the reverse direction, but since the index is independent of these collections of short multiplets by construction, we conclude it is invariant under all exactly marginal deformations of the theory. Furthermore, the index can be viewed as a supersymmetric partition function on $S^3 \times S^1$, in which case it also makes sense for theories without conformal invariance. This supersymmetric partition function can be shown to be invariant under RG flow, which enables us to calculate the index of an SCFT associated with an IR fixed point of an RG flow from the index of the UV theory, so long as the UV symmetries are non-anomalous along the flow and are identified correctly in the IR.

### 2.2.4 $\mathcal{N} = 2$ index

So far we have not defined the superconformal index concretely enough for our later purposes. We remedy this situation in the current section, focusing on the 4D $\mathcal{N} = 2$ index. We choose the index defining supercharge to be $\mathcal{Q} = \tilde{\mathcal{Q}}_{1\dot{-}}$.[10] and define the index as [26]

$$\mathcal{I}(p,q,t) = \mathrm{Tr}_{\mathcal{H}} \, (-1)^F p^{j_1+j_2-r} q^{j_2-j_1-r} t^{R+r} \prod_{i=1}^{r} (x_i)^{f_i} \ , \tag{2.24}$$

[8] This definition of the superconformal index is also suitable for non-conformal field theories in which case it is referred to as the supersymmetric index.

[9] Note however that there are certain short multiplets that are absolutely protected and cannot recombine at all, for $\mathcal{N} = 2$ SCFTs these include $\mathcal{E}$-type multiplets, $\hat{\mathcal{B}}_{\frac{1}{2}}$, $\hat{\mathcal{B}}_1$, $\mathcal{D}_0$, $\mathcal{D}_{\frac{1}{2}}$ and their conjugates [27].

[10] Different choices lead to equivalent indices for 4D $\mathcal{N} = 2$ theories.

| Field | $E$ | $j_1$ | $j_2$ | $R$ | $r$ | $\mathcal{I}(p,q,t)$ |
|---|---|---|---|---|---|---|
| $\phi$ | 1 | 0 | 0 | 0 | $-1$ | $pq/t$ |
| $\lambda_{1\pm}$ | $\frac{3}{2}$ | $\pm\frac{1}{2}$ | 0 | $\frac{1}{2}$ | $-\frac{1}{2}$ | $-p,\ -q$ |
| $\bar{\lambda}_{1\dot{+}}$ | $\frac{3}{2}$ | 0 | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $-t$ |
| $\bar{F}_{\dot{+}\dot{+}}$ | 2 | 0 | 1 | 0 | 0 | $pq$ |
| $\partial_{-\dot{+}}\lambda_{1+} + \partial_{+\dot{+}}\lambda_{1-} = 0$ | $\frac{5}{2}$ | 0 | $\frac{1}{2}$ | $\frac{1}{2}$ | $-\frac{1}{2}$ | $pq$ |
| $q$ | 1 | 0 | 0 | $\frac{1}{2}$ | 0 | $\sqrt{t}$ |
| $\bar{\psi}_{\dot{+}}$ | $\frac{3}{2}$ | 0 | $\frac{1}{2}$ | 0 | $-\frac{1}{2}$ | $-pq/\sqrt{t}$ |
| $\partial_{\pm\dot{+}}$ | 1 | $\pm\frac{1}{2}$ | $\frac{1}{2}$ | 0 | 0 | $p,\ q$ |

**Table 1:** Quantum numbers and index contributions of fields in $\mathcal{N}=2$ vector multiplets, half-hypermultiplets and derivatives [26].

where the fugacities $p,\ q,\ t$ are exponentials of the chemical potentials and the $x_i$ are fugacities counting flavor symmetry weights $f_i$ of the rank $r$ flavor group. For our chosen supercharge $\delta \equiv 2\{\mathcal{Q},\mathcal{Q}^\dagger\} = E-2j_2-2R+r$ and therefore only states satisfying

$$E - 2j_2 - 2R + r = 0\ , \tag{2.25}$$

contribute to the index.

With the help of the explicit formula (2.24) we can easily compute the superconformal index of $\mathcal{N}=2$ Lagrangian SCFTs. Table 1 lists the "single letter" index contributions of component fields in $\mathcal{N}=2$ vector and half-hypermultiplets. Note that two components of the derivative operator contribute to the index, so if we want to calculate the index of a vector multiplet or hypermultiplet, we need to include an infinite tower of states obtained by the action of these derivative operators on single letter fields. To account for this, we will multiply the single letter indices by the following factor

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} p^m q^n = \frac{1}{(1-p)(1-q)}\ . \tag{2.26}$$

The index of the $\mathcal{N}=2$ vector multiplet can now be assembled with the help of Table 1

$$\mathfrak{i}_v(p,q,t) = \frac{pq/t - p - q - t + pq + pq}{(1-p)(1-q)} = -\frac{p}{1-p} - \frac{q}{1-q} + \frac{pq/t - t}{(1-p)(1-q)}\ , \tag{2.27}$$

where the last $+pq$ term in the numerator accounts for the overcounting due to the equation of motion constraint $\partial_{-\dot{+}}\lambda_{1+} + \partial_{+\dot{+}}\lambda_{1-} = 0$ and its derivatives. Switching to the half-hypermultiplet ($\mathcal{N}=1$ chiral multiplet) we have

$$\mathrm{i}_{\frac{1}{2}h}(p,q,t) = \frac{\sqrt{t} - pq/\sqrt{t}}{(1-p)(1-q)} \ . \tag{2.28}$$

To account for products of fields and their derivatives we can use of the Plethystic Exponential defined as

$$\text{P.E.}[f(a_1,\ldots,a_p)] \equiv \exp\left[\sum_{n=1}^{\infty} \frac{1}{n} f(a_1^n,\ldots,a_p^n)\right] \ . \tag{2.29}$$

The full index contributions of the Lagrangian components are the Plethystic Exponentials of the single letter indices (2.27) and (2.28) [28]. In general vector and hypermultiplets come in representations of the gauge and flavor groups of the theory. Our discussion can easily accomodate these if we include group theory characters $\chi$ for these representations. For example the index of an $\mathcal{N}=2$ vector multiplet in the adjoint representation of the gauge group $G$ takes the form

$$\mathcal{I}_v(p,q,t,U) = \text{P.E.}\left[\mathrm{i}_v(p,q,t)\chi^G_{\text{adj}}(U)\right] \ , \tag{2.30}$$

where $U$ is an element of the gauge group and $\chi^G_{\text{adj}}(U)$ is the character of the adjoint representation of $G$. Similarly the index of a full hypermultiplet in representation $\mathcal{R}_G$ of the gauge and $\mathcal{R}_F$ of the flavour group $F$ is

$$\mathcal{I}_h(p,q,t,U,V) = \text{P.E.}\left[\mathrm{i}_{\frac{1}{2}h}(p,q,t)(\chi_{\mathcal{R}_G}(U)\chi_{\bar{\mathcal{R}}_F}(V) + \chi_{\bar{\mathcal{R}}_G}(U)\chi_{\mathcal{R}_F}(V))\right] \ , \tag{2.31}$$

with $V$ an element of the flavor group and bars denoting conjugate representations. Using the above ingredients we can finally write down the superconformal index of a conformal $\mathcal{N}=2$ SQCD, with the help of the following gauge integral

$$\mathcal{I}(V,p,q,t) = \tag{2.32}$$
$$\int [dU]\, \text{P.E.}\left[\mathrm{i}_{\frac{1}{2}h}(p,q,t)(\chi_{\mathcal{R}_G}(U)\chi_{\bar{\mathcal{R}}_F}(V) + \chi_{\bar{\mathcal{R}}_G}(U)\chi_{\mathcal{R}_F}(V)) + \mathrm{i}_v(p,q,t)\chi^G_{\text{adj}}(U)\right] \ ,$$

where the Haar measure $[dU]$ filters out contributions from operators that are not gauge invariant.

This gauging prescription is applicable more broadly in gauge theories where some subset of the hypermultiplets are replaced by isolated non-Lagrangian SCFTs. We will make use of this shortly in the next chapter. This concludes our discussion about the index in its full generality and we now turn to its special limits introduced in [26].

### 2.2.5 Limits of the index

Interesting limits of the index can be obtained by taking certain limits in some of the $p, q, t$ fugacities. These special indices count only a subset of operators that contribute to the full index and consequently some information about the spectrum is lost. On the other hand, those multiplets that do get counted enjoy more supersymmetry and therefore the corresponding limit is easier to compute. Our discussion closely follows [26].

#### Macdonald limit

The Macdonald limit[11] is defined by taking $p \to 0$, leaving two independent parameters $q$ and $t$. In this limit the definition of the index (2.24) reduces to

$$\mathcal{I}_M(q,t) = \mathrm{Tr}_M\,(-1)^F q^{\frac{1}{2}(E-2j_1-2R-r)} t^{R+r} \ , \tag{2.33}$$

where $\mathrm{Tr}_M$ means we are only tracing over the subspace of states satisfying

$$E + 2j_1 - 2R - r = 0 \ . \tag{2.34}$$

These states are $\frac{1}{4}$-BPS, being annihilated by two supercharges $\mathcal{Q}_{1+}$ and the usual $\tilde{Q}_{1\dot{-}}$. They can be found only in $\hat{\mathcal{C}}$, $\mathcal{D}$, $\bar{\mathcal{D}}$ and $\hat{\mathcal{B}}$-type multiplets. The single letter indices of half-hypermultiplets and vectormultiplets in the Macdonald limit reduce to

$$\mathrm{i}_M^{\frac{1}{2}h}(q,t) = \frac{\sqrt{t}}{1-q}, \qquad \mathrm{i}_M^{v}(q,t) = \frac{-t-q}{1-q} \ . \tag{2.35}$$

#### Hall-Littlewood limit

The Hall-Littlewood limit can be obtained from the Macdonald by also taking $q \to 0$, the trace formula for the index now becomes

$$\mathcal{I}_{HL}(t) = \mathrm{Tr}_{HL}\,(-1)^F t^{E-R} \ , \tag{2.36}$$

where we are tracing over states satisfying

$$E \pm 2j_1 - 2R - r = 0 \ . \tag{2.37}$$

These two constraints taken together with (2.25) mean that states contributing to the Hall-Littlewood limit satisfy

$$j_1 = 0 \ , \qquad j_2 = r \ , \qquad E = 2R + r \ . \tag{2.38}$$

[11] The first three limits are named after classes of symmetric polynomials relevant for their evaluation.

These states are annihilated by three supercharges $\mathcal{Q}_{1+}$, $\mathcal{Q}_{1-}$, $\tilde{Q}_{1\dot{-}}$ and sit in short multiplets $\mathcal{D}$ and $\hat{\mathcal{B}}$. Corresponding operators form the Hall-Littlewood antichiral ring, which is a consistent truncation of the $\mathcal{N}=1$ antichiral ring. The single letter indices of the half-hypermultiplets and vector multiplets take the simple form

$$\mathfrak{i}^{\frac{1}{2}h}_{HL}(t) = \sqrt{t}\ , \qquad \mathfrak{i}^{v}_{HL}(t) = -t\ . \tag{2.39}$$

**Schur limit**

The Schur limit of the superconformal index will play a central role in the following chapters due to its involvement in the 4D/2D chiral algebra correspondence discussed in the next section. It is obtained from the general index by setting $t=q$ which eliminates the fugacity $t$. The resulting index turns out to be independent of $p$ and is therefore a function of $q$ only. In the Schur limit, the trace formula becomes

$$\mathcal{I}_S(q) = \mathrm{Tr}\,(-1)^F q^{E-R}\ . \tag{2.40}$$

In addition to (2.25) states contributing to the Schur limit of the index also satisfy

$$r = j_2 - j_1\ . \tag{2.41}$$

Like the Hall-Littlewood limit, the Schur limit is also a special case of the Macdonald limit obtained by taking $t=q$, due to its independence of $p$. At the same time the Macdonald and Schur limits count the same set of operators called Schur operators, but the former achieves a more refined counting owing to the extra fugacity $t$.

There is a single conformal primary Schur operator living in every $\hat{\mathcal{C}}$, $\mathcal{D}$, $\bar{\mathcal{D}}$ and $\hat{\mathcal{B}}$-type short multiplet and they are the highest weight components of their Lorentz and $R$-symmetry representations [27]. The other Schur operators in these multiplets can be obtained from the conformal primaries by the action of $\partial_{+\dot{+}}$ which itself satisfies the Schur conditions (2.25) and (2.47). Schur operators in $\hat{\mathcal{B}}_R$ multiplets are also the superconformal primaries, whose vacuum expectation values parametrize the Higgs branch.

The single letter indices for the half-hypermultiplets and vector multiplets take the following form in the Schur limit

$$\mathfrak{i}^{\frac{1}{2}h}_{S}(q) = \frac{\sqrt{q}}{1-q}\ , \qquad \mathfrak{i}^{v}_{S}(q) = -\frac{2q}{1-q}\ , \tag{2.42}$$

the factor $1/(1-q)$ accounts for the derivative contributions.

### Coulomb branch limit

Finally, the Coulomb branch limit of the index is defined by $p, q, t \to 0$ with the ratio $\frac{pq}{t} \equiv T$ kept fixed. In this limit, the trace formula becomes

$$\mathcal{I}_C(T) = \mathrm{Tr}_C\,(-1)^F T^{j_1+j_2-r}\ , \tag{2.43}$$

where the trace is restricted to states satisfying

$$E + 2j_2 + 2R + r = 0\ . \tag{2.44}$$

which are annihilated by $\tilde{\mathcal{Q}}_{2\dot{+}}$ and $\tilde{\mathcal{Q}}_{1\dot{-}}$. These states reside in the chiral $\bar{\mathcal{E}}_{r(j_1,0)}$ and $\bar{\mathcal{D}}_{0(j_1,0)}$-type multiplets.[12] The single letter indices simplify considerably to

$$\mathfrak{i}_C^{\frac{1}{2}h}(T) = 0, \qquad\qquad \mathfrak{i}_C^{v}(T) = T\ . \tag{2.45}$$

As its name suggests, only the vector multiplets contribute to the Coulomb branch limit and information about flavor symmetry is lost.

Having introduced special limits of the index, we now turn to the chiral algebra correspondence and its close relation to Schur operators.

## 2.3 Chiral algebra correspondence

In this section we summarise the 4D/2D chiral algebra correspondence discovered in [27], which associates to any 4D $\mathcal{N} = 2$ SCFT a chiral algebra in two dimensions.[13] Since chiral algebras enjoy an infinite dimensional symmetry group, this relation affords us glimpses into the world of 4D $\mathcal{N} = 2$ SCFTs, even when they are strongly coupled. For example, the chiral algebra encodes information about the moduli spaces of SCFTs. The fact that it knows about the Higgs branch [30–35] does not come as a surprise, as it is related to the Schur sector of the SCFT which includes the Higgs branch operators as a subset. The Higgs branch can even be identified with the so-called associated variety of the chiral algebra [36]. Surprisingly, the chiral algebra also captures information about the Coulomb branch [37–40].

### 2.3.1 Chiral algebra from 4D $\mathcal{N} = 2$ SCFT

To construct the chiral algebra of a four-dimensional $\mathcal{N} = 2$ SCFT we first fix the chiral algebra plane $x_1 = x_2 = 0$ in $\mathbb{R}^4$ and introduce on it the complex coordinates $z = x_3 + ix_4$ and $\bar{z} = x_3 - ix_4$. We then pick out the nilpotent supercharge $\mathbb{Q} \equiv \mathcal{Q}^1 + \tilde{\mathcal{S}}^2$

[12] Exotic chiral operators with $j_1 > 0$ cannot exist in local SCFTs [20, 29].

[13] Chiral algebras are the set of symmetries of left-movers or right-movers of 2D CFTs. They are often called vertex operator algebras, especially in the mathematics literature.

and look for operators inserted at the origin that define nontrivial cohomology classes with respect to this supercharge

$$\{\mathbb{Q}, \mathcal{O}(0)] = 0 \ , \qquad \mathcal{O}(0) \neq \{\mathbb{Q}, \mathcal{O}'(0)] \ . \tag{2.46}$$

Such operators must satisfy the following relations among their quantum numbers

$$E = j_1 + j_2 + 2R \ , \qquad r = j_2 - j_1 \ . \tag{2.47}$$

These are the same conditions defining Schur operators (2.25) and (2.47) we have encountered earlier!

We have to be careful in defining these operators away from the origin, since translations in the $\bar{z}$ direction are not $\mathbb{Q}$-exact, and therefore take us out of the cohomology class defined by the Schur operator at the origin. To remedy this situation we have to prescribe a very specific $\bar{z}$ dependence using a translation operator twisted by a particular $SU(2)_R$ generator such that the twisted translation operator will be $\mathbb{Q}$-exact. There is a concise formula for the resulting twisted-translated operator in terms of the ordinary Schur operator defined at the point $(z, \bar{z})$ and other components from its $SU(2)_R$ representations

$$\mathcal{O}(z, \bar{z}) = u_{\mathcal{I}_1}(\bar{z}) \cdots u_{\mathcal{I}_{2k}}(\bar{z}) \mathcal{O}^{\mathcal{I}_1 \dots \mathcal{I}_{2k}}(z, \bar{z}) \ , \tag{2.48}$$

where $u_{\mathcal{I}}(\bar{z}) = (1, \bar{z})$ and we index spin $k$ $SU(2)_R$ representations $\mathcal{O}^{\mathcal{I}_1 \dots \mathcal{I}_{2k}}$ using $2k$ binary indices $\mathcal{I}_i = 1, 2$. Since now the dependence on $\bar{z}$ is $\mathbb{Q}$-exact, cohomology classes of twisted-translated Schur operators depend holomorphically on their insertion points while their correlation functions are meromorphic, giving us operators and correlation functions of a 2D chiral algebra.

### 2.3.2 Notable elements of the chiral algebra correspondence

In this section we introduce the most important elements of the map $\chi$ between objects in 4D $\mathcal{N} = 2$ SCFTs and their chiral algebras arising from this construction. Most importantly the SCFT $\to$ chiral algebra map is not empty since any local 4D $\mathcal{N} = 2$ SCFT has a stress tensor multiplet $\hat{\mathcal{C}}_{0(0,0)}$ which contains a Schur operator. This operator is the highest $SU(2)_R$ weight component of the $SU(2)_R$ current $J^{11}_{+\dot{+}}$. The corresponding operator in the 2D chiral algebra is the stress tensor $T$

$$\chi \left[ J^{11}_{+\dot{+}} \right] = -\frac{1}{2\pi^2} T \ . \tag{2.49}$$

This can be seen by computing the OPE of the twisted-translated $J^{11}_{+\dot{+}}$ with itself, which is determined by $\mathcal{N} = 2$ superconformal invariance in terms of the central charge

$c_{4d}$ [27]. Ignoring $\mathbb{Q}$-exact operators, the resulting OPE is the well-known 2D stress tensor OPE

$$T(z)T(0) \sim \frac{c_{2d}/2}{z^4} + \frac{2T(0)}{z^2} + \frac{\partial T(0)}{z} , \tag{2.50}$$

with the 2D central charge identified as

$$c_{2d} = -12c_{4d} . \tag{2.51}$$

We thus learn that the chiral algebra of a 4D $\mathcal{N} = 2$ SCFT necessarily contains a Virasoro subalgebra generated by modes of $T$. This last relation also implies that the chiral algebra associated with a unitary SCFT must itself be non-unitary.

If the SCFT enjoys a continuous flavor symmetry then the corresponding conserved current $J_{\alpha\dot{\alpha}}$ sits in a $\hat{\mathcal{B}}_1$ multiplet. The Schur operator of this multiplet is the highest $SU(2)_R$ component of the moment-map $M^{IJ}$ transforming in the adjoint representation of the flavor group. The self-OPE of the twisted-translated Schur operator $M^{11}$ is determined in terms of the flavor central charge $k_{4d}$ and the structure constants of the Lie-algebra of the flavor group. This reduces (modulo $\mathbb{Q}$-exact terms) to the self-OPE of the affine current in 2D

$$J^A(z)J^B(0) \sim \frac{k_{2d}\delta^{AB}}{z^2} + \sum_C i f^{ABC} \frac{J^C(0)}{z} , \tag{2.52}$$

with the level of the affine algebra given by

$$k_{2d} = -\frac{k_{4d}}{2} . \tag{2.53}$$

We conclude that the flavor symmetry of the 4D SCFT enhances to affine symmetry in the 2D chiral algebra

$$\chi\left[M^{11}\right] = \frac{1}{2\sqrt{2}\pi^2} J . \tag{2.54}$$

The final piece of the correspondence we mention here concerns the superconformal index of the 4D SCFT. As we saw earlier, the Schur limit of the index is counting Schur operators[14] and because these are in one-to-one correspondence with operators of the chiral algebra, it is natural to expect this limit to be related to the partition function of the chiral algebra. We will now see how this relation works out precisely. The torus partition function of the chiral algebra is defined as the following trace over its local operators

$$Z(x, q) = \mathrm{Tr}\, x^{M_\perp} q^{L_0} , \tag{2.55}$$

the transverse spin and holomorphic dimension (eigenvalue of the $L_0$ operator) can be

[14] Actually we also saw that the Macdonald limit counts Schur operators as well and there are indeed results on how one can obtain the more general Macdonald index from the chiral algebra [34, 41, 42].

| 4D Schur sector | 2D chiral algebra |
|---|---|
| stress tensor multiplet | Virasoro symmetry |
| flavor current multiplet | affine symmetry |
| Schur limit of superconformal index | partition function |
| $\partial_{+\dot{+}}$ | $\partial_z$ |

**Table 2:** Summary of notable elements of 4D/2D chiral algebra correspondence.

expressed in terms of 4D quantum numbers as

$$M^{\perp} = j_1 - j_2 \ , \qquad\qquad h = \frac{E + j_1 + j_2}{2} = E - R \ , \tag{2.56}$$

where we used the Schur relation (2.47). We set $x = -1$ and identify the fermion number $F$ in $(-1)^{j_1 - j_2} = (-1)^F$. The torus partition function then becomes

$$Z(-1, q) = \mathrm{Tr}\,(-1)^{j_1 - j_2} q^{L_0} = \mathrm{Tr}\,(-1)^F q^{E-R} \ , \tag{2.57}$$

we recognise the last expression as the Schur limit of the superconformal index introduced in (2.40)! Under the chiral algebra map then, the Schur limit of the superconformal index becomes the torus partition function of the chiral algebra

$$\chi\left[\mathcal{I}_S(q)\right] = Z(-1, q) \ . \tag{2.58}$$

We will use this relation extensively in the following three chapters to study non-Lagrangian SCFTs known as Argyres-Douglas theories. Before moving on to introducing these theories, let us digress briefly about various characters of chiral algebras.

The torus partition function is also the character of the vacuum module and due to modularity, these often satisfy differential equations called linear modular differential equations (LMDEs).[15] In fact characters of rational conformal field theories (RCFTs) can be classified by listing LMDEs that produce characters with nonnegative integer coefficients in their $q$-expansions. So far this strategy seems to be viable only for RCFTs with a small number of characters, see [44, 45] and more recently [46]. One can also wonder about the role non-vacuum characters might play in the chiral algebra correspondence. Remarkably, these turn out to be related to surface defects in the 4D SCFT [47–51].

[15] For a recent review of LMDEs in the context of 4D $\mathcal{N} = 2$ SCFTs and their relation to the identification of the Higgs branch as the associated variety of the chiral algebra, see [43].

## 2.4 Argyres-Douglas theories

Argyres-Douglas theories are the main focus of this thesis, in this section we describe their most important properties and discuss known ways of constructing them.

The first AD theory was found in [52], as the low-energy theory at a singular point on the Coulomb branch of $\mathcal{N} = 2$ $SU(3)$ SYM, with the help of the Seiberg-Witten solution to $\mathcal{N} = 2$ gauge theories discovered shortly beforehand. The special property of this singular point is that dyons with nonzero Dirac pairing turn massless there. To explain the consequence of this, we define the Dirac pairing of two particles with electric and magnetic charges $(n, m)$ and $(n', m')$ as

$$nm' - mn' \ . \tag{2.59}$$

This is invariant under electromagnetic duality transformations that act on charge vectors by elements of $SL(2, \mathbb{Z})$. Consequently, if two dyons exist with nonzero pairing then it is impossible to go to a duality frame where only electrically charged particles exist, that is characterized by zero pairing. Since we do not know of a way to write down a manifestly Lorentz invariant Lagrangian for such theories, we call them non-Lagrangian. Another peculiar feature of AD theories, which we take to be their defining property is that they contain Coulomb branch operators with fractional scaling dimensions. To see this for the original AD theory we can look at its Seiberg-Witten curve [53]

$$x^2 = z^3 + mz + u \ . \tag{2.60}$$

The scaling dimensions of the parameters appearing in the curve can be found by setting the dimension of the Seiberg-Witten differential $\lambda = x\,\mathrm{d}z$ equal to one, and requiring that each term is homogeneous in scaling dimension. This leads to the following dimensions

$$[x] = \frac{3}{5}, \quad [z] = \frac{2}{5}, \quad [m] = \frac{4}{5}, \quad [u] = \frac{6}{5} \ . \tag{2.61}$$

Parameter $u$ has scaling dimension $6/5$ and corresponds to the single Coulomb branch operator of the theory. In a Lagrangian gauge theory, Coulomb branch dimensions are given by the degrees of Casimirs of the gauge group and are therefore integers. It follows that the existence of fractional dimension Coulomb branch operators is a direct manifestation of the non-Lagrangian nature of these theories.

Soon after the discovery of AD theories, more examples were found in $SU(2)$ gauge theories with enough hypermultiplets, and were shown to be interacting SCFTs in [54]. This was generalised for the cases of $SU(N)$ and $SO(N)$ gauge theories and was given an ADE classification in [55, 56], see also [57] for some corrections. This type of construction using the Seiberg-Witten curve of the parent theory, taking a singular limit to obtain the SW curve and through that the Coulomb data of the AD SCFT facilitated the discovery of AD theories, but it only offers information about their Coulomb branch. A family of AD theories indexed by two ADE gauge groups $(G, G')$ were constructed in [58], using type IIB String Theory compactified on a Calabi-Yau threefold with the pair of ADE singularities.[16] The construction we will make most use of is however the class $\mathcal{S}$ construction, which we will detail in the next section.

### 2.4.1 Review of Class $\mathcal{S}$ theories

Theories of class $\mathcal{S}$ [62, 63] are 4D $\mathcal{N} = 2$ SCFTs obtained as compactifications of 6D $(2, 0)$ SCFTs on punctured Riemann surfaces. Although the 6D $(2, 0)$ theory is non-Lagrangian and not well understood, many physical properties of a class $\mathcal{S}$ theory are encoded in the geometry of the Riemann surface $\mathcal{C}$. For example, we can construct the Seiberg-Witten curve from the Hitchin integrable system defined on $\mathcal{C}$, or we can calculate the Schur index from correlators of a topological quantum field theory (TQFT) defined on the Riemann surface [30, 64–66]. Although not all 4D $\mathcal{N} = 2$ SCFTs are in class $\mathcal{S}$, it is still a very rich set of theories and we quickly review their construction in the next section.

### 2.4.2 Class $\mathcal{S}$ construction

The class $\mathcal{S}$ construction [14, 47, 63] starts with a choice of 6D $(2, 0)$ theory of type $\mathfrak{g} = ADE$, which is a strongly coupled non-Lagrangian SCFT. This theory is then compactified on a Riemann surface with punctures coming from codimension-two defects of the parent 6D theory. The punctures carry flavor symmetries and come in two different types, regular and irregular [53, 63, 67, 68] the latter of which are necessary to construct class $\mathcal{S}$ Argyres-Douglas theories. The punctures correspond to boundary conditions of the Higgs field $\phi$ which constitutes a solution to Hitchin's equation on $\mathcal{C}$. We can recover the Seiberg-Witten curve of a class $\mathcal{S}$ theory from the spectral curve of the Hitchin system:

$$\det\left(x - \varphi(z)\right) = 0 \ , \tag{2.62}$$

where $\varphi$ is the holomorphic part of the Higgs field.

We can understand why the Hitchin system comes into play by noting that the Hitchin moduli space is the Higgs branch of the 5D maximal super Yang-Mills (MSYM) theory obtained from compactifying the parent 6D theory on a circle transverse to the Riemann surface. If we further compactify the 5D MSYM on $\mathcal{C}$, we get a 3D $\mathcal{N} = 4$ theory which is the mirror of the direct $S^1$ reduction of the 4D class $\mathcal{S}$ theory,

[16] More recently in [59–61], the authors found certain $\mathcal{N} = 1$ preserving deformations of $\mathcal{N} = 2$ Lagrangian SCFTs that trigger RG flows to some of the $(G, G')$ type theories. In principle, this connection allows us to calculate the full superconformal index, which has been checked against the known Schur indices of these theories.

and therefore its Higgs branch is identified with the Coulomb branch of the direct reduction [69].

Regular punctures [62, 63] correspond to the Higgs field with first order pole near the singularity, with the coefficients of the pole identified as the mass parameters of the flavor symmetry induced by the regular puncture. S-duality has a very nice geometric interpretation for class $\mathcal{S}$ theories, as different S-duality frames of an SCFT with punctured Riemann surface $\mathcal{C}$ correspond to different pants decompositions of $\mathcal{C}$ into thrice punctured spheres. The basic building blocks of class $\mathcal{S}$ are then the theories associated with thrice punctured spheres, often referred to as trinions, any other class $\mathcal{S}$ theory can be engineered from their exactly marginal gaugings. Geometrically, the gauge coupling is identified with the complex structure moduli of the punctured Riemann surface. Regular punctures are classified by Young Tableaux and there is no upper limit to the genus of the Riemann surface or the number of regular punctures we can use to build class $\mathcal{S}$ theories from.

Irregular punctures [53, 63] correspond to solutions with higher order poles in the Higgs field and are necessary to end up with fractional scaling dimension Coulomb branch operators and dimensionful coupling constants, two characteristic properties of AD theories.[17] The latter are encoded in the coefficients of the higher order poles of the Higgs field, while as in the regular case, the coefficient of the first order pole contains the mass parameters. Compared to Riemann surfaces with only regular punctures, the ones with irregular punctures are much more restricted. If we want to get an SCFT then the only allowed Riemann surface is the Riemann sphere and there can either be an irregular singularity alone or together with a regular singularity.[18] Even though the allowed number of irregular punctures is thus very limited, the richness of this type of puncture is demonstrated by the fact that they can realize any theory engineered using regular punctures on the Riemann sphere [53]. Classification of irregular singularities is done via a series of Young Tableaux $Y_n \subseteq Y_{n-1} \cdots \subseteq Y_1$ corresponding to poles of the Higgs field, where each Young Tableau is obtained from the previous by partitioning of its columns. We will encounter irregular punctures throughout the coming chapters.

[17] An important exception to this lore was found recently in [70] for class $\mathcal{S}$ theories engineered using type $A_{2n}$ $(2,0)$ theory with $\mathbb{Z}_2$ twisted regular punctures. Many of the AD theories considered in this thesis can be constructed this way.

[18] Multiple irregular punctures break $U(1)_R$ and lead to asymptotically free theories.

# Chapter 3

# $\mathcal{N} = 2$ S-duality Revisited

In this chapter we study an exotic Argyres-Douglas theory $\mathcal{T}_{3,\frac{3}{2}}$ involved in the AD counterpart of the famous Argyres-Seiberg S-duality (AS-duality). In order to compute the Schur limit of its superconformal index we need to invert a gauge integral in an index relation which follows from the invariance of the index under S-duality. An inversion formula borrowed from the study of elliptic hypergeometric integrals was applied successfully for the original AS-duality, but it is highly nontrivial that it can also be applied in our case, where some of the free fields are replaced by the $(A_1, D_4)$ AD theories. What nevertheless makes a modification of this method work is the uncanny resemblance between the indices of $(A_1, D_4)$ and free fields, hinting at a possible connection between the two.

Having obtained the index of $\mathcal{T}_{3,\frac{3}{2}}$ using the inversion formula, we can study its spectrum and we soon discover that $\mathcal{T}_{3,\frac{3}{2}}$ is actually a combination of a more elementary AD theory $\mathcal{T}_X$ and free hypermultiplets. From the Schur index of $\mathcal{T}_X$ we also make a simple conjecture for its chiral algebra, which we support by a bootstrap argument.[19] We use this in Sec. 3.6 to conjecture a simple closed-form expression for the Schur index of $\mathcal{T}_X$. In general the index can be viewed as the partition function of the theory on $S^3 \times S^1$ and this form of the index allows us to take the $q \to 1$ limit, which corresponds to the shrinking of the radius of the $S^1$ factor. The result should then agree with the partition function of the proposed $S^1$ reduction of $\mathcal{T}_X$ and indeed we find agreement between the two. In Sec. 3.7 we discuss some operator relations in the Hall-Littlewood chiral ring of $\mathcal{T}_X$ and their consequences for the Schur sector. These relations follow from our chiral algebra ansatz and the Hall-Littlewood limit of the index that we compute in Appendix D using the 3D mirror of $\mathcal{T}_X$. We also comment on the Witten anomaly of $\mathcal{T}_X$. The proof of an important index identity for $(A_1, D_4)$, details of the

[19] $\mathcal{T}_X$ has also found its place among the rank-two instanton SCFTs considered recently in [71]. The authors of that paper find the same chiral algebra using bootstrap approach on a minimal ansatz of generators descending from the Higgs branch chiral ring. They also arrive at the same result independently using a free field realization.

inversion formula and computations related to the $q \to 1$ limit of the index are relegated to separate appendices.

This chapter is based on [1].

## 3.1 Introduction

Four-dimensional superconformal field theories often admit exactly marginal deformations, we refered to these deformation spaces as conformal manifolds in the previous chapter. In the context of theories with $\mathcal{N} \geq 2$ supersymmetry, one can easily obtain examples with exactly marginal deformations by coupling a vector multiplet to precisely enough matter so that the one-loop beta function vanishes. A canonical example of this phenomenon occurs in $SU(N)$ $\mathcal{N} = 4$ Super Yang-Mills. At the level of the Lie algebra and the local operators, this theory is self-dual:[20] as we vary the exactly marginal gauge coupling, $\tau$, towards a strong-coupling cusp on the conformal manifold, an $S$-dual weakly coupled $SU(n)$ $\mathcal{N} = 4$ SYM theory emerges. A similar story holds in $SU(2)$ $\mathcal{N} = 2$ gauge theory with four fundamental flavors [11].

On the other hand, the $S$-duality in $SU(3)$ $\mathcal{N} = 2$ gauge theory with six fundamental flavors is dramatically different [17]. As one takes the gauge coupling to infinity, Argyres and Seiberg found that, instead of getting a weakly coupled $S$-dual description in terms of another $SU(3)$ gauge theory with fundamental matter, one instead finds a dual consisting of an $SU(2)$ theory coupled to a doublet of hypermultiplets and an $SU(2) \subset \mathfrak{e}_6$ factor of the global symmetry of the Minahan-Nemeschansky $E_6$ SCFT [73].

The message of [17] is clear: sometimes, starting from vanilla building blocks, the "matter" that appears via $\mathcal{N} = 2$ $S$-duality is not standard matter (i.e., hypermultiplets) but is instead a strongly coupled isolated SCFT[21] whose global symmetry (or a proper subgroup thereof) is weakly gauged. Moreover, $S$-duality can be a machine for generating exotic isolated theories.

This latter point was driven home in [62]. Indeed, Gaiotto generalized [17] to higher-rank gauge theories and, in the process, found an infinite number of new isolated SCFTs—the so-called $T_N$ theories—at strong-coupling cusps on the resulting conformal manifolds.[22] Since a $T_N$ theory has $SU(N)^3$ global symmetry[23] and the following central charge corresponding to each such $SU(N)$ flavor current algebra (and hence

[20]See [72] for a discussion of subtleties at the level of the gauge group and the line operators.

[21]By "isolated," we mean a theory that lacks an exactly marginal deformation.

[22]The $T_3$ case is just the $E_6$ SCFT of [73], and the $T_2$ case is eight free half-hypermultiplets. However, the $T_N$ SCFTs with $N \geq 4$ are new isolated theories.

[23]The $T_3$ case has an enhanced $E_6 \supset SU(3)^3$ global symmetry, but the discussion below applies to this theory as well. A similar discussion holds for the $T_2$ theory, which has $Sp(4) \supset SU(2)^3$ global symmetry.

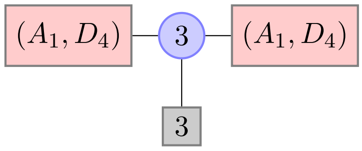



**Figure 2:** The quiver diagram describing the simplest (i.e., lowest rank) AD generalization of Argyres-Seiberg duality in the $SU(3)$ duality frame. The total flavor symmetry is $U(3)$. In [75], this theory was called the "$\mathcal{T}_{3,2,\frac{3}{2},\frac{3}{2}}$" SCFT.

1-loop beta function contribution upon gauging)

$$k^{T_N}_{SU(N)_i} = 2N \ , \quad i = 1, 2, 3 \ , \tag{3.1}$$

one can always find a non-trivial conformal manifold by taking two $T_N$ theories and gauging a diagonal $SU(N)$. Indeed, the contributions from the $T_N$ theories in (3.1) cancel those of the $SU(N)$ gauge fields

$$\beta^{1-\text{loop}}_{SU(N)} = -4N + 2N + 2N = 0 \ . \tag{3.2}$$

One can then proceed to construct a conformal manifold consisting only of arbitrarily many $T_N$ theories and conformal gauge fields.

While the above set of theories is quite vast, the $T_N$ theories (and their cousins) are somewhat special: their $\mathcal{N}=2$ chiral primaries have integer scaling dimensions.[24] The underlying reason is that these theories emerge in a duality with a Lagrangian theory.[25] On the other hand, the most generally allowed values for the scaling dimensions, $\Delta_i$, of $\mathcal{N}=2$ chiral operators are widely believed to be $\Delta_i \in \mathbb{Q}$, and non-integer rational values are indeed realized in Argyres-Douglas theories as we have seen before in Chapter 2. These theories cannot emerge in an $\mathcal{N}=2$ $S$-duality with a Lagrangian theory.

Motivated by a desire to understand $\mathcal{N}=2$ $S$-duality more broadly, it is then natural to ask what is the minimal (which we will define to be lowest rank[26]) AD generalization of Argyres-Seiberg (i.e., non self-similar) duality [75]. Since the starting point cannot be a Lagrangian theory, one must engineer such a conformal manifold from a weakly coupled gauging of a global symmetry of a collection of AD building blocks (potentially with additional hypermultiplets). An answer, using general consistency conditions and the class $\mathcal{S}$ Argyres-Douglas theories in [53], was given in [75] and is

[24] By $\mathcal{N}=2$ chiral primaries, we mean superconformal primaries that are annihilated by all the anti-chiral Poincaré supercharges of $\mathcal{N}=2$ SUSY.

[25] We called these $\mathcal{E}_r$-type operators earlier, by the rules of [22] these cannot disappear from the spectrum or, by the discussion in [74], have their dimensions renormalized as we vary $\tau$, so the $T_N$ $\mathcal{N}=2$ chiral ring generators must correspond to some subset of the gauge Casimirs of a Lagrangian theory.

[26] By rank, we mean the complex dimension of the Coulomb branch.

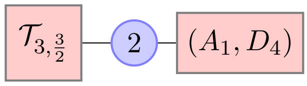



**Figure 3:** The quiver diagram describing the theory dual to the one in Fig. 2. The $SU(3) \subset U(3)$ symmetry is furnished by the $\mathcal{T}_{3,\frac{3}{2}}$ theory while the $U(1) \subset U(3)$ symmetry is furnished by the $(A_1, D_4)$ SCFT. In [75], this theory was called the "$\mathcal{T}_{3,2,\frac{3}{2},\frac{3}{2}}$" SCFT.

reproduced in Fig. 2 (there, this theory was referred to as the "$\mathcal{T}_{3,2,\frac{3}{2},\frac{3}{2}}$" SCFT). This theory is constructed by gauging the diagonal $SU(3)$ symmetry of three fundamental flavors and a pair of $(A_1, D_4)$ SCFTs (the $(A_1, D_4)$ theory, originally discussed in [54], has $SU(3)$ flavor symmetry and a single $\mathcal{N}=2$ chiral ring generator of dimension $3/2$). The resulting global symmetry is $U(3)$ and is furnished by the three fundamental flavors.

The $S$-dual frame of this theory is given in Fig. 3 and consists of an $SU(2)$ gauge theory coupled to an $(A_1, D_4)$ factor and a more exotic AD theory called the $\mathcal{T}_{3,\frac{3}{2}}$ SCFT [75] which has flavor symmetry $G \supset SU(3) \times SU(2)$.[27] Therefore, in rough analogy with Argyres-Seiberg duality, the strongly coupled $(A_1, D_4)$ theory plays the role of the hypermultiplets on the $SU(2)$ side of the duality and the $\mathcal{T}_{3,\frac{3}{2}}$ theory plays the role of the $E_6 = T_3$ theory.

However, upon closer inspection, the analogy with Argyres-Seiberg duality seems to break down. Indeed, the anomalies of the $\mathcal{T}_{3,\frac{3}{2}}$ theory were computed in [75] and found to be

$$k_{SU(2)}^{\mathcal{T}_{3,\frac{3}{2}}} = 5 \ , \quad k_{SU(3)}^{\mathcal{T}_{3,\frac{3}{2}}} = 6 \ , \quad c^{\mathcal{T}_{3,\frac{3}{2}}} = \frac{9}{4} \ , \quad a^{\mathcal{T}_{3,\frac{3}{2}}} = 2 \ . \tag{3.3}$$

Using these symmetries, one cannot construct conformal manifolds built only out of arbitrary numbers of $\mathcal{T}_{3,\frac{3}{2}}$ SCFTs and conformal gauge fields. The reason is that the contribution to the $SU(2)$ beta function in (3.3) is too large and the required $SU(2)$ gauging would be infrared free. This state of affairs is quite unlike the $E_6 = T_3$ case described above, where an arbitrary number of such theories can be concatenated by gauging enough diagonal symmetries.

Still, there are some puzzles in the above picture. To begin with, the flavor symmetry group of the $\mathcal{T}_{3,\frac{3}{2}}$ theory is not obvious. One standard way to find such symmetries for SCFTs that, like the $\mathcal{T}_{3,\frac{3}{2}}$ is of class $\mathcal{S}$ is to construct the Hitchin system corresponding to the theory [53, 63]. In particular, in the case of the $\mathcal{T}_{3,\frac{3}{2}}$ SCFT, one can

[27] This latter theory first appeared in the classification of [53] (using the nomenclature of this thesis, $\mathcal{T}_{3,\frac{3}{2}}$ is a "Type III" theory with Young diagrams $[2,2,2], [2,2,2], [2,2,1,1]$).

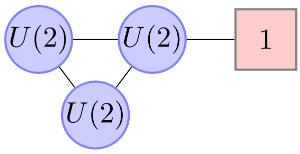



**Figure 4:** The quiver diagram describing the mirror of the $S^1$ reduction of $\mathcal{T}_{3,\frac{3}{2}}$.

construct the corresponding Higgs field $\varphi$ using the methods in [53]

$$\varphi(z) = zM_1 + M_2 + \frac{1}{z}M_3 + \mathcal{O}(z^{-2}) \ , \tag{3.4}$$

where we have expanded around a third-order pole at $z = \infty$ ($\varphi$ is non-singular at all other points $z \in \mathcal{C} = \mathbb{CP}^1$), and the $M_i$ are the following diagonal traceless matrices

$$\begin{aligned} M_1 &= \text{diag}\left(\tilde{a}_1, \tilde{a}_1, \tilde{a}_2, \tilde{a}_2, \tilde{a}_3, \tilde{a}_3\right) \ , \quad M_2 = \text{diag}\left(\tilde{b}_1, \tilde{b}_1, \tilde{b}_2, \tilde{b}_2, \tilde{b}_3, \tilde{b}_3\right) \ , \\ M_3 &= \text{diag}\left(\tilde{m}_1, \tilde{m}_1, \tilde{m}_2, \tilde{m}_2, \tilde{m}_3, \tilde{m}_4\right) \ . \end{aligned} \tag{3.5}$$

The flavor symmetries are then read off by studying the independent parameters appearing as coefficients of the simple pole, i.e., the entries of $M_3$.[28] This traceless matrix has three degrees of freedom which correspond to the Cartans of $SU(3) \times SU(2)$. Therefore, according to this description, $G_{\mathcal{T}_{3,\frac{3}{2}}} = SU(3) \times SU(2)$. One reaches the same conclusion by constructing the Seiberg-Witten curve via the spectral curve $\det\left(x - \varphi(z)\right) = 0$, and looking at the mass parameters (i.e., the simple poles in the SW 1-form, $\lambda = xdz$).

On the other hand, one often computes flavor symmetries of strongly interacting 4D $\mathcal{N} = 2$ theories by taking their $S^1$ reductions and studying the mirror theory (which may sometimes be described by a Lagrangian that flows to the same 3D $\mathcal{N} = 4$ SCFT). Now, the $\mathcal{T}_{3,\frac{3}{2}}$ theory has a proposed Lagrangian mirror for its $S^1$ reduction given in Fig. 4 (following the rules in [53]) that predicts flavor symmetry $G^{3d}_{\mathcal{T}_{3,\frac{3}{2}}} = SU(3) \times SU(2)^2$. Indeed, IR dimension-one monopole operators in this theory describe the enhancement of the manifest $U(1)^3$ topological symmetry to $SU(3) \times SU(2)^2$ [75]. In particular, there is a free monopole operator in the IR that gives rise to an additional $SU(2)$ factor.[29] By

[28] This data gives us the Cartans of the flavor symmetry. By studying various limits of the Hitchin system, we can often identify the full flavor symmetry by matching onto Hitchin sub-systems with known flavor symmetries.

[29] This result is somewhat counterintuitive since the rules derived in [76] for the case of linear quivers suggest that the presence of a free monopole operator can be detected by looking at each gauge node in the quiver and counting the number of local flavors. If this number reaches a certain threshold, then the theory produces a free monopole after one turns on the corresponding gauge coupling(s) and flows to the IR (the theory is then referred to as “ugly” in the nomenclature of [76]). However, it is straightforward to check that the quiver in Fig. 4 should have no free monopoles by these tests and no accidental superconformal $R$ symmetries. The resolution to this puzzle is that the free monopole depends on the global topology of the quiver—it has non-trivial flux through each gauge node—and so

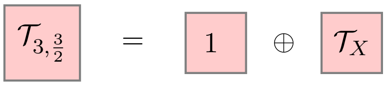



**Figure 5:** The factorized form of the $\mathcal{T}_{3,\frac{3}{2}}$ SCFT into a decoupled free hypermultiplet and the interacting $\mathcal{T}_X$ SCFT.

mirror symmetry [77], one expects, upon performing an $S^1$ reduction, the enhancement of $G_{\mathcal{T}_{3,\frac{3}{2}}} \to SU(3) \times SU(2)^2$ with a decoupled hypermultiplet.

A priori, there are various possible resolutions to the different predictions for $G_{\mathcal{T}_{3,\frac{3}{2}}}$. First, it could be that the extra $SU(2)$ factor is an accidental symmetry at energies $E \ll R^{-1}$ (where $R$ is the radius of the compactification circle). Second, it could be that the 4D description around (3.4) from the M5 brane simply misses some flavor symmetries.[30] Finally, it could be that neither description gets the correct symmetries.

We claim the 3D quiver of Fig. 4 captures the full flavor symmetry and the 4D description around (3.4) does not. In particular, we will argue that the $\mathcal{T}_{3,\frac{3}{2}}$ SCFT splits into a free hypermultiplet and an interacting theory, $\mathcal{T}_X$, as in Fig. 5 and that the $SU(2)$ symmetry detected around (3.4) corresponds to a diagonal subgroup of the $SU(2)^2 \subset G_{\mathcal{T}_{3,\frac{3}{2}}}$ factor. Happily, the interacting $\mathcal{T}_X$ theory then has ($\mathcal{N} = 2$) flavor symmetry $G_{\mathcal{T}_X} = SU(3) \times SU(2)$ and the following anomalies[31]

$$k_{SU(2)}^{\mathcal{T}_X} = 4 \ , \quad k_{SU(3)}^{\mathcal{T}_X} = 6 \ , \quad c^{\mathcal{T}_X} = \frac{13}{6} \ , \quad a^{\mathcal{T}_X} = \frac{47}{24} \ . \tag{3.6}$$

In particular, we can now, in more direct analogy with the $E_6 = T_3$ theory, construct conformal manifolds just from arbitrarily many $\mathcal{T}_X$ theories and conformal gauge fields.[32] On the other hand, we need to be careful when constructing theories by gauging the $SU(2)$ factor since it has a $\mathbb{Z}_2$ Witten anomaly [78]! Indeed, as argued in [75], the (diagonal) $\mathcal{T}_{3,\frac{3}{2}}$ $SU(2)$ factor is anomaly free. However, since a single hypermultiplet has a Witten anomaly, the $\mathcal{T}_X$ theory must have a non-trivial compensating anomaly.

In order to substantiate our claim in Fig. 5 and also to further examine the analogy between the $\mathcal{T}_X$ theory and the $T_N$ theories, we must go beyond the simple description around (3.4). To that end, we will focus on the Schur sector of the various component

the linear quiver tests of [76] do not apply.

[30] A similar phenomenon occurs in some theories with only regular punctures.

[31] Somewhat intriguingly, as an $\mathcal{N} = 1$ theory, the flavor symmetry is $SU(3) \times SU(2) \times U(1)$. Note that since the $U(1)$ symmetry comes from the $\mathcal{N} = 2$ $U(1)_R \times SU(2)_R$ symmetry, it is chiral (although the $SU(3) \times SU(2)$ factors are not by the general analysis of [21]). We are not aware of another method in field or string theory to impose a minimality condition and find $SU(3) \times SU(2) \times U(1)$ as a set of symmetries. However, note that these are genuine (global) symmetries and not gauge symmetries as in the Standard Model.

[32] Since now we can build an infinite linear quiver of $\mathcal{T}_X$ theories where we alternate gauging $SU(2)$ and $SU(3)$ flavor symmetry factors.

theories in our duality which contains a wealth of information and is often exactly solvable, due to the chiral algebra correspondence we saw earlier in Chapter 2.

In order to get a handle on the Schur sector, we will compute the Schur limit of the superconformal index that captures contributions only from operators in this sector. For our starting point in Fig. 2, this computation can easily be carried out using the results of [30, 38]. Invariance of the Schur index under $S$-duality guarantees that we then also have the index for the theory in Fig. 3.[33]

Obtaining the index of the $\mathcal{T}_X$ theory itself is somewhat more delicate. However, using a recent conjecture in [31] (proven in [79] and reviewed in Appendix A), we are able to find the Schur index of $\mathcal{T}_X$ from the index of the quiver in Fig. 3 using the inversion theorem in [80]. Our use of the result in [80] is in the same spirit that it was used by the authors of [81] to determine the index of the $E_6$ SCFT (however, there are some technical differences, because our $SU(2)$ duality frame involves an additional strongly interacting factor).

In order to check our index computation and also to gain more insight into the $\mathcal{T}_X$ theory, we bootstrap its chiral algebra, $\chi(\mathcal{T}_X)$, (and hence by the chiral algebra correspondence we find its Schur operators) using techniques described in [82]. In particular, we show that there is a unique consistent chiral algebra with the (minimal) number of generators required, via the correspondence in [27], for compatibility with our inversion result and the anomalies in (3.6). Then, using arguments closely related to those in [82], we argue for an exact expression for the vacuum character of $\chi(\mathcal{T}_X)$ in terms of certain "diagonal" $\widehat{SU(2)}_{-2}\times\widehat{SU(3)}_{-3}$ Affine Kac-Moody (AKM) characters. By the correspondence of [27], this gives us a simple closed-form expression for the Schur index of the $\mathcal{T}_X$ theory and allows us to recover the $S^3$ partition function of the proposed 3D mirror in Fig. 4 by taking the $q\to 1$ limit of this quantity.

As we will see, our expression for the Schur index in terms of AKM characters reveals a much deeper connection with the $T_N$ theories: the "structure constants" that emerge are precisely those of the $T_2$ theory (although the AKM characters we sum over are different, they are in one-to-one correspondence with those we sum over in the $T_2$ case). We explore these connections in greater detail below and also comment on some consequences of the non-trivial Witten anomaly of the $\mathcal{T}_X$ theory for the 2D/4D chiral algebra correspondence.

In the next section, we will apply the chiral algebra correspondence to argue for the factorization in Fig. 5. Along the way, we also make use of the results in [30, 38].

[33] Moreover, the consistency of the resulting picture we will find below bolsters the claimed duality in Fig. 2 and Fig. 3 beyond the checks that were performed in [75] at the level of the SW curves and dimensional reductions.

## 3.2 A chiral algebra argument for $\mathcal{T}_{3,\frac{3}{2}} = \mathcal{T}_X \oplus$ hyper

To understand why the $\mathcal{T}_{3,\frac{3}{2}}$ theory factorizes, note that a simple consequence of the duality discussed in the introduction is that the spectrum of gauge invariant operators arising from the quiver in Fig. 2 must match the spectrum of such operators arising from the quiver in the dual frame in Fig. 3. In particular, the $SU(3)$ side of the theory clearly has dimension three and $SU(2)_R$ weight $\frac{3}{2}$ baryons

$$B = \epsilon^{ijk} Q_i^a Q_j^b Q_k^c \ , \quad \tilde{B} = \epsilon_{ijk} \tilde{Q}_a^i \tilde{Q}_b^j \tilde{Q}_c^k \ , \tag{3.7}$$

that are charged under the baryonic $U(1) \subset U(3)$ factor of the flavor symmetry. Moreover, we have

$$\left[\tilde{Q}_{2\dot{-}}, B\right] = \left[Q_-^1, B\right] = \left[\tilde{Q}_{2\dot{-}}, \tilde{B}\right] = \left[Q_-^1, \tilde{B}\right] = 0 \ , \tag{3.8}$$

and so these degrees of freedom are Schur operators of type $\hat{\mathcal{B}}_{\frac{3}{2}}$, by (2.56) such operators are in turn related to 2D chiral algebra primaries $\mathcal{B}$ and $\tilde{\mathcal{B}}$ of holomorphic scaling dimension $h = E - R = \frac{3}{2}$.

As a result, the $SU(2)$ side of the duality must also have operators $B$ and $\tilde{B}$. Since the $(A_1, D_4)$ factor in this duality frame is responsible for the baryonic symmetry, $\mathcal{B}$ and $\tilde{\mathcal{B}}$ must either be Schur operators of the $(A_1, D_4)$ sector or composite gauge-invariant operators built from Schur operators of this sector and Schur operators of at least one other sector. However, we know the Schur sector of the $(A_1, D_4)$ theory exactly: it corresponds, via the chiral algebra map described in Sec. 2.3, to the $\widehat{SU(3)}_{-\frac{3}{2}}$ AKM chiral algebra [30,38,83,84][34] generated by the AKM current $J^I_{SU(3)}$ ($I = 1, \cdots, 8$ is an adjoint index of $SU(3)$).

Therefore, $\chi\left[(A_1, D_4)\right]$ has no operators with the quantum numbers of $\mathcal{B}$ and $\tilde{\mathcal{B}}$ (since $J^I_{SU(3)}$ has $h = 1$, there are no operators with $h = \frac{3}{2}$ in the $\widehat{SU(3)}_{-\frac{3}{2}}$ vacuum module). As a result, we must construct $B$ and $\tilde{B}$ as composites of the holomorphic moment map of the $(A_1, D_4)$ theory, $\mu^I_{SU(3)}$, with a field of dimension one (and $h = 1/2$).[35] In other words, we must have a sector consisting of a hypermultiplet, $Q_i$ (with $i = 1, 2$), charged under the gauged $SU(2)$ (recall that the hypermultiplet has $Sp(1) \simeq SU(2)$ flavor symmetry) from which we can construct

$$B = \mu^i_{SU(3)} Q_i \ , \quad \tilde{B} = \tilde{\mu}^i_{SU(3)} Q_i \ , \tag{3.9}$$

where $\mu^i_{SU(3)}$ and $\tilde{\mu}^i_{SU(3)}$ are the two doublets descending from the eight $\mu^I_{SU(3)}$ moment

[34] See also the beautiful generalization in [32].

[35] In fact, the baryons map to generators of the chiral algebra related to the theory in Figs. 2 and 3. Note that, in accord with the bound in [19], this chiral algebra has at least three generators, since there are also multiple generators with $h = 1$ as well.

maps under the decomposition of $SU(3)$ into representations of the $SU(2)$ gauge group (we have $\mathbf{8} = \mathbf{1} + 2\times\mathbf{2} + \mathbf{3}$). In particular, we see that the $\mathcal{T}_{3,\frac{3}{2}}$ SCFT splits into a free hyper and another theory which we call $\mathcal{T}_X$ (as in Fig. 5).[36] Moreover, as discussed in the introduction, since the $\mathcal{T}_{3,\frac{3}{2}}$ theory does not have a Witten anomaly for its $SU(2)$ global symmetry subgroup but the free hypermultiplet does, the $SU(2)$ global symmetry subgroup of the $\mathcal{T}_X$ theory has a Witten anomaly. We will see an interesting consequence of this fact below. This discussion also derives the result in (3.6) from (3.3).

In the next section, we begin a deeper exploration of the $\mathcal{T}_X$ theory. To do so, we first construct the Schur index of the theory. After finding this index, we will conjecture a chiral algebra, $\chi(\mathcal{T}_X)$, that reproduces it and then use bootstrap techniques to confirm our conjecture.

## 3.3 The Schur index of $\mathcal{T}_X$ from $S$-duality and inversion

In order to get more detailed information about the $\mathcal{T}_X$ theory, we compute its Schur index using the $S$-duality described in Fig. 2 and Fig. 3. Indeed, since the index is invariant under $S$-duality, the Schur indices of the theories in these two figures must agree. On the $SU(3)$ side of the duality, it is easy to compute the Schur index as follows

$$\begin{aligned}\mathcal{I}_{SU(3)}(q,s,z_1,z_2) &= \oint d\mu_{SU(3)}(x_1,x_2)\times\mathcal{I}_{\text{vect}}(q,x_1,x_2)\times\mathcal{I}_{\text{flavors}}(q,x_1,x_2,s,z_1,z_2)\times\\ &\times\ \mathcal{I}_{(A_1,D_4)}(q,x_1,x_2)^2\ ,\end{aligned}\tag{3.10}$$

where the measure of integration is the $SU(3)$ Haar measure, $\mathcal{I}_{\text{flavors}}$ is the index of the three fundamental flavors, $\mathcal{I}_{(A_1,D_4)}$ is the index of the $(A_1,D_4)$ theory, and $\mathcal{I}_{\text{vect}}$ is the vector multiplet index (see Appendix B for detailed expressions). The fugacities, $s$ and $(z_1,z_2)$, are for $U(1)\subset U(3)$ and $SU(3)\subset U(3)$ flavor subgroups, respectively. All terms appearing in the integrand of (3.10) have known closed-form expressions ($\mathcal{I}_{(A_1,D_4)}$ was computed in [30,38]). Now, on the $SU(2)$ side of the duality, we have

$$\mathcal{I}_{SU(2)}(q,s,z_1,z_2)=\oint d\mu_{SU(2)}(e)\times\mathcal{I}_{\text{vect}}(q,e)\times\mathcal{I}_{(A_1,D_4)}(q,e,s)\times\mathcal{I}_{\mathcal{T}_{3,\frac{3}{2}}}(q,e,z_1,z_2)\ ,\tag{3.11}$$

where $\mathcal{I}_{\mathcal{T}_{3,\frac{3}{2}}}$ is the Schur index of the $\mathcal{T}_{3,\frac{3}{2}}$ theory. From the general discussion in the previous section and Fig. 5, we must have

$$\mathcal{I}_{\mathcal{T}_{3,\frac{3}{2}}}(q,e,z_1,z_2)=\mathcal{I}_{\mathcal{T}_X}(q,e,z_1,z_2)\times\mathcal{I}_{\text{hyper}}(q,e)\ ,\tag{3.12}$$

[36] One may also derive this result using facts about the moduli spaces of vacua for the theories in our duality. However, our arguments at the level of the chiral algebra provide a stronger consistency check of the duality in [75] as well as of the picture we propose in Fig. 5.

where the second factor on the RHS is the Schur index of a free hypermultiplet, and the first factor is the index of the $\mathcal{T}_X$ SCFT.

In order to compute the index in (3.12), we will use an inversion procedure based on the theorem in [80] to extract it from the expression in (3.11). Roughly the same basic procedure was first used in [81] to extract the index of the $E_6$ SCFT from Argyres-Seiberg duality. However, there are some technical differences (due to the fact that our $SU(2)$ duality frame has an additional strongly interacting factor) in our use of [80] that are reviewed in Appendix B. One important precondition for our inversion procedure involves the use of a conjectured form for $\mathcal{I}_{(A_1,D_4)}(q,x_1,x_2)$ due to Xie-Yan-Yau (XYY) [31] (proved in [79] and reviewed in Appendix A) that is compatible with its known form in [30, 38]

$$\mathcal{I}_{(A_1,D_4)}(q,x_1,x_2) = P.E.\left[\frac{q}{1-q^2}\chi_{\text{Adj}}(x_1,x_2)\right] \ , \tag{3.13}$$

where we defined the Plethystic Exponential in (2.29). Indeed, the surprising fact that the index of the strongly interacting $(A_1, D_4)$ SCFT in (3.13) is related to the index of a free adjoint hypermultiplet by the rescaling $q \to \sqrt{q}$ allows us to use the inversion theorem of [80] (as in [81], we will a posteriori justify the assumptions used in applying this theorem by finding a consistent symmetry structure for our index). One surprising fact we will uncover later on is that, when appropriately re-written, $\mathcal{I}_{\mathcal{T}_X}$ will also be closely related to a Schur index for free fields.

Applying the procedure in Appendix B, we find that the Schur index of the $\mathcal{T}_{3,\frac{3}{2}}$ theory can be written as

$$\mathcal{I}_{\mathcal{T}_{3,\frac{3}{2}}}(q,w,z_1,z_2) = \frac{1}{(w^{\pm 2}q;q)}\left[\frac{1}{1-w^2}\mathcal{I}_{SU(3)}(q,wq,z_1,z_2) + \frac{w^2}{w^2-1}\mathcal{I}_{SU(3)}(q,\frac{q}{w},z_1,z_2)\right] \ , \tag{3.14}$$

where $(a;q)$ denotes the $q$-Pochhammer symbol

$$(a;q) = \prod_{n=0}^{\infty}(1-aq^n) \ , \tag{3.15}$$

and we also use the condensed notation

$$(a^{\pm};q) \equiv (a;q)(a^{-1};q) \ . \tag{3.16}$$

Expanding (3.14) perturbatively in $q$ we obtain

$$\begin{aligned}\mathcal{I}_{\mathcal{T}_{3,\frac{3}{2}}}(q,w,z_1,z_2) &= 1+\chi_1 q^{\frac{1}{2}} + (2\chi_2+\chi_{1,1})q + 2(\chi_1+\chi_3+\chi_1\chi_{1,1})q^{\frac{3}{2}} + (4+3\chi_2+ \\ &+ 3\chi_4+3\chi_{1,1}+3\chi_2\chi_{1,1}+\chi_{2,2})q^2 + (8\chi_1+5\chi_3+3\chi_5+7\chi_1\chi_{1,1}+ \\ &+ 4\chi_3\chi_{1,1}+\chi_1\chi_{3,0}+\chi_1\chi_{0,3}+2\chi_1\chi_{2,2})q^{\frac{5}{2}} + (6+15\chi_2+6\chi_4+\end{aligned}$$

$$+ \quad 4\chi_6 + 10\chi_{1,1} + 12\chi_2\chi_{1,1} + 5\chi_4\chi_{1,1} + 3\chi_{3,0} + \chi_2\chi_{3,0} + 3\chi_{0,3} +$$
$$+ \quad \chi_2\chi_{0,3} + 3\chi_{2,2} + 4\chi_2\chi_{2,2} + \chi_{3,3})q^3 + \mathcal{O}(q^{\frac{7}{2}}) \ , \tag{3.17}$$

where $\chi_\lambda \equiv \chi_\lambda(w)$ is the character of the spin $\frac{\lambda}{2}$ representation of $SU(2)$ and $\chi_{\lambda_1,\lambda_2} \equiv \chi_{\lambda_1,\lambda_2}(z_1, z_2)$ is the character of the $SU(3)$ representation with Dynkin labels $\lambda_{1,2} \in \mathbb{Z}_{\geq 0}$.

One check of (3.14) and (3.17) is that they are compatible with the factorization we argued for in Sec. 3.2 and explained at the level of the index in (3.12). In particular, we see a free hypermultiplet at $\mathcal{O}(q^{\frac{1}{2}})$. Moreover, the total global symmetry of the $\mathcal{T}_{3,\frac{3}{2}}$ theory is then, as explained in the introduction, $SU(2)^2 \times SU(3)$ with one $SU(2)$ factor coming from the free hypermultiplet.[37] Although this enhancement is not quite as dramatic as the $E_6$ enhancement of flavor symmetry observed in the example studied in [81], we will find a much deeper statement about the (hidden) symmetries of this theory (and hence the consistency of our picture) by bootstrapping the chiral algebra associated with $\mathcal{T}_X$ below.

As a first step towards this goal, we arrive at the index of the $\mathcal{T}_X$ theory by dividing both sides of (3.12) by the free hypermultiplet contribution

$$\begin{aligned}\mathcal{I}_{\mathcal{T}_X}(q, w, z_1, z_2) \ = \ & 1 + (\chi_2 + \chi_{1,1})q + \chi_1\chi_{1,1}q^{\frac{3}{2}} + (2 + \chi_2 + \chi_4 + 2\chi_{1,1} + \chi_2\chi_{1,1} + \\ + \ & \chi_{2,2})q^2 + (\chi_1 + 2\chi_1\chi_{1,1} + \chi_3\chi_{1,1} + \chi_1\chi_{0,3} + \chi_1\chi_{3,0} + \\ + \ & \chi_1\chi_{2,2})q^{\frac{5}{2}} + (2 + 4\chi_2 + \chi_4 + \chi_6 + 5\chi_{1,1} + 3\chi_2\chi_{1,1} + \chi_4\chi_{1,1} + \\ + \ & 2\chi_{3,0} + 2\chi_{0,3} + 2\chi_{2,2} + 2\chi_2\chi_{2,2} + \chi_{3,3})q^3 + \mathcal{O}(q^{\frac{7}{2}}) \ ,\end{aligned} \tag{3.18}$$

which has, as promised, $SU(2) \times SU(3)$ global symmetry (we see currents in the adjoint of this symmetry group at $O(q)$, and the index organizes into characters of this symmetry).

In the next section, we use this expansion to conjecture the generators of the associated chiral algebra, $\chi(\mathcal{T}_X)$. We then bootstrap this chiral algebra and show that it is consistent with Jacobi identities. Moreover, we will argue that it is the unique such chiral algebra with the generators we conjecture and the anomalies required from the discussion in the introduction.[38]

[37] Note that, on the $SU(2)$ side of our duality, we gauge the diagonal $SU(2) \subset SU(2)^2$ to construct the theory in Fig. 3.

[38] We will also see that, for example, the central charge of the chiral algebra is fixed to be $c_{2d} = -26$ given our generators and AKM levels. Similarly, the AKM levels are fixed given our generators and $c_{2d} = -26$ (here we assume that the 2D chiral algebra is related to a unitary 4D SCFT by the chiral algebra correspondence).

## 3.4 A chiral algebra conjecture

From the simple expansion presented in (6.8), we can immediately conjecture the generators of the corresponding chiral algebra. Indeed, using the map in (2.57), (6.8) is also an expansion for the character of the vacuum module of the chiral algebra we want to find.

The only possible contributions in the vacuum module at $\mathcal{O}(q)$ must come from AKM currents, which, in this case, are for $\widehat{SU(2)}_{-2} \times \widehat{SU(3)}_{-3}$. We have used (3.6) and (2.53) to fix the levels of the AKM algebras to the so-called critical levels (these are $k = -h^\vee$, where $h^\vee$ is the dual Coxeter number). As in the case of the $T_N$ theories (with the exception of the $T_3 = E_6$ theory which has enhanced $E_6 \supset SU(3)^3$ flavor symmetry and the $T_2$ theory which has $Sp(4) \supset SU(2)^3$ symmetry and no AKM currents as generators), this discussion means that the holomorphic stress tensor of the chiral algebra must be an independent generator, since the Sugawara stress tensor is not normalizable (note that from (3.6) and (2.51) we have $c = -26$ for the Virasoro subalgebra[39]). Looking at $\mathcal{O}(q^{\frac{3}{2}})$, we see that there must be at least one operator, $\mathcal{O}_{aI}$, transforming in the $\mathbf{2} \times \mathbf{8}$ representation of the global symmetry (since all the other generators are integer dimensional).[40] This operator is mapped to an AKM primary, $\chi[\mathcal{O}_{aI}] = W_{aI}$. Therefore, the minimal conjecture for $\chi\left[\mathcal{T}_X\right]$ is the following

**Conjecture:** The chiral algebra, $\chi\left[\mathcal{T}_X\right]$, is generated by a stress tensor, $T$ (with $c = -26$), AKM currents, $J^A_{SU(2)}$ and $J^I_{SU(3)}$ (with $A = 1, \cdots, 3$ and $I = 1, \cdots, 8$) for $\widehat{SU(2)}_{-2} \times \widehat{SU(3)}_{-3}$, and an $h = \frac{3}{2}$ AKM primary, $W_{aI}$ (with $a = 1, 2$ and $I = 1, \cdots, 8$), transforming in the $\mathbf{2} \times \mathbf{8}$ representation of $SU(2) \times SU(3)$.

Note that this conjecture is consistent with the simplicity of AD theories: to get the chiral algebra of $\mathcal{T}_X$, one needs to add only a single additional generator (really 16 generators if one counts all the allowed $a, I$ pairs) beyond the universal ones required by 4D symmetries. Indeed, this algebra is considerably simpler than those of the interacting $T_N$ theories (even the $T_3 = E_6$ theory has a larger number of generators by virtue of its large global symmetry).

We will give convincing evidence for this conjecture in Sec. 3.5, where we will show there is a unique consistent chiral algebra satisfying this conjecture. For now, we also give some powerful circumstantial evidence in favor of our proposal. In particular, if this conjecture is correct, then all contributions appearing in (6.8) can be generated by plethystic exponentials of our generators modulo constraints. Assuming our conjecture

[39] Amusingly, this value is the same as the $c$ anomaly for the $bc$ ghost system.

[40] This operator must be of type $\hat{\mathcal{B}}_{\frac{3}{2}}$. The only other Schur multiplets of the appropriate statistics that can appear at $\mathcal{O}(q^{\frac{3}{2}})$ are $\mathcal{D}_{0(0,\frac{1}{2})} \oplus \bar{\mathcal{D}}_{0(\frac{1}{2},0)}$. However, these operators have the wrong multiplicity and, on general grounds, should not be present in this theory [20] (note that they also satisfy free field equations of motion and so presumably should not appear on such grounds as well).

is correct, we find some natural operator relations at low order in $q$

- A singlet relation at $O(q^2)$. As we will see in greater detail below, we expect that

$$\mathrm{Tr} J^2_{SU(3)} \sim \mathrm{Tr} J^2_{SU(2)} \ , \tag{3.19}$$

  where we will fix the non-zero constant of proportionality in the next section. The motivation for this relation is that the $\widehat{SU(2)}_{-2} \times \widehat{SU(3)}_{-3}$ subalgebras of $\chi(\mathcal{T}_X)$ are at the critical level. Therefore, in their respective modules, the LHS and RHS of (3.19) separately vanish. However, it is natural to expect that, as in the case of the $T_N$ theories [82], one linear combination of these operators becomes non-null in the full chiral algebra and therefore remains as a non-trivial operator.

- At $\mathcal{O}(q^{\frac{5}{2}})$ we have two operator relations with quantum numbers $\mathbf{2} \times \mathbf{8}$.

- At $\mathcal{O}(q^3)$ we have many operator relations. One important set of relations are the singlets of the form

$$\mathrm{Tr} J^3_{SU(3)} = \mathrm{Tr} J^3_{SU(2)} = W^{aI}_{\frac{3}{2}} W_{\frac{3}{2} aI} = 0 \ . \tag{3.20}$$

  The first relation again follows from the fact that the flavor symmetry is at the critical level and is a non-trivial statement, while the last two relations are a simple consequence of bosonic statistics.

## 3.5 Bootstrapping the Chiral Algebra of $\mathcal{T}_X$

One strong piece of evidence in favor of our conjecture in the previous section is that there exists a (unique) set of operator product expansions (OPEs) among the generators described there that is consistent with Jacobi identities

$$[\mathcal{O}_1(z_1)\,[\mathcal{O}_2(z_2)\mathcal{O}_3(z_3)]] - [\mathcal{O}_3(z_3)\,[\mathcal{O}_1(z_1)\mathcal{O}_2(z_2)]] - [\mathcal{O}_2(z_2)\,[\mathcal{O}_3(z_3)\mathcal{O}_1(z_1)]] = 0 \ , \tag{3.21}$$

where we take $|z_2 - z_3| < |z_1 - z_3|$, $[\cdots]$ is the singular part of the OPE of the operators enclosed, and we have assumed the $\mathcal{O}_i$ are all bosonic (as will be the case for $\chi(\mathcal{T}_X)$). To understand this statement, let us consider the most general OPEs among the generators. The non-vanishing singular parts of the OPEs among the stress tensor and the $SU(2) \times$

$SU(3)$ currents are completely fixed by Ward identities to take the form

$$\begin{aligned}
T(z)T(0) &\sim \frac{c_{2d}}{2z^4}+\frac{2T}{z^2}+\frac{\partial T}{z}\ , \\
T(z)J^A_{SU(2)}(0) &\sim \frac{J^A_{SU(2)}}{z^2}+\frac{\partial J^A_{SU(2)}}{z}\ , \\
T(z)J^I_{SU(3)}(0) &\sim \frac{J^I_{SU(3)}}{z^2}+\frac{\partial J^I_{SU(3)}}{z}\ , \\
J^A_{SU(2)}(z)J^B_{SU(2)}(0) &\sim \frac{k^{SU(2)}_{2d}\delta^{AB}}{2z^2}+\frac{i\epsilon^{ABC}J^C_{SU(2)}}{z}\ , \\
J^I_{SU(3)}(z)J^J_{SU(3)} &\sim \frac{k^{SU(3)}_{2d}\delta^{IJ}}{2z^2}+\frac{if^{IJK}J^K_{SU(3)}}{z}\ ,
\end{aligned} \tag{3.22}$$

where $f_{IJK}$ is the structure constant of $SU(3)$ and, as discussed in the previous section, $c_{2d}=-26$, $k^{SU(2)}_{2d}=-2$ and $k^{SU(3)}_{2d}=-3$. Moreover, since there is no generator with $h=1/2$, ${W_a}^I$ has to be a primary of the Virasoro and $\widehat{SU(2)}_{-2}\times\widehat{SU(3)}_{-3}$ algebras. This fact implies the following singular parts of the OPEs:

$$\begin{aligned}
T(z){W_a}^I(0) &\sim \frac{3{W_a}^I}{2z^2}+\frac{\partial {W_a}^I}{z}\ , \\
J^A_{SU(2)}(z){W_a}^I(0) &\sim \frac{\sigma^A_{ab}W^{bI}}{2z}\ , \\
J^I_{SU(3)}(z){W_a}^J(0) &\sim \frac{f^{IJK}W_{aK}}{z}\ ,
\end{aligned} \tag{3.23}$$

where the $\sigma^A$ are Pauli matrices.

On the other hand, the OPE between ${W_a}^I$ and ${W_b}^J$ is not fixed by the symmetries. Therefore, we adopt the following general ansatz for the singular parts of this OPE:

$$\begin{aligned}
{W_a}^I(z){W_b}^J(0) \sim{}& \frac{\epsilon_{ab}\delta^{IJ}}{z^3}+\frac{1}{z^2}\left(\frac{a_1}{2}\delta^{IJ}\sigma^A_{ab}\,J_{SU(2)A}+\epsilon_{ab}\,(a_2\,f^{IJK}+a_3\,d^{IJK})J_{SU(3)K}\right) \\
&+\frac{1}{z}\Bigg[\epsilon_{ab}\delta^{IJ}\left(a_4\,T+a_5\,J^A_{SU(2)}J_{SU(2)A}+a_6\,J^K_{SU(3)}J_{SU(3)K}\right) \\
&\quad+\frac{a_7}{2}\,\delta^{IJ}\sigma^A_{ab}\,J'_{SU(2)A}+a_8\,\epsilon_{ab}\,f^{IJK}\,J'_{SU(3)K}+\frac{a_9}{2}\,\sigma^A_{ab}\,f^{IJK}J_{SU(2)A}J_{SU(3)K} \\
&\quad+\epsilon_{ab}(a_{10}\,f^{IJK}+a_{11}\,d^{IJK})d_{KLM}J^L_{SU(3)}J^M_{SU(3)}+2a_{12}\,\epsilon_{ab}J^{(I}_{SU(3)}J^{J)}_{SU(3)}\Bigg]\ ,
\end{aligned} \tag{3.24}$$

where $d^{IJK}$ is the totally symmetric tensor of $SU(3)$ normalized so that $d^{IJK}d_{IJK}=\frac{40}{3}$, and the ${W_a}^I$ are normalized so that the coefficient of $\epsilon_{ab}\delta^{IJ}/z^3$ is one.[41] The twelve

[41] Note that the coefficient of $\epsilon_{ab}\delta^{IJ}/z^3$ is non-vanishing because otherwise ${W_a}^I$ is null. Therefore, this normalization is always possible.

coefficients, $a_1, \cdots, a_{12}$, are free parameters to be fixed in such a way that the Jacobi identities are satisfied. Note that (3.24) is the most general OPE written in terms of the generators, $T$, $J^A_{SU(2)}$, $J^I_{SU(3)}$ and ${W_a}^I$.[42]

To fix the above constants and test the consistency of what we have written, we impose the various Jacobi identities among the generators. In particular, the Jacobi identities among $\mathcal{O}$, ${W_a}^I$, and ${W_b}^J$ for $\mathcal{O} \in \left\{T, J^A_{SU(2)}, J^I_{SU(3)}\right\}$ imply that

$$a_1 = 1 \ , \quad a_2 = a_9 = -\frac{2i}{3} \ , \quad a_3 = a_{10} = 0, \quad a_4 = -\frac{1}{4} \ , \quad a_6 = \frac{2-3a_5}{12} \ , \\ a_7 = a_{11} = \frac{1}{2} \ , \quad a_8 = -\frac{i}{3} \ , \quad a_{12} = -\frac{1}{12} \ . \tag{3.25}$$

Note that this condition fixes all the OPE coefficients except for $a_5$. Moreover, it turns out that, with $a_6 = (2-3a_5)/12$ imposed, the undetermined parameter $a_5$ is only coupled to a null operator. Indeed, under the condition $a_6 = (2-3a_5)/12$, the only $a_5$-dependent term in (3.24) is

$$a_5 \left( J^A_{SU(2)} J_{SU(2)A} - \frac{1}{4} J^K_{SU(3)} J_{SU(3)K} \right) \ . \tag{3.26}$$

Since the OPEs of this operator with the generators only involve operators of holomorphic dimension larger than or equal to its own dimension, (3.26) is a null operator. Therefore, we set $a_5 = 0$ in the rest of this section.

Let us now look at the Jacobi identities among ${W_a}^I$, ${W_b}^J$, and ${W_c}^K$. With the condition (3.25), they are automatically satisfied up to the following operators:

$$\sigma^A_{ab} J_{SU(2)A} W^{bI} + \frac{if^{IJK}}{2} J_{SU(3)J} W_{aK} \ , \quad d^{IJK} J_{SU(3)J} W_{aK} \ . \tag{3.27}$$

Since the OPEs of these operators with the generators of the chiral algebra only involve operators of holomorphic dimensions larger than or equal to their own dimensions, the above two operators are both null. This means that (3.25) is consistent with all the Jacobi identities among the generators. The existence of such a consistent $WW$ OPE is strong evidence for our chiral algebra conjecture in the previous section.

Another interesting observation is that the chiral algebra generated by $T$, ${W_a}^I$, and $J^A_{SU(2)}$, $J^I_{SU(3)}$ at the critical levels exist if and only if the Virasoro central charge is $c_{2d} = -26$. Indeed, when we do the above analysis with $c_{2d}$ unfixed, we see that the Jacobi identities among the generators imply $c_{2d} = -26$. Similarly, if we take $c_{2d} = -26$ with the levels of the AKM algebras unfixed, we can show that the Jacobi identities imply that $k_{SU(2)} = -2$ and $k_{SU(3)} = -3$.[43]

[42] In particular, note that $J^{[I}_{SU(3)} J^{J]}_{SU(3)}$ is vanishing and therefore does not appear as an independent term.

[43] This last statement is true as long as the 2D chiral algebra is related to a unitary 4D SCFT by the

We have seen there are at least three null operators up to $h=\frac{5}{2}$. The first one is shown in (3.26) and is a singlet of $SU(2)\times SU(3)$ with $h=2$. The second and third null operators are shown in (3.27) and are in the $\mathbf{2}\times\mathbf{8}$ representation of $SU(2)\times SU(3)$ with $h=\frac{5}{2}$. These three null operators are perfectly consistent with the 4D operator relations discussed in Sec. 3.4.

Finally, we note that the following normal-ordered product

$$J^I_{SU(3)}W^a_I \neq 0 \ , \tag{3.28}$$

does not vanish. On the other hand, as we will see below when we discuss the HL chiral ring, there is a non-trivial operator relation for the 4D $\hat{\mathcal{B}}_R$ ancestors of these operators. However, as we will explain in greater detail below, this statement is consistent with (3.28) because of the $SU(2)_R$ mixing described in 2.48 which induces a non-trivial $\hat{\mathcal{C}}_{\frac{1}{2}(0,0)}$ component for the chiral algebra normal-ordered product.[44]

Given this chiral algebra, we will argue that its vacuum character has a surprisingly simple exact expression in terms of certain $\widehat{SU(2)}_{-2}\times\widehat{SU(3)}_{-3}$ characters. This expansion will turn out to be remarkably similar to the expansion one finds for the $T_2$ theory (although the precise characters we sum over are different). In addition to pointing to some mysterious connections between AD theories and $T_N$ SCFTs, we are able to use this formula to take the $q\to 1$ limit and make contact with the $S^3$ partition function of the 3D quiver appearing in Fig. 4.

## 3.6 Re-writing the index in terms of AKM characters

Since $\chi\left[\mathcal{T}_X\right]$ has AKM symmetry, it is reasonable to organize the index in terms of AKM representations. In particular, we claim that (6.8) can be re-written as follows

$$I_{\mathcal{T}_X}(q,w,z_1,z_2)=\sum_{\lambda=0}^{\infty}q^{\frac{3}{2}\lambda}P.E.\left[\frac{2q^2}{1-q}+2q-2q^{\lambda+1}\right]\mathrm{ch}^{SU(2)}_{R_\lambda}(q,w)\mathrm{ch}^{SU(3)}_{R_{\lambda,\lambda}}(q,z_1,z_2)\ , \tag{3.29}$$

where $\mathrm{ch}^{SU(2)}_{R_\lambda}$ and $\mathrm{ch}^{SU(3)}_{R_{\lambda,\lambda}}$ are AKM characters with highest-weight states transforming in representations of $SU(2)$ and $SU(3)$ characterized by Dynkin labels $\lambda$ and $\lambda_1=\lambda_2=\lambda$ respectively.

In fact, (3.29) is a completely explicit formula, since AKM characters of $\widehat{su}(N)$ at

chiral algebra correspondence

[44]Therefore, the Schur operator sitting in this $\hat{\mathcal{C}}_{\frac{1}{2}(0,0)}$ multiplet does not map to a generator of the chiral algebra. This situation is quite similar to what happens in, say, the chiral algebra of the $T_3=E_6$ theory, where the stress tensor is not a new generator of $\chi(E_6)$ due to the $SU(2)_R$ twisting of the moment maps and the mixing in of the $\hat{\mathcal{C}}_{0(0,0)}$ multiplet in the corresponding normal-ordered product.

the critical level have the following simple closed-form expression (e.g., see [82])

$$\mathrm{ch}_{R_{\vec{\lambda}}}(\boldsymbol{x}) = \frac{\text{P.E.}[\frac{q}{1-q}\chi_{adj}(\boldsymbol{x})]\chi_{R_{\vec{\lambda}}}(\boldsymbol{x})}{q^{\langle\vec{\lambda},\rho\rangle}\text{P.E.}[\sum_{j=1}^{N-1}\frac{q^{j+1}}{1-q}]\dim_q R_{\vec{\lambda}}} , \tag{3.30}$$

where $\vec{\lambda}$ is a vector containing the $N-1$ Dynkin labels characterizing the $SU(N)$ quantum numbers of the highest-weight state, $\rho$ is the Weyl vector, $\langle\cdot,\cdot\rangle$ is the standard inner product,[45] and the $q$-dimension is defined as

$$\dim_q R_{\vec{\lambda}} = \prod_{\alpha\in\Delta_+}\frac{\left[\langle\vec{\lambda}+\rho,\alpha\rangle\right]_q}{[\langle\rho,\alpha\rangle]_q} , \tag{3.32}$$

where $\Delta_+$ denotes the set of positive roots, and the $q$-deformed number is given by

$$[x]_q = \frac{q^{-\frac{x}{2}}-q^{\frac{x}{2}}}{q^{-\frac{1}{2}}-q^{\frac{1}{2}}} . \tag{3.33}$$

Amusingly, we can give an argument in favor of (3.29) that parallels the discussion in [82] for the $T_N$ case. The first term, $q^{\frac{3}{2}\lambda}$, is related to the dimension of the non-trivial AKM primary, $W_I^a$, and the dimensions of its products. The plethystic exponential "structure constants"

$$P.E.\left[\frac{2q^2}{1-q}+2q-2q^{\lambda+1}\right] , \tag{3.34}$$

have a simple interpretation as well. Indeed, the first term adds in normal-ordered products of the stress tensor and its derivatives with the other operators in the theory (note that these operators vanish in the AKM modules at the critical level) and also adds in normal-ordered products of the $h=2$ state built out of Casimirs of currents orthogonal to (3.26) with other operators in the theory (since this linear combination should not be null in the full chiral algebra). The second term in (3.34) adds back in the level one modes of these two operators, and the final term subtracts relations (for $\lambda=0$, this relation is required by the invariance of the vacuum under these modes).

We have also conducted many highly non-trivial checks of (3.29). For example, we have checked that, perturbatively in $q$, (3.29) coincides with the expression in (6.8) to very high order. Non-perturbatively in $q=e^{-\beta}$ we have also performed various checks.

[45] For $SU(N)$, we have $\langle\vec{\lambda},\rho\rangle=\sum_{i,j}\lambda_iF^{ij}\rho_j=\sum_{i,j}\lambda_iF^{ij}$ (where we have used that $\rho=(1,\cdots,1)$ in the last step) and $F^{ij}$ is the quadratic form matrix (i.e., the inverse of the Cartan matrix). In the cases of interest, this inner product reduces to

$$\langle\lambda,\rho\rangle_{SU(2)}=\frac{1}{2}\lambda_1 , \quad \langle\vec{\lambda},\rho\rangle_{SU(3)}=\lambda_1+\lambda_2 . \tag{3.31}$$

For example, it is straightforward to see that

$$\lim_{\beta\to 0}\log\mathcal{I}_{\mathcal{T}_X}(q,w,z_1,z_2)=\frac{5\pi^2}{3\beta}+\cdots\ . \tag{3.35}$$

This behavior is consistent with the expected Cardy-like scaling discussed in [85–87][46]

$$\lim_{\beta\to 0}\log\mathcal{I}(q,\mathbf{x})=-\frac{8\pi^2}{3\beta}(a-c)+\cdots=\frac{\pi^2}{3\beta}\mathrm{dim}_{\mathbb{Q}}\mathcal{M}_H+\cdots\ , \tag{3.36}$$

where, the last equality holds by $U(1)_R$ 't Hooft anomaly matching in theories with genuine Higgs branches (i.e., moduli spaces where, at generic points, the theory just has free hypermultiplets). In the case of the $\mathcal{T}_X$ theory, we expect there to be a genuine Higgs branch since the mirror of the $S^1$ reduction of the $\mathcal{T}_{3,\frac{3}{2}}$ theory in Fig. 4 has a genuine Coulomb branch (the result in (3.35) can also be taken as further evidence for the proposal in Fig. 4).

An even more interesting non-perturbative in $q$ check of our above discussion is to take the $\beta\to 0$ limit of (3.29), drop the divergent piece in (3.35), and study the resulting $S^3$ partition function, $Z_{S^3}$. As we review in greater detail in Appendix C, using the prescription in [88] we obtain

$$\begin{aligned}\lim_{\beta\to 0}\mathcal{I}_{\mathcal{T}_X}(q,w,z_1,z_2)\ &=\ \mathrm{Div.}\times\int_{-\infty}^{\infty}\frac{dm}{\sinh 2\pi m\sinh\pi m}\,\frac{\sin\pi m(\zeta_1-\zeta_2)\sin\pi m(2\zeta_1+\zeta_2)}{\sinh\pi(\zeta_1-\zeta_2)\sinh\pi(2\zeta_1+\zeta_2)}\\ &\quad\times\ \frac{\sin\pi m(2\zeta_2+\zeta_1)\sin 2\pi m\zeta}{\sinh\pi(2\zeta_2+\zeta_1)\sinh 2\pi\zeta}\ ,\end{aligned} \tag{3.37}$$

where the "Div." factor is the flavor-independent divergent piece in (3.35), $w=e^{-i\beta\zeta}$, $z_k=e^{-i\beta\zeta_k}$, and the summation over $\lambda$ in (3.29) becomes an integral over $m$. On the other hand, we can compute the partition function of the mirror of the quiver in Fig. 4, given in Fig. 18 of Appendix C, (or of the original quiver in Fig. 4 itself) and divide out by the contribution of a decoupled hypermultiplet to obtain

$$\begin{aligned}Z_{S^3}^{\mathrm{quiver}}\ &=\ \mathrm{Div.}\times\frac{1}{2}\int_{-\infty}^{\infty}dx_1dx_2\frac{\sinh^2(\pi(x_1-x_2))e^{2\pi i\eta(x_1+x_2)}}{\cosh\pi(x_1-x_2-m')\cosh\pi(x_2-x_1-m')}\\ &\quad\times\ \frac{1}{\cosh\pi m'\cosh\pi(x_1-m_1)\cosh\pi(x_2-m_1)\cosh\pi(x_1-m_2)}\\ &\quad\times\ \frac{1}{\cosh\pi(x_2-m_2)\cosh\pi(x_1+m_1+m_2)\cosh\pi(x_2+m_1+m_2)}\ .\end{aligned} \tag{3.38}$$

A direct calculation carried out in further detail in Appendix C reveals that (up to an

[46] Such behavior holds for theories whose $S^3$ partition function (upon performing an $S^1$ reduction) is finite. On the other hand, we are not aware of any $\mathcal{N}=2$ SCFT counterexamples to this behavior. Moreover, this scaling has been observed in many classes of strongly interacting $\mathcal{N}=2$ SCFTs [30,66].

unimportant overall constant)

$$\lim_{\beta\to 0}\left(\text{Div.}^{-1}\times\mathcal{I}_{\mathcal{T}_X}\right)=Z_{S^3}^{\text{quiver}}\ , \tag{3.39}$$

when we identify $m_i \leftrightarrow \zeta_i$ and $m' \leftrightarrow \zeta$.[47] This result is a strong check of our discussion and also of the proposal in [53, 89].

In the next section we move on and discuss the HL limit of the index and some additional predictions for the Schur sector of $\mathcal{T}_X$. Before doing so, let us make a few brief comments on what we have found in this section

- The structure constants given in (3.34) that multiply the AKM characters in (3.29) are precisely those of the free $T_2$ theory [82]. While the set of modules we sum over is "diagonal," it is not the same set of modules we sum over for the $T_2$ theory (although the modules are in one-to-one correspondence). We will explain this observation in Chapter 7, based on the recent findings of [70]. It is quite remarkable that all the component Schur indices in our duality described in Fig. 2 and 3 are so closely related to those of free fields. We will explore this relation for the case of $(A_1, D_4)$ in Chapter 4. Moreover, the form of the partition function in (3.29) suggests simple generalizations to other (hypothetical) SCFTs.

- We have found strong evidence in favor of the quiver given in Fig. 4 for the mirror of the $S^1$ reduction of the $\mathcal{T}_{3,\frac{3}{2}}$ theory. Note, however, that the corresponding mirror for the $S^1$ reduction of the $\mathcal{T}_X$ theory contains 3D monopole mass terms

$$\delta W_{\mathcal{N}=2}=m\varphi_+\mathcal{O}_+ + m\varphi_-\mathcal{O}_-\ , \tag{3.40}$$

where $\mathcal{O}_\pm$ are the monopoles in the UV theory that map to the free (twisted) hypermultiplet according to the discussion in Footnote 29, and $\varphi_\pm$ are fields we add by hand in order to reproduce the IR SCFT that the $\mathcal{T}_X$ theory reduced on a circle flows to. This situation is quite unlike what happens for the mirrors of many of the dimensional reductions of the AD theories discussed in [53, 89] (see also the discussions in [37, 66, 90]).

[47] The fact that there are no imaginary FI parameters turned on is consistent with the 4D $U(1)_R$ symmetry flowing to the Cartan of the 3D $SU(2)_L \subset SO(4)_R$. This statement is also consistent (at least as far as the $\mathcal{N}=2$ chiral operators of the $\mathcal{T}_X$ theory are concerned) with the $SU(2)$ quantization condition discussed in [37].

## 3.7 A remark on the Hall-Littlewood chiral ring of $\mathcal{T}_X$ and the Schur sector

In this section, we briefly discuss the Hall-Littlewood (HL) chiral ring of the $\mathcal{T}_X$ theory in order to tease out some additional information about the Schur sector of the $\mathcal{T}_X$ SCFT. Based on our discussion above, the HL ring is generated by the following 4D Schur operators

$$\mu^A_{SU(2)} \in \hat{\mathcal{B}}_1 \ , \quad \mu^I_{SU(3)} \in \hat{\mathcal{B}}_1 \ , \quad \mathcal{O}^a_I \in \hat{\mathcal{B}}_{\frac{3}{2}} \ , \tag{3.41}$$

where $A$ and $I$ are adjoint indices of $SU(2)$ and $SU(3)$ respectively, and $a$ is a fundamental index of $SU(2)$.

In (3.28), we saw that $W^a_I = \chi\left[\mathcal{O}^a_I\right]$ and $J^I_{SU(3)} = \chi\left[\mu^I_{SU(3)}\right]$ had a non-trivial normal-ordered product in the $\mathbf{2}\times\mathbf{1}$ channel of $SU(2)\times SU(3)$. On the other hand, as we show in Appendix D, the HL limit of the $\mathcal{T}_X$ index has the following expansion

$$\begin{aligned}\mathcal{I}^{\mathcal{T}_X}_{HL}(t,w,z_1,z_2) &= 1+(\chi_2+\chi_{1,1})t+\chi_1\chi_{1,1}t^{\frac{3}{2}}+(1+\chi_4+\chi_{1,1}+\chi_2\chi_{1,1}+\chi_{2,2})t^2+ \\ &+ (\chi_1\chi_{1,1}+\chi_3\chi_{1,1}+\chi_1\chi_{3,0}+\chi_1\chi_{0,3}+\chi_1\chi_{2,2})t^{\frac{5}{2}}+\mathcal{O}(t^3) \ . \end{aligned}\tag{3.42}$$

Note that, compared with the Schur index in (6.8), the HL index is missing a contribution of the form $\chi_1$ at $\mathcal{O}(t^{\frac{5}{2}})\sim\mathcal{O}(q^{\frac{5}{2}})$ (recall that the power of the fugacity in the HL limit of the index is also given by $h=E-R$). The only apparent explanation, given our generators and the above discussion, is that there is a relation in the HL ring of the form

$$\mu^I_{SU(3)}\mathcal{O}^a_{\frac{3}{2}\ I}=0 \ . \tag{3.43}$$

In order to reconcile this relation with (3.28), we conjecture that the theory has a $\hat{\mathcal{C}}_{\frac{1}{2}(0,0)}$ multiplet with Schur operator, $\hat{\mathcal{O}}^{111}_{++}$, and that this operator appears in the $SU(2)_R$ twisted OPE of the $\mu^I_{SU(3)}$ and $\mathcal{O}^a_I$ operators (in the sense described in (2.48)) so that

$$\mu^I_{SU(3)}(z,\bar{z})\mathcal{O}^a_I(0)\supset\hat{\mathcal{O}}^{111}_{+\dot{+}}(0) \ . \tag{3.44}$$

At the level of component (untwisted OPEs), we have

$$J^{4d,I}_{SU(3)}(x)\mathcal{O}^a_I(0)\supset\frac{x_{-\dot{-}}}{x^2}\hat{\mathcal{O}}^{111}_{+\dot{+}}(0) \ , \tag{3.45}$$

where the operator on the far left of this inclusion is the $R=0$ partner of the holomorphic moment map, $\mu^I_{SU(3)}$. It is straightforward to check that such mixing is compatible with $\mathcal{N}=2$ superconformal Ward identities and that therefore $\hat{\mathcal{O}}^{111}_{+\dot{+}}$ maps to a normal ordered product of generators of $\chi(\mathcal{T}_X)$.[48] This discussion is analogous to what happens

[48] Often one must use highly non-trivial superspace techniques to determine which short multiplets are allowed by $\mathcal{N}=2$ superconformal symmetry to appear in the OPE of two short multiplets (e.g., see [91,92]). However, in our case, a more pedestrian approach suffices to show that (3.45) is allowed.

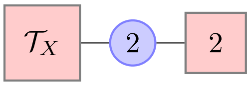



**Figure 6:** The above SCFT is inconsistent because of the $SU(2)$ anomaly of the $\mathcal{T}_X$ theory. It would be interesting to study how this inconsistency is manifested in the chiral algebra setting.

in the OPE of moment maps in the rank one theories discussed in [27] (there the 2D interpretation of the corresponding OPE is that the stress tensor is a Sugawara stress tensor; in the case of the $\mathcal{T}_X$ theory, the conclusion is quite different).

In the next section we will switch gears and focus on the implication of the non-vanishing Witten anomaly of $SU(2) \supset G_{\mathcal{T}_X}$ for the 2D/4D chiral algebra correspondence.

## 3.8 Witten's anomaly and the chiral algebra

One of the deepest questions in the 4D/2D chiral algebra correspondence of [27] is to understand which chiral algebras in 2D are part of a "swampland" of theories that cannot be related to consistent (and unitary) 4D $\mathcal{N}=2$ SCFTs. One example of a constraint all chiral algebras that are not part of this swampland must obey (unless they are part of the special set of chiral algebras related to a finite subset of free SCFTs in 4D with sufficiently few fields) follows from the analysis in [91]

$$c_{2d} \leq -\frac{22}{5} \ . \tag{3.49}$$

We would like to point out that another constraint chiral algebras outside the swampland must obey is that they are not related to 4D $\mathcal{N}=2$ SCFTs that have

Indeed, we can show that such terms exist in free SCFTs. To that end, consider a free hypermultiplet

$$q^i = \begin{pmatrix} Q \\ \tilde{Q}^\dagger \end{pmatrix} \ , \quad q^{\dagger i} = \tilde{q}^i = \begin{pmatrix} \tilde{Q} \\ -Q^\dagger \end{pmatrix} \ , \tag{3.46}$$

where $i$ is an $SU(2)_R$ spin-half index. Let us construct $\hat{\mathcal{B}}_1$ and $\hat{\mathcal{B}}_{\frac{3}{2}}$ multiplets of the form $q^{(i}\tilde{q}^{j)}$ and $q^{(i}q^j\tilde{q}^{k)}$ respectively (where "$(\cdots)$" denotes symmetrization of the enclosed indices). This theory has a $\hat{\mathcal{C}}_{\frac{1}{2}(0,0)}$ multiplet with a primary of the form $\epsilon_{ij}q^i\tilde{q}^j q^k$. The associated Schur operator is (up to an overall normalization)

$$\mathcal{O}^{111}_{++} \sim (\tilde{Q}\partial_{+\dot{+}}Q - Q\partial_{+\dot{+}}\tilde{Q})Q \ . \tag{3.47}$$

We then see that (3.45) is allowed by supersymmetry since a trivial computation in free field theory reveals that (at separated points)

$$\langle (QQ^\dagger - \tilde{Q}\tilde{Q}^\dagger)(x) QQ\tilde{Q}(y)(\tilde{Q}^\dagger\partial_{+\dot{+}}Q^\dagger - Q^\dagger\partial_{+\dot{+}}\tilde{Q}^\dagger)Q^\dagger)(0)\rangle \neq 0 \ . \tag{3.48}$$

a gauge symmetry with a Witten anomaly [78].[49] Indeed, the corresponding 4D theory is inconsistent. Interestingly, our $\mathcal{T}_X$ theory allows us to construct an infinite number of pathological SCFTs by gauging the $SU(2)$ global symmetry (of course, we can also construct infinitely many conformal manifolds that are consistent and have no Witten anomaly; note that the $\mathcal{T}_X$ theory on its own is also perfectly consistent since the $SU(2)$ symmetry is global).

A simple example of such a pathological theory is given in Fig. 6. To construct this SCFT, we gauge a diagonal $SU(2)$ flavor symmetry of the $T_2$ and $\mathcal{T}_X$ theories (where the $\mathcal{T}_X$ contribution is the anomalous $SU(2)$ factor and not a subgroup of $SU(3)$). The putative chiral algebra of this pathological theory can be constructed using the gauging prescription described in [27]. If this BRST procedure leads to a consistent chiral algebra, then (modulo the caveat described in footnote 49) it necessarily lies in the chiral algebra swampland. Using the expression for the $T_2$ index given in [82] and our expression in (3.29), it is straightforward to verify that the naive index of the pathological theory is[50]

$$\begin{aligned}\mathcal{I}(q, y_1, y_2, z_1, z_2) &= \sum_\lambda q^{2\lambda} P.E.\left[\frac{2q^2}{1-q} + 2q - 2q^{1+\lambda}\right] \mathrm{ch}^{SU(2)}_{R_\lambda}(q, y_1)\mathrm{ch}^{SU(2)}_{R_\lambda}(q, y_2)\times \\ &\times \quad \mathrm{ch}^{SU(3)}_{R_{\lambda,\lambda}}(q, z_1, z_2)\ , \end{aligned} \tag{3.50}$$

where $y_{1,2}$ are $SO(4)$ fugacties, and $z_{1,2}$ are the $SU(3)$ fugacities introduced above.

It would be interesting to understand how (or even if!) this pathology is manifested in the 2D setting. One possibility is that such chiral algebras (like the one whose vacuum character is given in (3.50)) are somehow pathological (or perhaps the non-trivial representations of these chiral algebras are pathological). Another possibility is that the chiral algebras and their modules are perfectly consistent at the level of 2D QFT but still detect the pathology of the 4D theory. While we have not fully investigated this question, we suspect the latter possibility holds (we should also note that, in principle, it could be that the chiral algebra and its representations are perfectly consistent and also do not detect the 4D pathology). We hope to return to this question soon.[51]

[49] However, it is conceivable that two different 4D $\mathcal{N} = 2$ SCFTs might have the same chiral algebra (although we are not aware of any such examples). Therefore, we cannot immediately rule out the (perhaps remote) possibility that one might have a 2D chiral algebra that is related both to a well-defined 4D SCFT and a pathological one of the type described here.

[50] We are making this statement at the naive level of operator counting. Note that the $Z_{S^1\times S^3}$ partition function (which differs from the index by certain pre-factors) may have additional pathologies.

[51] It may be possible to use some of the theories described in [93, 94] to study this question as well.

# Chapter 4

# A Non-Unitary Surprise

In this chapter we pick up on the curious connection between $(A_1, D_4)$ and free fields noted in Chapter 3. We start with the chiral algebra of $(A_1, D_4)$, which is a non-unitary AKM algebra and find that its unrefined characters are related to those of a unitary AKM algebra. Furthermore, this unitary chiral algebra is known to have a construction in terms of free fermions. This leads us to conjecture in Sec. 4.3, that the 4D parent theory of this unitary chiral algebra is a free but non-unitary SCFT. We verify this conjecture directly by carrying out the steps of the chiral algebra construction starting from the Lagrangian of the non-unitary SCFT. In Sec. 4.4 we extend this analysis to an infinite family of AD theories of which $(A_1, D_4)$ is just the first representative. Surprisingly then, this sequence of relations gives us a Lagrangian description of certain observables in a subsector of an infinite series of non-Lagrangian AD theories.

This chapter is based on [2].

## 4.1 Introduction

Free fields in two spacetime dimensions are versatile: operators, correlation functions, and partition functions of interacting conformal field theories can often be constructed algebraically from free bosons via the Coulomb gas formalism, and the simplest unitary minimal model—the Ising model—has a free Majorana fermion underlying it (see [95] for a review). Free fields in higher dimensions seem less powerful: in order to have something useful to say about an interacting CFT, one must usually labour to connect such free fields to the CFT in question through a suitably "smooth" path in the space of couplings.[52]

However, one may hope to overcome these obstacles in $d > 3$ spacetime dimensions whenever there are relations between quantum field theories in $d$ dimensions and QFTs

[52] We will not make this notion of smoothness precise here, except to say that, at the very least, there should not be any accidental symmetries along the resulting renormalization group flow that obscure the observable one would like to compute.

in 2D. As we saw earlier in the case of 4D superconformal field theories with at least $\mathcal{N}=2$ supersymmetry the sector of Schur operators of the 4D SCFT is isomorphic to a 2D chiral algebra. In the following we will exploit this correspondence, in particular the equality between the Schur index of the 4D SCFT and the torus partition function of the associated chiral algebra (2.57). Note that both of these quantities can be refined by additional flavor fugacities (i.e., fugacities for symmetries that commute with $\mathcal{N}=2$ SUSY in 4D), but such modifications will not play a role in our discussion below.

While we believe that many of the ideas we will present are quite broadly applicable (with suitable modifications), in this chapter we specialize to a particular infinite set of strongly coupled SCFTs whose simplest member is the $(A_1, D_4)$ theory we encountered in the previous chapter. In this class, the manipulations we use are particularly simple.

The Schur index for the $(A_1, D_4)$ theory was computed in [30, 37, 38, 84] and was shown to equal the vacuum character of $\widehat{su(3)}_{-\frac{3}{2}}$ (as conjectured in [96]). More recently, the authors of [31] proposed that this unflavored Schur index takes the following simple form

$$I_S^{(A_1,D_4)}(q) = q^{\frac{1}{3}}\text{P.E.}\left(8\frac{q}{1-q^2}\right) \equiv q^{\frac{1}{3}}\text{Exp}\left(8\sum_{n=1}^{\infty}\frac{1}{n}\frac{q^n}{1-q^{2n}}\right) , \tag{4.1}$$

and this formula was proven in [79] (see also the discussion in [32]) to be equivalent to the vacuum character of $\widehat{su(3)}_{-\frac{3}{2}}$. Interestingly, under the rescaling $q \to q^{\frac{1}{2}}$, (4.1) reduces to

$$I_S^{(A_1,D_4)}(q^{\frac{1}{2}}) = q^{\frac{1}{6}}\text{P.E.}\left(8\frac{q^{\frac{1}{2}}}{1-q}\right) = \left(I_S^{\text{half-hyper}}\right)^8 , \tag{4.2}$$

where according to (2.42) the righthand side (RHS) is just the index of eight free half-hypermultiplets (i.e., the $T_2$ theory [62]) or, equivalently in 2D, the vacuum character of four symplectic bosons.

While the derivation in [79] proved (4.1)[53] along with various generalizations we will encounter below, we would like to give a physical argument for why this index is so closely related to the index of free fields. One hint comes from our study in the previous chapter that shows the $(A_1, D_4)$ theory playing a role in a particular $S$-duality that is reminiscent of the role played by free hypermultiplets in the $S$-duality of [17]. Moreover, by thinking of (4.2) as a manifestation of a weak-strong "duality"[54] we speculated that this connection might be related to modularity.

As we will see below, this intuition is morally correct, although the free fields that are more closely related to modularity are actually non-unitary (wrong statistics) rather than the unitary fields appearing on the RHS of (4.2). A strong indication that this idea

[53]More precisely, the authors of [79] proved that the vacuum character of $\widehat{su(3)}_{-\frac{3}{2}}$ is given by the RHS of (4.1).

[54]The relation in (4.2) is not a duality in the truest sense of the word since it is a relation between the Schur sectors of two different theories.

is correct comes from noting that (4.1) satisfies the modular differential equation [36]

$$\left(D_q^{(2)} - 40\mathbb{E}_4\right) I_S^{(A_1,D_4)} = \left(D_q^{(2)} - 40\mathbb{E}_4\right) \chi_0^{\widehat{su(3)}_{-\frac{3}{2}}} = 0 \,, \tag{4.3}$$

where $D_q^{(2)}$ is a modular differential operator, and $\mathbb{E}_4$ is an Eisenstein series (we refer the interested reader to [36] for more details). The characters of $\widehat{so(8)}_1$ satisfy the same modular differential equation [44] and are included in the $\widehat{so(2n)}_1$ series whose characters are given by [95]:

$$\begin{aligned}
\chi_0^{\widehat{so(2n)}_1} &= \frac{1}{2}\left(\frac{\theta_3^n + \theta_4^n}{\eta^n}\right) \,, \\
\chi_{\frac{1}{2}}^{\widehat{so(2n)}_1} &= \frac{1}{2}\left(\frac{\theta_3^n - \theta_4^n}{\eta^n}\right) \,, \\
\chi_{\frac{N(N+1)}{4},1}^{\widehat{so(2n)}_1} &= \chi_{\frac{N(N+1)}{4},2}^{\widehat{so(2n)}_1} = \chi_{\frac{N(N+1)}{4}}^{\prime\widehat{so(2n)}_1} = \frac{1}{2}\frac{\theta_2^n}{\eta^n} \,,
\end{aligned} \tag{4.4}$$

where $q = e^{2\pi i\tau}$, $\eta$ is the Dedekind eta function, and $\theta_i$ are the Jacobi theta functions. Since $\widehat{so(8)}_1$ is unitary and has a representation in terms of eight free Majorana fermions, it is reasonable to imagine that the 4D ancestor of this theory is a non-unitary free theory (recall that, in the chiral algebra correspondence, $c_{4d} = -\frac{1}{12}c_{2d}$, so $c_{4d} < 0$ in this case). Clearly, these free fields then reproduce some of the observables in the Schur sector of the $(A_1, D_4)$ SCFT.

## 4.2 Modular $S$-transformations and an AKM relation

In order to understand the modular properties of (4.1), it is useful to re-write it as follows

$$I_S^{(A_1,D_4)} = 2^{-4}\frac{\theta_2^4}{\eta^4} \,. \tag{4.5}$$

Under a modular $S$-transformation, we have

$$\theta_2\left(-\frac{1}{\tau}\right) = \sqrt{-i\tau}\,\theta_4(\tau) \,, \quad \eta\left(-\frac{1}{\tau}\right) = \sqrt{-i\tau}\,\eta(\tau) \,. \tag{4.6}$$

In particular, we see that applying a modular $S$-transformation to (4.5) yields

$$\mathcal{S}\left(I_S^{(A_1,D_4)}\right) = 2^{-4}\frac{\theta_4^4}{\eta^4} = 2^{-4}q^{-\frac{1}{6}}\ \text{P.E.}\left(-\frac{8q^{\frac{1}{2}}}{1-q}\right) \,. \tag{4.7}$$

We immediately recognize the expression on the RHS as also counting (with a $(-1)^F$ weighting) the $\widehat{so(8)}_1$ fields generated by acting on the $\widehat{so(8)}_1$ vacuum with the $h = \frac{1}{2}$ Majorana fermions in the $\mathbf{8_v}$ representation, $\psi^I$ (where $I = 1, \cdots, 8$) [95] (hence, this theory is related to eight decoupled Ising models). These fields have the following singular OPE

$$\psi^I(z)\psi^J(w) \sim \frac{\delta^{IJ}}{z-w} \ . \tag{4.8}$$

At the level of characters, we have the relation

$$\begin{aligned} \mathcal{S}\Big(\chi_0^{\widehat{su(3)}_{-\frac{3}{2}}}\Big) &= -\frac{1}{2}\Big(\chi_0^{\widehat{su(3)}_{-\frac{3}{2}}} + \chi_{-\frac{1}{2},1}^{\widehat{su(3)}_{-\frac{3}{2}}} + \chi_{-\frac{1}{2},2}^{\widehat{su(3)}_{-\frac{3}{2}}} \\ &- \chi_{-\frac{1}{2},3}^{\widehat{su(3)}_{-\frac{3}{2}}}\Big) = 2^{-4}\Big(\chi_0^{\widehat{so(8)}_1} - \chi_{\frac{1}{2},v}^{\widehat{so(8)}_1}\Big) \ , \end{aligned} \tag{4.9}$$

where, in the second equality, we have used our observation above and, in the first equality, we have used the modular $S$ matrix acting on the characters of the four admissible representations of $\widehat{su(3)}_{-\frac{3}{2}}$

$$S_{\widehat{su(3)}_{-\frac{3}{2}}} = -\frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{pmatrix} \ . \tag{4.10}$$

There are four admissible representations of $\widehat{su(3)}_{-\frac{3}{2}}$ (the vacuum and three $h = -\frac{1}{2}$ representations) and four representations of $\widehat{so(8)}_1$ (the vacuum and three $h = \frac{1}{2}$ representations), but in the latter case all four corresponding unrefined characters are finite, while in the former case only two linear combinations of unrefined characters are finite (the vacuum and the linear combination of $h = -\frac{1}{2}$ characters, $\chi_{-\frac{1}{2}}^{\prime\widehat{su(3)}_{-\frac{3}{2}}}$, appearing in (4.9)). However, all the unrefined $h = \frac{1}{2}$ characters of $\widehat{so(8)}_1$ are equal (we denote the corresponding character $\chi_{\frac{1}{2}}^{\prime\widehat{so(8)}_1}$), and we find the bijection of finite unrefined characters[55]

$$\chi_0^{\widehat{su(3)}_{-\frac{3}{2}}} \sim \chi_{\frac{1}{2}}^{\prime\widehat{so(8)}_1} \ , \quad \chi_{-\frac{1}{2}}^{\prime\widehat{su(3)}_{-\frac{3}{2}}} \sim \chi_0^{\widehat{so(8)}_1} \ , \tag{4.11}$$

where the relations hold up to overall constants (see [45] for character relations between other pairs of unitary and non-unitary theories).

## 4.3 A 4D interpretation

We would like to give a 4D interpretation for the unitary $\widehat{so(8)}_1$ theory described in the previous section by using the relation discovered in [27] (although, apriori, it is not

[55] Ultimately, the fact that there are only two characters transforming amongst each other under modular transformations in the case of our two AKM algebras is a consequence of the Jacobi quartic identity $\theta_3(\tau)^4 = \theta_2(\tau)^4 + \theta_4(\tau)^4$.

clear such an interpretation must exist). As discussed above, this theory should be non-unitary since

$$c_{\widehat{so(8)}_1} = 4 \ \Rightarrow \ c_{4d} = -\frac{1}{3} \ . \tag{4.12}$$

Moreover, from the results in (2.57) and (4.7), we see that an obvious candidate for our 4D theory is a collection of 8 half-hypermultiplets with wrong statistics (i.e., a "ghost" $T_2$ theory).[56] Indeed, the $a$ and $c$ anomalies for such a theory are just minus the corresponding anomalies for the $T_2$ theory since the wrong statistics leads to an insertion of a factor of $-1$ in any quantum loop. In particular, we have

$$c_{4d} = -8 \times c_{\rm half-hyper} = -\frac{1}{3} \ , \quad a_{4d} = -8 \times a_{\rm half-hyper} = -\frac{1}{6} \ . \tag{4.13}$$

Note that $a_{4d} - c_{4d}$ is then consistent with the $q \to 1$ "Cardy" limit of the index [30, 36, 85, 86], and the full (unrefined) Schur index is precisely what we want (see the previous footnote).

To get a map of operators, the chiral algebra correspondence requires us to take 4D Schur operators, fix them in a plane (with coordinates $z, \bar{z}$), and then twist the global $\bar{z}$ conformal transformations with $su(2)_R$. Working in the cohomology of a particular supercharge, $\mathbb{Q}$, then gives a map to 2D chiral algebra operators. This procedure is naturally implemented in the operator product expansion.

For the case at hand, we can build all Schur operators as arbitrary (non-vanishing) products of the $su(2)_R$ highest weight anti-commuting scalars of the non-unitary free hypermultiplets, $q^I$, and their derivatives. These fields are organized as $q^i = Q^i$ and $q^{i+4} = \tilde{Q}^i$ (with $i = 1, \cdots, 4$) and live in the following $su(2)_R$ doublets

$$\begin{pmatrix} Q^i \\ \tilde{Q}^{i\dagger} \end{pmatrix} \ , \quad \begin{pmatrix} \tilde{Q}^i \\ -Q^{i\dagger} \end{pmatrix} \ . \tag{4.14}$$

We can write a simple Lagrangian for this non-unitary theory (note that the spinors in the hypermultiplets commute while the scalars anti-commute)

$$\mathcal{L} = -\int d^4\theta \left( q^{I\dagger} \Omega_{IJ} q^J \right) = \int d^4\theta \left( \tilde{Q}^{i\dagger} \delta_{ij} Q^j - Q^{i\dagger} \delta_{ij} \tilde{Q}^j \right) \ , \tag{4.15}$$

where we have defined

$$\Omega \equiv \begin{pmatrix} 0_{4\times4} & 1_{4\times4} \\ -1_{4\times4} & 0_{4\times4} \end{pmatrix} \ . \tag{4.16}$$

Related Lagrangians have been considered in different contexts in [97, 98].

The non-vanishing singular OPEs are then (in an appropriate normalization to

[56] To get just the $\widehat{so(8)}_1$ vacuum module, we should restrict to composite Schur operators built from an even number of hypermultiplet scalars.

eliminate a common overall constant factor)

$$\tilde{Q}^{i\dagger}(x)Q^j(0) \sim \frac{\delta^{ij}}{x^2} \;, \quad Q^{i\dagger}(x)\tilde{Q}^j(0) \sim -\frac{\delta^{ij}}{x^2} \;. \tag{4.17}$$

According to the discussion in [27], we should twist the hypermultiplets with vectors $u_i = (1, \bar{z})$ having $su(2)_R$ indices $i = 1, 2$ as in (2.48). In particular, we have twisted-translated fields

$$\begin{aligned} Q'^i(z,\bar{z}) &= Q^i(z,\bar{z}) + \bar{z}\tilde{Q}^{i\dagger}(z,\bar{z}) \;, \\ \tilde{Q}'^i(z,\bar{z}) &= \tilde{Q}^i(z,\bar{z}) - \bar{z}Q^{i\dagger}(z,\bar{z}) \;, \end{aligned} \tag{4.18}$$

with the following singular OPEs

$$Q'^i(z,\bar{z})Q'^j(0,0) \sim \frac{\delta^{ij}}{z} \;, \quad \tilde{Q}'^i(z,\bar{z})\tilde{Q}'^j(0,0) \sim \frac{\delta^{ij}}{z} \;. \tag{4.19}$$

Passing to $\mathbb{Q}$ cohomology gives the same OPEs as above (the identity operator is $\mathbb{Q}$-closed but clearly cannot be $\mathbb{Q}$-exact). In particular, we reproduce the free Majorana OPEs of (4.8).

The theory also has conserved currents sitting as level-two descendants in multiplets with Schur operators of the form

$$\mu^{ij} = iQ^iQ^j \;, \quad \tilde{\mu}^{ij} = i\tilde{Q}^i\tilde{Q}^j \;, \quad \mu'^{ij} = iQ^i\tilde{Q}^j \;, \tag{4.20}$$

where $i, j = 1, \cdots, 4$. More covariantly, we can define these operators to form part of a 28-dimensional adjoint representation with $\mu^{IJ} = iq^Iq^J$ and $I, J = 1, \cdots, 8$ (this operator is anti-symmetric in $I$ and $J$). The charges arising from real currents sitting as descendants of linear combinations of the above satisfy an $so^*(8) \simeq so(6,2)$ Lie algebra, which is a real form of $so(8, \mathbb{C})$. On the other hand, the operators in (4.20) are related to currents that are not real. However, these currents give rise to charges that act in accordance with the reality condition in two dimensions

$$\begin{aligned} \mu^{ij} &: \; \delta Q^i \sim -Q^j \;, \delta Q^j \sim Q^i \;, \delta\tilde{Q}^{i\dagger} \sim -\tilde{Q}^{j\dagger} \;, \delta\tilde{Q}^{j\dagger} \sim \tilde{Q}^{i\dagger} \;, \\ \tilde{\mu}^{ij} &: \; \delta\tilde{Q}^i \sim \tilde{Q}^j \;, \delta\tilde{Q}^j \sim -\tilde{Q}^i \;, \delta Q^{i\dagger} \sim Q^{j\dagger} \;, \delta Q^{j\dagger} \sim -Q^{i\dagger} \;, \\ \mu'^{ij} &: \; \delta\tilde{Q}^j \sim Q^i \;, \delta Q^i \sim -\tilde{Q}^j \;, \delta\tilde{Q}^{i\dagger} \sim Q^{j\dagger} \;, \delta Q^{j\dagger} \sim -\tilde{Q}^{i\dagger} \;. \end{aligned}$$

Relabeling the moment maps with an adjoint index of $so(8)$, we obtain the following twisted OPE

$$\mu^A(z,\bar{z})\mu^B(0) \sim \frac{\delta^{AB}}{z^2} + if^{AB}{}_C\frac{\mu^C(0,0)}{z} + \{\mathbb{Q}, \cdots\} \;, \tag{4.21}$$

where $f^{AB}{}_C$ are the structure constants of $so(8)$. Dropping the $\mathbb{Q}$-exact terms then leads to the standard $\widehat{so(8)}_1$ current-current OPE. As a result, we see that a generalization of the procedure of [27] applied to our non-unitary 4D theory yields the desired unitary theory in 2D.

## 4.4 Infinitely many generalizations

One can imagine generalizing our discussion above in many directions. Here we choose the simplest direction: the $(A_1, D_4)$ theory is part of an infinite family of SCFTs called the $D_2[SU(2N+1)]$ theories [99, 100] (where $D_2[SU(3)] \equiv (A_1, D_4)$). The corresponding chiral algebras were found in [31] and were argued to be $\widehat{su(2N+1)}_{-\frac{2N+1}{2}}$. The generalization of (4.1) is

$$I_S^{D_2[SU(2N+1)]}(q) = q^{\frac{N(N+1)}{6}} \text{P.E.}\left(4N(N+1)\frac{q}{1-q^2}\right) , \tag{4.22}$$

and one finds that, upon taking $q \to q^{\frac{1}{2}}$, the index (4.22) reduces to the index of $4N(N+1)$ free half-hypermultiplets.

For $N > 1$, the modular properties of the theory are somewhat different. For example, the modular differential equation in (4.3) for the $N = 1$ case becomes third order for all $N > 1$. However, we can proceed as before and write

$$I_S^{D_2[SU(2N+1)]} = 2^{-2N(N+1)} \frac{\theta_2^{2N(N+1)}}{\eta^{2N(N+1)}} . \tag{4.23}$$

Then, performing a modular $S$-transformation yields

$$\begin{aligned} \mathcal{S}\left(I_S^{D_2[SU(2N+1)]}\right) &= 2^{-2N(N+1)} \frac{\theta_4^{2N(N+1)}}{\eta^{2N(N+1)}} \\ &= 2^{-2N(N+1)} q^{-\frac{N(N+1)}{12}} \text{ P.E.}\left(-\frac{4N(N+1)q^{\frac{1}{2}}}{1-q}\right) . \end{aligned} \tag{4.24}$$

This result generalizes the $N = 1$ result discussed above, since we recognize (4.24) as also counting (with a $(-1)^F$ weighting) the $\widehat{so(4N(N+1))}_1$ fields generated by acting on the $\widehat{so(4N(N+1))}_1$ vacuum with the $h = \frac{1}{2}$ Majorana fermion in the vector representation, $\psi^I$ (its singular self-OPE is the obvious generalization of (4.8) with $I = 1, \cdots, 4N(N+1)$).

The $\widehat{so(4N(N+1))}_1$ algebra has four representations for all $N$: the vacuum, the $h = 1/2$ representation discussed above, and two $h = N(N+1)/4$ representations. The latter two representations have identical unrefined characters which we denote as $\chi^{\widehat{so(4N(N+1))}_1}_{\frac{N(N+1)}{4}}$ (for $N = 1$, the last three unrefined characters are identical). On the other hand, the $\widehat{su(2N+1)}_{-\frac{2N+1}{2}}$ algebra has three finite (linear combinations of) unrefined characters that transform into each other under modular transformations: one starting with $h=0$ (the vacuum), one starting with $h=\frac{2-N(N+1)}{4}$, and one starting with $h=-\frac{N(N+1)}{4}$. It is straightforward to check that

$$\begin{aligned}
\chi_0^{\widehat{su(2N+1)}_{-\frac{2N+1}{2}}} &\sim \chi_{\frac{N(N+1)}{4}}^{\prime\widehat{so(4N(N+1))}_1}\ ,\\
\chi_{\frac{2-N(N+1)}{4}}^{\prime\widehat{su(2N+1)}_{-\frac{2N+1}{2}}} &\sim \chi_{\frac{1}{2}}^{\widehat{so(4N(N+1))}_1}\ ,\\
\chi_{-\frac{N(N+1)}{4}}^{\prime\widehat{su(2N+1)}_{-\frac{2N+1}{2}}} &\sim \chi_0^{\widehat{so(4N(N+1))}_1}\ .
\end{aligned} \tag{4.25}$$

These results are simple consequences of the fact that our two chiral algebras satisfy the same modular differential equation for all $N$.

The 4D generalization of the $N=1$ case is straightforward. For example, we have that

$$c_{\widehat{so(4N(N+1))}_1}=2N(N+1)\Rightarrow c_{4d}=-\frac{N(N+1)}{6}\ . \tag{4.26}$$

This anomaly is precisely what we expect for $4N(N+1)$ half-hypers with wrong statistics (i.e., $N(N+1)/2$ "ghost" $T_2$ theories). Similarly, $a_{4d}$ and the superconformal index are compatible with this interpretation. In particular, our 4D Lagrangian is just the obvious generalization of (4.15)

$$\mathcal{L}=-\int d^4\theta\left(q^{I\dagger}\Omega_{IJ}q^J\right)=\int d^4\theta\left(\tilde{Q}^{i\dagger}\delta_{ij}Q^j-Q^{i\dagger}\delta_{ij}\tilde{Q}^j\right)\ , \tag{4.27}$$

where now $I=1,\cdots,4N(N+1)$. Note that the real flavor currents in 4D generate an $so^*(4N(N+1))$ algebra, but the $N(2N-1)$ Schur operators that are the generalizations of (4.20) give rise to the $\widehat{so(4N(N+1))}_1$ AKM algebra in 2D.[57]

[57]To get just the $\widehat{so(4N(N+1))}_1$ vacuum module, we should restrict to composite Schur operators built from an even number of $q^I$.

# Chapter 5

# Rationalizing CFTs and Anyonic Imprints on Higgs Branches

In this chapter we continue our exploration of Argyres-Douglas theories through the study of their chiral algebras. In contrast to the previous chapter, our aim here is not to find a free field representation, but to study RG flows between the AD theories $(A_1, D_p)$ and $(A_1, A_{p-3})$ for positive odd $p$. These RG flows have the natural interpretation of closing a regular puncture on the Riemann surface associated with the class $\mathcal{S}$ construction of $(A_1, D_p)$ theories.

Even though our goals are ultimately different, our tools will be similar to those of Chapter 4, and we start with a character relation, this time between the logarithmic chiral algebras of $(A_1, D_p)$ and a series of rational chiral algebras. This flavorless relation has the advantage that both chiral algebras have the same number of flavor fugacities and we manage to turn on discrete flavor fugacities in Sec. 5.3.1. Motivated by this correspondence, we analyze the modular data of the modular tensor categories (MTC) underlying the chiral algebras of these two series of AD theories. We find that some pieces of this data called quantum dimensions are related via the action of Galois conjugation. We close the chapter with some implications of this result and connections to other theories.

This chapter is based on [3], detailed proof of the relation between quantum dimensions can be found in Appendix E.

## 5.1 Introduction

Quantum field theory in lower dimensions generally seems richer and less rigid than QFT in higher dimensions. For example, in D $<$ 7 dimensions we readily find many examples of interacting conformal field theories, while the situation looks somewhat murkier in D $\geq$ 7 (however, see [101]). As another example, 2D CFTs readily admit non-supersymmetric exactly marginal deformations, while the situation in D $> 2$ seems far more constrained. In some sense, the relative richness of lower dimensions is to be expected: we can compactify higher dimensional QFTs, and the geometry and topology of the compactifications then enrich the resulting lower-dimensional theories.

Given this picture, we may expect that when a direct algebraic link exists (without going through a compactification) between certain QFTs in higher dimensions and a subset of QFTs in lower dimensions, this subset of lower dimensional QFTs will be "small" in comparison with the full space of lower dimensional theories.

One concrete playground in which to test this idea is given by the 4D/2D chiral algebra correspondence in [27][58]: Schur operators in 4D $\mathcal{N}=2$ SCFTs are related to sets of meromorphic currents generating non-unitary 2D chiral algebras. While the resulting space of 2D chiral algebras is quite large (e.g., see [1,4,19,27,30–33,38,82,103–107])—reflecting the diversity of 4D $\mathcal{N}=2$ SCFTs[59]—it is a highly constrained subspace within the space of 2D chiral algebras (e.g., see [19,36,91] for some constraints). More simply, if we start from unitary 4D theories, then the corresponding 2D theories should be non-unitary chiral algebras with a hidden notion of unitarity.

Motivated by these ideas and a duality discussed in [75], in Chapter 4 we embarked on a program to relate the (typically) logarithmic theories that appear via the correspondence in [27][60] with a more special set of 2D theories: the unitary rational conformal field theories (RCFTs). These theories, which include the well-known $(m, m+1)$ (where $m \geq 3$) Virasoro minimal models as well as various affine Kac-Moody theories and even many of the more complicated higher-spin $W$-algebra theories (e.g., see [108] for a review), form a very "small" subspace in the space of 2D CFTs.

More precisely, in Chapter 4 we studied an infinite class of 4D $\mathcal{N}=2$ SCFTs called the $D_2[SU(2n+1)]$ theories [99,100]. The corresponding chiral algebras are the logarithmic $\widehat{su(2n+1)}_{-\frac{2n+1}{2}}$ AKM theories (see [30,37] for the $n=1$ case and [31,33] for $n \geq 1$). We then showed that the finite linear combinations of unrefined characters for admissible[61] representations of $\widehat{su(2n+1)}_{-\frac{2n+1}{2}}$ coincide (up to overall constants) with unrefined characters of the free $\widehat{so(4n(n+1))}_1$ theories. For example, in the case of $n=1$, $D_2[SU(3)]$, we have (up to an overall constant that has been dropped) (4.11)

[58] Similar ideas can also be pursued using the more restricted theories in [102].

[59] It is not clear that the chiral algebra and its representations uniquely specify the 4D theory, so there may be some coarse-graining involved in this correspondence. Note that even in 2D CFT itself, the left and right chiral algebras and their representations are not always sufficient to specify a 2D CFT (e.g., we can have different permutation modular invariants).

[60] Note that these theories are sometimes non-unitary but rational. For example, the $(A_1, A_{p-3})$ SCFTs with odd $p \geq 5$, which will appear again below, have chiral algebras corresponding to those of the $(2, p)$ Virasoro minimal models.

[61] For an introduction to these types of representations, see [95]. Roughly speaking, they are highest weight representations that transform linearly into each other under modular transformations.

$$\chi_0(q)^{\widehat{su(3)}_{-\frac{3}{2}}} \sim \chi_{\frac{1}{2}}^{\widehat{so(8)}_1}(q) \ , \quad \chi_{-\frac{1}{2}}^{\widehat{su(3)}_{-\frac{3}{2}}}(q) \sim \chi_0^{\widehat{so(8)}_1}(q) \ , \tag{5.1}$$

where $\chi_0^{\widehat{su(3)}_{-\frac{3}{2}}}(q)$ and $\chi_0^{\widehat{so(8)}_1}(q)$ are the vacuum characters of $\widehat{su(3)}_{-\frac{3}{2}}$ and $\widehat{so(8)}_1$ respectively, $\chi_{\frac{1}{2}}^{\widehat{so(8)}_1}(q)$ is the character for a dimension $1/2$ primary of $\widehat{so(8)}_1$ (there are three such primaries, and their unrefined characters are all equal), and $\chi_{-\frac{1}{2}}^{\widehat{su(3)}_{-\frac{3}{2}}}(q)$ is a finite linear combination of characters for the three primaries with scaling dimension $-1/2$. In these relations, the non-unitary vacuum is mapped to a unitary primary with largest scaling dimension, and a linear combination of the smallest scaling dimension non-unitary primaries is mapped to the unitary vacuum. Given this matching, a main result in Chapter 4 was to find a 4D interpretation of the $\widehat{so(8)}_1$ chiral RCFT (and similarly for $\widehat{so(4n(n+1))}_1$).

While we will briefly return to this 4D interpretation below, our goals in the present chapter are different:

- First, we straightforwardly generalize the correspondence in Chapter 4 between logarithmic theories descending from 4D via the chiral algebra correspondence and 2D chiral RCFTs to include flavor fugacities as refinements. For simplicity (and because of their more interesting Higgs branches), we will mainly focus on a slightly different class of 4D $\mathcal{N}=2$ theories, the so-called $(A_1, D_p)$ theories with $p \in \mathbb{Z}_{\text{odd}}$.[62] However, we will return to the particular theories in Chapter 4 toward the end of this chapter.

- Second, we will study the topological quantum field theories—or, in a more mathematical language, the modular tensor categories (MTCs)[63]—underlying the 2D chiral RCFTs, and we will show that these MTCs contain seeds of the IR physics that result from certain Higgs branch RG flows in 4D. In all the examples we will consider, these MTCs are MTCs associated with Chern-Simons theories.

At a naive level, one can see an apparently suggestive topological link between the admissible representations of the logarithmic $\widehat{su(3)}_{-\frac{3}{2}}$ chiral CFT and the representations of $\widehat{so(8)}_1$ by constructing the naive fusion rules for the logarithmic theory that follow from applying Verlinde's formula to the modular $S$-matrix for the admissible

[62]We follow the naming conventions of [58].

[63]We will describe the relevant aspects of MTCs in somewhat more detail below. Roughly speaking, MTCs consist of a fusion algebra (in this case a commutative multiplication operation) specified by the action on various simple elements (i.e., elements that are not sums of other elements), a set of matrices, $F$, that implement associativity and satisfy a set of polynomial equations called the "pentagon" equations, and a set of braiding matrices, $R$, that, together with the $F$ matrices satisfy the so-called "hexagon" equations (e.g., see [109–111]). Moreover, the associated $S$ and $T$ matrices are non-degenerate (and hence the theory is modular).

representations. Indeed, labeling the four admissible representations in this theory as $1, a, b, c$ (where 1 is the vacuum, and $a, b, c$ are dimension $-1/2$ representations), one finds (dropping the trivial $1 \otimes x = x$ for $x = 1, a, b, c$)

$$a \otimes a = 1 \ , \quad a \otimes b = -c \ , \quad a \otimes c = -b \ , \quad b \otimes b = 1 \ , \quad b \otimes c = -a \ , \quad c \otimes c = 1 \ . \tag{5.2}$$

Up to some signs, which reflect the fact that these are not the actual fusion rules of the theory (e.g., see [112–115]),[64] we find the fusion rules for $\mathbb{Z}_2 \times \mathbb{Z}_2$. Still, we might be tempted to interpret these signs as being related in some way to a projective representation of $\mathbb{Z}_2 \times \mathbb{Z}_2$. More formally, we may write

$$x \otimes y = \omega(x, y) \cdot z \ , \tag{5.3}$$

where $\omega(x, y) \in H^2(\mathbb{Z}_2 \times \mathbb{Z}_2, U(1)) = \mathbb{Z}_2$ is a 2-cocycle[65] with

$$\omega(a, b) = \omega(b, a) = \omega(a, c) = \omega(c, a) = \omega(b, c) = \omega(c, b) = -1 \ , \tag{5.5}$$

and all other $\omega = 1$. In fact, our $\omega$ is trivial in $H^2(\mathbb{Z}_2 \times \mathbb{Z}_2, U(1))$ (i.e., it is a 2-coboundary[66]) and so we are naively led to interpret the simple elements as leading to a genuine representation of $\mathbb{Z}_2 \times \mathbb{Z}_2$.

While the above analysis is suggestive of a link with $\mathbb{Z}_2 \times \mathbb{Z}_2$ fusion rules, we can make a more direct connection by noting that the $\widehat{su(3)}_{-\frac{3}{2}}$ theory is related, at the level of unrefined characters, to the $\widehat{so(8)}_1$ theory via (5.1). This latter theory has genuine $\mathbb{Z}_2 \times \mathbb{Z}_2$ fusion rules! The underlying MTC is just a theory of abelian anyons with a one-form $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry (see [116] for a discussion of one-form symmetries) generated by these anyons.[67]

A different link to abelian anyons appeared in the interesting paper [120] for the

[64] One issue is that, properly speaking, the admissible modules are not closed under fusion. To find a set of representations that are (conjecturally) closed under fusion one should consider so-called (generalized) "relaxed" highest weight modules and their images under spectral flow. We thank Simon Wood for a discussion on this point.

[65] In other words, $\omega$ satisfies

$$\omega(h, k) \cdot \omega(g, hk) = \omega(g, h) \cdot \omega(gh, k) \ , \quad \omega(1, g) = 1 \ , \quad \forall g, h, k \in \mathbb{Z}_2 \times \mathbb{Z}_2 \ . \tag{5.4}$$

[66] This statement amounts to the fact that $\omega(x, y) = \omega(x)\omega(y)\omega(xy)^{-1}$ with $\omega(1) = 1$ and $\omega(a) = \omega(b) = \omega(c) = -1$.

[67] At the level of the underlying MTC, one way to describe the full set of results in Chapter 4 is that we associate the two independent MTCs with $\mathbb{Z}_2 \times \mathbb{Z}_2$ fusion rules—the $Spin(8)_1$ MTC and the toric code MTC (e.g., see [117, 118] for a discussion of these MTCs)—with the $D_2[SU(2n+1)]$ SCFTs. In particular, if $n(n+1) = 0 \pmod 4$, then we associate the toric code MTC with the 4D theory. On the other hand, if $n(n+1) = 2 \pmod 4$, then we associate the $Spin(8)_1$ MTC with the theory. Note that the number of admissible representations in $\widehat{su(2n+1)}_{-\frac{2n+1}{2}}$ is $2^{2n}$ [119], so this is not, in general, a one-to-one map of admissible representations to simple elements.

$(A_1, A_3) \simeq (A_1, D_3)$ SCFT and formed some of the motivation for the work described in this chapter. There the authors studied new TQFTs coming from AD theories and noted that by "flipping the sign" of a simple object in their $(A_1, A_3)$ TQFT, one obtains an MTC with $\mathbb{Z}_3$ fusion rules. In the present context, the naive fusion rules of the admissible representations of the $\widehat{su(2)}_{-\frac{4}{3}}$ chiral algebra associated with the $(A_1, A_3) \simeq (A_1, D_3)$ theory [30] are

$$a \otimes b = -1 \ , \quad a \otimes a = b \ , \quad b \otimes b = -a \ , \tag{5.6}$$

which, up to two signs, are just $\mathbb{Z}_3$ fusion rules.[68] By solving the hexagon and pentagon equations, it is easy to check that there are two independent unitary MTCs with such $\mathbb{Z}_3$ fusion rules[69] (see also [117]): $SU(3)_1$ and $(E_6)_1$. Therefore, it is natural to wonder if there is an associated RCFT whose characters are related to those of the $\widehat{su(2)}_{-\frac{4}{3}}$ theory in a way that parallels the relation in (5.1).

In fact, an old result of Mukhi and Panda [45] shows the following proportionality of unrefined characters

$$\chi_0^{\widehat{su(2)}_{-\frac{4}{3}}}(q) \sim \chi_{\frac{1}{3}}^{\widehat{su(3)}_1}(q) \ , \quad \chi_{-\frac{1}{3}}^{\widehat{su(2)}_{-\frac{4}{3}}}(q) \sim \chi_0^{\widehat{su(3)}_1}(q) \ , \tag{5.7}$$

where $\chi_0^{\widehat{su(2)}_{-\frac{4}{3}}}(q)$ is the vacuum character of $\widehat{su(2)}_{-\frac{4}{3}}$, $\chi_{-\frac{1}{3}}^{\widehat{su(2)}_{-\frac{4}{3}}}(q)$ is a finite linear combination of the characters corresponding to the two dimension $-1/3$ representations of $\widehat{su(2)}_{-\frac{4}{3}}$, $\chi_0^{\widehat{su(3)}_1}(q)$ is the vacuum character of $\widehat{su(3)}_1$, and $\chi_{\frac{1}{3}}^{\widehat{su(3)}_1}(q)$ is the character of a dimension $1/3$ representation of $\widehat{su(3)}_1$ (there are two such representations, and their unrefined characters are equal). As in (5.1), the non-unitary vacuum is mapped to a unitary primary with largest scaling dimension, and a linear combination of the smallest scaling dimension non-unitary primaries is mapped to the unitary vacuum. Therefore, we see that the $\widehat{su(3)}_1$ theory is the desired theory related to an MTC with $\mathbb{Z}_3$ fusion rules.

It will be somewhat more useful to think about the $\widehat{su(3)}_1$ characters in terms of the $D$-type modular invariant of $\widehat{su(2)}_4$ [121, 122], which we will denote as $\tilde{D}_4$.[70] This theory can be obtained from $\widehat{su(2)}_4$ by gauging the $\mathbb{Z}_2$ symmetry.[71] In particular, one

[68] As in the $(A_1, D_4)$ case, it is easy to check that these two signs give rise to a 2-coboundary. This statement is consistent with the fact that $H^2(\mathbb{Z}_3, U(1)) = \emptyset$. In particular, by formally taking $a \to -a$ in (5.6) we recover $\mathbb{Z}_3$ fusion rules.

[69] There are infinitely many CFTs associated with each of these MTCs since we can take any theory satisfying these fusion rules and tensor in arbitrarily many $(\hat{e}_8)_1$ RCFTs.

[70] We add the tilde on top of $\tilde{D}_4$ to distinguish this $D$ from the one appearing in the related $(A_1, D_3)$ 4D $\mathcal{N} = 2$ SCFT.

[71] At the level of the underlying MTC, this procedure corresponds to the evocatively named "anyon condensation" [123, 124] (see also the recent [125]) and leaves over an MTC with $\mathbb{Z}_3$ fusion rules consisting of anyons having trivial braiding with the anyons generating the $\mathbb{Z}_2$ one-form symmetry in

finds [45]

$$\chi_0^{\widehat{su(2)}_{-\frac{4}{3}}}(q) \sim \chi_{\frac{1}{3}}^{\tilde{D}_4}(q) = \chi_2^{\widehat{su(2)}_4}(q) \ , \quad \chi_{-\frac{1}{3}}^{\widehat{su(2)}_{-\frac{4}{3}}}(q) \sim \chi_0^{\tilde{D}_4}(q) = \chi_0^{\widehat{su(2)}_4}(q) + \chi_4^{\widehat{su(2)}_4}(q) \ , \tag{5.8}$$

where the $\widehat{su(2)}_4$ characters appearing on the RHS of the above expressions are indexed by an $su(2)$ Dynkin label subscript.

The interpretation in terms of $\widehat{su(2)}_4$ is particularly useful, since now there is a canonical way in which we can try to turn on flavor fugacities in (5.8) (the number of fugacities on the LHS and RHS match). As we will see below, there is a discrete subset of fugacities we can turn on so that the characters of $\tilde{D}_4$ are equal to those of $\widehat{su(2)}_{-\frac{4}{3}}$ up to overall $q$-independent functions. These $q$-independent functions generalize the constants of proportionality we suppressed in writing (5.8). As we will see, a similar story holds for the more general $\widehat{su(2)}_{2(1-p)/p}$ chiral algebras corresponding to the $(A_1, D_p)$ theories with $p \in \mathbb{Z}_{\rm odd}$ and the $\mathbb{Z}_2$ gauging of $\widehat{su(2)}_{2(p-1)}$, $\tilde{D}_{p+1}$.

The existence of such a matching set of fugacities then motivates us to study RG flows onto the Higgs branch of our $(A_1, D_p)$ theories from the perspective of the related 2D rational chiral algebras and their representations. For $\widehat{su(2)}_{\frac{2(1-p)}{p}}$, the 2D avatar of the 4D Higgs branch RG flow is just quantum Drinfeld-Sokolov (qDS) reduction [49] (see also [27, 126] for earlier discussion in other theories).

Instead of performing qDS on the unitary side, we will show that the MTCs underlying our unitary theories "know" about certain quantum dimensions (or expectation values for Wilson loop operators) in the non-unitary MTCs related to the IR Higgs branch theories. More precisely, we will argue that these quantum dimensions can be computed after performing a suitable "Galois conjugation" [127] (see also [128, 129]) that takes the unitary RCFT data and makes it non-unitary.

The plan of this chapter is as follows. In the next section, we review the $(A_1, D_p)$ theories and place them in a slightly larger context. We also describe how the chiral algebra correspondence is applied to these theories. We then review the 2D logarithmic / rational correspondence of characters in [45] and introduce non-trivial flavor fugacities. In the following section we describe how to see topological aspects of the 4D RG flow in the 2D chiral RCFT. Along the way we review relevant aspects of MTCs and Galois conjugation. We conclude with some comments on generalizations of our analysis.

the $SU(2)_4$ MTC.

## 5.2 The 4D theories and their associated non-unitary chiral algebras

Our primary theories of interest are the so-called $(A_1, D_p)$ theories with $p \in \mathbb{Z}_{\text{odd}}$. These are 4D Argyres-Douglas theories[72] of class $\mathcal{S}$ whose construction we now describe.

To get the $(A_1, D_p)$ theory from the $A_1$ 6D $(2,0)$ theory, we perform a twisted compactification on a twice-punctured $\mathbb{CP}^1$. One puncture is an irregular puncture and one is a "full" regular puncture. The "full" regular puncture supports the $su(2)$ flavor symmetry of the theory, while the irregular puncture does not have any flavor symmetry associated with it. This picture is useful for us because it gives rise to a natural set of RG flows in 4D: by turning on an expectation value for the moment map operator in the multiplet corresponding to the $su(2)$ flavor symmetry, we can Higgs the regular puncture. In so doing, we go onto the one-quaternionic dimensional Higgs branch of the theory.[73] Moreover, the remaining irregular singularity supports an $(A_1, A_{p-3})$ theory. There is also a decoupled axion-dilaton hypermultiplet for spontaneous conformal symmetry breaking. As a result, our flow is, up to the decoupled hypermultiplet which we drop[74]

$$(A_1, D_p) \to (A_1, A_{p-3}) \ . \tag{5.9}$$

The latter $(A_1, A_{p-3})$ SCFTs have no Higgs branches or flavor symmetry themselves and are again strongly interacting Argyres-Douglas theories (the $p = 5$ case is the original theory in [52]).

In order to unify the results of this chapter with Chapter 4, it will be useful to slightly generalize the theories we are studying and consider the $(A^N_{N-1}[p-N], F)$ SCFTs with $p$ and $N$ co-prime integers (e.g., see [31, 33]). The above discussion was for the case of $N = 2$. In particular

$$(A_1, D_p) \sim (A^2_1[p-2], F) \ , \quad (A_1, A_{p-3}) \sim A^2_1[p-2] \ . \tag{5.10}$$

However, the pattern for general $N$ is similar: these theories are compactifications of the $A_{N-1}$ 6D $(2,0)$ theory on a $\mathbb{CP}^1$ with an irregular and "full" regular puncture. This latter puncture supports an $su(N)$ flavor symmetry with level

$$k^{4d}_{su(N)} = \frac{2N(p-1)}{p} \ , \tag{5.11}$$

while the irregular puncture does not have any flavor symmetry associated with it. We

[72] The $p = 3$ case originally appeared in [17] generalizing the earlier work in [52].

[73] Note that we define the Higgs branch to be the moduli space on which only the $su(2)_R \subset su(2)_R \times u(1)_R$ UV superconformal $R$ symmetry is broken. We do not necessarily mean a branch of moduli space on which there are only free hypermultiplets at generic points.

[74] For further details, see [31, 33, 49].

can again fully Higgs the regular puncture and obtain the following RG flow (where again we drop decoupled free hypermultiplets) to a theory with just an irregular puncture

$$(A_{N-1}^N[p-N], F) \to A_{N-1}^N[p-N] \ . \tag{5.12}$$

The $A_{N-1}^N[p-N]$ theory is again an interacting SCFT only if $p > N$.[75] The central charges of the theories appearing in the above flow are [31,33]

$$c_{(A_{N-1}^N[p-N],F)} = \frac{p-1}{12}(N^2-1) \ , \quad c_{(A_{N-1}^N[p-N],F)} = \frac{(N-1)(p-1-N)(p+(p-N)N)}{12p} \ . \tag{5.13}$$

We can also consider more general RG flows than (5.12) in which we only partially Higgs the regular puncture (and break the associated global symmetry group to some more general subgroup). In these cases, we can have more complicated theories in the IR. These flows will play a role when we return to discuss the theories in [2].

The 2D chiral algebras corresponding to the $(A_{N_1}^N[p-N], F)$ and $A_{N-1}^N[p-N]$ SCFTs, were found to be [31,33,38]

$$\chi[(A_{N-1}^N[p-N], F)] = \widehat{su(N)}_{N\frac{1-p}{p}} \ , \quad \chi[A_{N-1}^N[p-N]] = W_{N-1}(N,p) \ , \tag{5.14}$$

where $W_{N-1}(N,p)$ is the chiral algebra of the $A_{N-1}$ $W$-algebra minimal model. In particular, for the case of $N=2$, $W_1(2,p)$ is just the algebra of the $(2,p)$ Virasoro minimal model. Interestingly, the indices for the UV theories take a particularly simple form [31,33][76]

$$\mathcal{I}_{S,(A_{N-1}^N[p-N],F)} = \text{P.E.}\left(\frac{q-q^p}{(1-q)(1-q^p)}\chi_{\text{adj}}\right) \ , \tag{5.15}$$

where $\chi_{\text{adj}}$ is an adjoint character for $su(N)$. Indeed, this result has been mathematically proven (assuming the correspondence in (5.14)) for so-called "boundary admissible" theories in 2D [32,79] (this class of theories includes the $\widehat{su(N)}_{N\frac{1-p}{p}}$ theories).

## 5.3 From logarithmic theories to RCFT

For much of the remainder of the chapter, we will be concerned with the case of $N=2$. In particular, the relevant logarithmic chiral algebras will be

$$\chi[(A_1, D_p)] = \chi[(A_1^2[p-2], F)] = \widehat{su(2)}_{\frac{2(1-p)}{p}} \ , \tag{5.16}$$

[75] This statement is not an if and only if: the theory with $p=3$ and $N=2$ is trivial.

[76] See [30,38] for earlier work on subsets of these theories (and also closely related work in [66,84,130]).

with positive $p \in \mathbb{Z}_{\rm odd}$. As we briefly mentioned in the introduction, unrefined characters for these chiral algebras and their admissible representations were studied in [45], where the authors found an interesting connection with unrefined characters of the rational and unitary $\widehat{su(2)}_{2(p-1)}$ algebras and representations.

These latter AKM algebras have $2p-1$ primaries, $\Phi_\ell$ (here $\ell \in \{0,1,\cdots,2(p-1)\}$ is an $su(2)$ Dynkin label), with conformal dimensions

$$h(\Phi_\ell) = \frac{\ell(\ell+2)}{8p} \ . \tag{5.17}$$

The $\Phi_\ell$ satisfy the following fusion algebra [131]

$$\Phi_{\ell_1} \otimes \Phi_{\ell_2} = \sum_{\ell=|\ell_1-\ell_2|}^{\min(|\ell_1+\ell_2|,4(p-1)-\ell_1-\ell_2)} \Phi_\ell \ . \tag{5.18}$$

Note that the $\Phi_{2(p-1)}$ field satisfies $\mathbb{Z}_2$ fusion rules, $\Phi_{2(p-1)} \otimes \Phi_{2(p-1)} = \Phi_0$, and is associated with the non-trivial element of $\mathbb{Z}_2$ (here $\Phi_0 = 1$). More precisely, the associated topological defect (see [132] for a recent discussion) implements the $\mathbb{Z}_2$ symmetry of the $\widehat{su(2)}_{2(p-1)}$ CFT. Equivalently, we can think of $\Phi_{2(p-1)}$ as corresponding to the abelian anyon in the related Chern-Simons theory (e.g., see the classic works [109, 133]) that generates the $\mathbb{Z}_2$ one-form symmetry.

As discussed in the introduction, the particular theories that the authors studied in [45] are actually $\mathbb{Z}_2$ oribfolds of the $\widehat{su(2)}_{2(p-1)}$ theories. We label these theories as $\tilde{D}_{p+1}$ (or $\widehat{psu(2)}_{2(p-1)} \simeq \widehat{so(3)}_{2(p-1)}$), and they are the chiral parts of the $D$-type modular invariants in [121, 122]. Gauging the $\mathbb{Z}_2$ symmetry projects out fields that are not invariant under the action of the corresponding topological defect, i.e. those fields satisfying

$$\frac{S_{2(p-1),\ell}}{S_{1,\ell}} \neq 1 \ , \quad S_{\ell_1,\ell_2} = \frac{1}{\sqrt{p}} \sin\left[\frac{(\ell_1+1)(\ell_2+1)\pi}{2p}\right] \ , \tag{5.19}$$

where $S_{\ell_1,\ell_2}$ is the modular $S$-matrix of $\widehat{su(2)}_{2(p-1)}$ [131]. This projection immediately eliminates the (half-integer spin) odd $\ell$ fields. Next, one organizes primaries into representations of a larger chiral algebra by associating each representation with the orbit under fusion with $\Phi_{2(p-1)}$ and treating fixed points separately. There is one fixed point under this fusion since $\Phi_{2(p-1)} \otimes \Phi_{p-1} = \Phi_{p-1}$, and so one associates $|\mathbb{Z}_2| = 2$ representations of the enlarged chiral algebra with this representation, $\Phi^i_{D,p-1}$ where $i = 1, 2$. In other words, our theory after $\mathbb{Z}_2$ gauging is just given in terms of the following representations of the original theory[77]

$$\Phi_{D,\ell} = \Phi_\ell \oplus \Phi_{2(p-1)-\ell} \ , \quad \ell \in \{0, 2, 4, \cdots, p-3\} \ , \quad \Phi^i_{D,p-1} = \Phi^i_{p-1} \ . \tag{5.20}$$

As a result, there are $(p+3)/2$ representations and $(p+1)/2$ independent characters since the characters for $\Phi^i_{p-1}$ are equal

$$\chi_{D,p-1,1}(q) = \chi_{D,p-1,2}(q) = \chi_{p-1}^{\widehat{su(2)}_{2(p-1)}}(q) \ . \tag{5.21}$$

On the other hand, the $\widehat{su(2)}_{\frac{2(1-p)}{p}}$ algebra has $p$ admissible representations, $\hat{\Phi}_j$, with $j = 0, 1, 2, \cdots, p-1$ and scaling dimensions

$$h(\hat{\Phi}_j) = -\frac{j}{2}\left(\frac{p-j}{p}\right) \ . \tag{5.22}$$

In the limit that we turn off flavor fugacities, all the corresponding characters except the vacuum character are divergent. However, the following linear combinations of non-unitary characters are finite

$$\chi_{-,0}(q) \equiv \chi_0(q) \ , \quad \chi_{-,j} \equiv \chi_j(q) - \chi_{p-j}(q) \ , \quad j = 1, 2, \cdots, \frac{p-1}{2} \ . \tag{5.23}$$

Clearly, there are $(p+1)/2$ such characters, which matches the number of independent characters in the unitary case.

Given these sets of characters on the unitary and non-unitary sides, one of the main results of [45] is that, up to overall constants, we have

$$\chi_{D,p-1-2j}(q) \sim \chi_{-,j}^{\widehat{su(2)}_{\frac{2(1-p)}{p}}}(q) \ . \tag{5.24}$$

In other words, the unrefined character for the Dynkin label $p-1-2j$ primary of $\widehat{su(2)}_{2(p-1)}$ is proportional to the unrefined character of the $j^{\text{th}}$ non-unitary primary.

In the next subsection we will briefly expand on this result and introduce discrete flavor fugacities for $su(2)$. This matching then motivates us to study RG flows onto the 4D Higgs branch from the perspective of the unitary 2D theory.

### 5.3.1 Flavoring the correspondence

Let us consider turning on the $su(2)$ flavor fugacity, $y$, in the above correspondence. For simplicity, we will limit ourselves to $j = 0$. This case is the most immediately interesting from the 4D perspective since the $j = 0$ non-unitary character is the 4D

[77]In the condensed matter literature, the corresponding Chern-Simons MTC is said to have undergone anyonic condensation.

Schur index of the $(A_1, D_p)$ SCFT (see (2.57) and (5.14)).

For generic $y \in u(1)$, the refined characters are related in relatively complicated ways. However, it is straightforward to show that the two characters agree up to a $q$-independent function of $y$ when $y$ is a $(p+1)^{\rm st}$ root of unity.[78] More precisely, we have

$$\chi_{D,p-1}(q,y) = \chi_{su(2),p-1}(y) \cdot \chi_0^{\widehat{su(2)}_{\frac{2(1-p)}{p}}}(q,y) \ , \quad y = y_k = e^{\frac{2\pi i k}{p+1}} \ , \tag{5.25}$$

where $\chi_{su(2),p-1}(y) = \sum_{i=-\frac{p-1}{2}}^{\frac{p-1}{2}} y^i$ is a spin $(p-1)/2$ character of $su(2)$. At the discrete points $y_k = e^{\frac{2\pi i k}{p+1}} \neq 1$, we have $\chi_{su(2),p-1}(y_k) = (-1)^{1+k}$ while $\chi_{su(2),p-1}(y_0) \equiv \chi_{su(2),p-1}(1) = p$, and so

$$\chi_{D,p-1}(q,y_k) = \begin{cases} (-1)^{1+k} \cdot \chi_{-,0}^{\widehat{su(2)}_{\frac{2(1-p)}{p}}}(q,y_k) \ , & \text{if } 1 \leq k \leq p \\ p \cdot \chi_{-,0}^{\widehat{su(2)}_{\frac{2(1-p)}{p}}}(q,y_k) \ , & \text{if } k = 0 \ . \end{cases} \tag{5.26}$$

To prove (5.26), we start by writing the explicit forms of the two characters. For the non-unitary vacuum character, we have [45, 95, 134]

$$\chi_{-,0}^{\widehat{su(2)}_{\frac{2(1-p)}{p}}}(q,y) = \frac{\Theta_p^{(2p)}(q,y^{\frac{1}{p}}) - \Theta_{-p}^{(2p)}(q,y^{\frac{1}{p}})}{\Theta_1^{(2)}(q,y) - \Theta_{-1}^{(2)}(q,y)} \ , \tag{5.27}$$

where $y = e^{-2\pi i z}$, and

$$\Theta_j^{(k)}(q,x) = x^{\frac{j}{2}} q^{\frac{j^2}{4k}} \sum_{n\in\mathbb{Z}} x^{kn} q^{kn^2+nj} \ . \tag{5.28}$$

Similarly, the rational character is given by [45, 95, 134]

$$\chi_{(p-1)_D}(q,y) = \frac{\Theta_p^{(2p)}(q,y) - \Theta_{-p}^{(2p)}(q,y)}{\Theta_1^{(2)}(q,y) - \Theta_{-1}^{(2)}(q,y)} \ . \tag{5.29}$$

In particular, the denominators in (5.27) and (5.29) agree. The numerators are closely related as well. Indeed, the numerator of (5.27) is

$$\Theta_p^{(2p)}(q,y^{\frac{1}{p}}) - \Theta_{-p}^{(2p)}(q,y^{\frac{1}{p}}) = \sum_{n\in\mathbb{Z}} \left(y^{2n+\frac{1}{2}} - y^{-\left(2n+\frac{1}{2}\right)}\right) q^{\frac{p}{2}\left(2n+\frac{1}{2}\right)^2} \ , \tag{5.30}$$

[78] This statement holds somewhat more generally.

while the numerator of (5.29) is

$$\Theta_p^{(2p)}(q,y) - \Theta_{-p}^{(2p)}(q,y) = \sum_{n\in\mathbb{Z}} \left(y^{p\left(2n+\frac{1}{2}\right)} - y^{-p\left(2n+\frac{1}{2}\right)}\right) q^{\frac{p}{2}\left(2n+\frac{1}{2}\right)^2} . \tag{5.31}$$

Asking that the characters be proportional to each other up to a function that is independent of $q$ requires that we choose values of $y$ such that the ratio

$$r(y,n) = \frac{y^{p\left(2n+\frac{1}{2}\right)} - y^{-p\left(2n+\frac{1}{2}\right)}}{y^{2n+\frac{1}{2}} - y^{-\left(2n+\frac{1}{2}\right)}} , \tag{5.32}$$

is independent of $n$. This condition is satisfied when $y = y_k$. To verify this statement, first suppose $k \neq 0$. Then, the numerator and denominator in (5.32) do not vanish, and

$$r(y,n) = \frac{\sin\left(2\pi\frac{(2n+\frac{1}{2})kp}{p+1}\right)}{\sin\left(2\pi\frac{(2n+\frac{1}{2})k}{p+1}\right)} = \frac{\sin\left(2\pi k\left(2n+\frac{1}{2}\right) - 2\pi\frac{\left(2n+\frac{1}{2}\right)k}{p+1}\right)}{\sin\left(2\pi\frac{(2n+\frac{1}{2})k}{p+1}\right)} = (-1)^{1+k} . \tag{5.33}$$

If $k=0$, then we find $r(y_0,n) \equiv \lim_{y\to 1} r(y,n) = p$ as desired (the characters themselves do not degenerate, because the denominators in (5.27) and (5.29) also vanish at the same order).

The simple relations in (5.25) and (5.26) for $y \neq 1$ suggest that $\tilde{D}_{p+1}$ should know something about the Higgs branch of the $(A_1, D_p)$ SCFT. Indeed, from the 4D perspective, we can learn about the index of the Higgs branch theory by considering poles in the flavor fugacity, $y$ [47].[79]

## 5.4 MTCs and the RG flow

In section 5.2, we saw that there were interesting RG flows emanating from the $(A_1, D_p)$ fixed points that take us onto their Higgs branches

$$(A_1, D_p) \to (A_1, A_{p-3}) . \tag{5.34}$$

In writing (5.34), we have dropped a decoupled hypermultiplet containing Goldstone bosons and their superpartners. Since moving onto the Higgs branch requires breaking flavor symmetry, and since we showed in the previous section that $\tilde{D}_{p+1}$ knows about certain (discretely) flavored observables in the $(A_1, D_p)$ SCFT, one might be tempted to guess that we can learn about the 4D Higgs branch using the 2D chiral RCFT.

We will see that this intuition is indeed correct, although perhaps not in the most

[79] In fact, we will see that for general $p$ we most directly learn something about the 4D theory in the presence of a surface defect.

obvious way one might first imagine. Indeed, as a first guess, one might try to perform qDS reduction on the $\tilde{D}_{p+1}$ theory, since this reduction applied to the 2D chiral algebra of the $(A_1, D_p)$ theory gives the 2D chiral algebra of the $(A_1, A_{p-3})$ theory (e.g., see the discussion in [49]). Instead, we will describe a simpler connection.

The idea is to consider some of the most basic data in the Chern-Simons theories underlying the $\tilde{D}_{p+1}$ theories: the $S^3$ expectation values of Wilson loops, $W^i_{D,p-1}$, corresponding to the highest-spin primaries, $\Phi^i_{D,p-1}$ (we will see that the answer does not depend on $i$)

$$\langle W^i_{D,p-1} \rangle = \frac{S_{0,(p-1)_i}}{S_{0,0}} \ , \tag{5.35}$$

where $S_{a,b}$ is the modular $S$-matrix for $\tilde{D}_{p+1}$. We will show that this data can be related—via Galois conjugation—to the expectation value of a Wilson loop in the TQFT underlying the chiral part of the $(2,p)$ Virasoro minimal model (i.e., the 2D theory for the IR $(A_1, A_{p-3})$ SCFT in the sense of [27]). More precisely, the expectation value in question is for the Wilson loop corresponding to the lowest scaling dimension primary, $\phi_{(1,(p-1)/2)}$.

In order to understand these statements and their implications, we review basic aspects of MTCs and Galois conjugation in the next subsection. We then move on to discuss the action of the RG flow on (5.35).

### 5.4.1 MTC / TQFT basics

Roughly speaking, to the representations of any 2D rational chiral algebra, we can associate a corresponding MTC (or 3D TQFT depending on one's preference) [109,133]. In our cases of interest, these MTCs are MTCs associated to Chern-Simons theories. The general data that defines an MTC is a set of simple objects with corresponding commutative fusion rules, a set of $F$ matrices that implement associativity and satisfy "pentagon" equations, and a set of braiding or $R$ matrices that satisfy, together with the $F$ matrices, the so-called "hexagon" equations [109]. The MTC is modular because it has associated with it non-degenerate $S$ and $T$ matrices. Since our MTCs arise from representations of 2D rational chiral algebras, the resulting simple objects are in one-to-one correspondence with the representations of these chiral algebras. In a Chern-Simons theory, one thinks of these simple objects as tracing out Wilson lines in some representation of the gauge group. As can be seen by studying their braiding properties, these objects are generally anyonic.

For us in what follows, the most important data in the MTC will be the $S$ and $T$ matrices. The $T$ matrices we use consist of the twists (unnormalized by the standard RCFT prefactor, $e^{-2\pi i \frac{c}{24}}$)

$$T_{i,j} = \delta_{ij}\theta_i = \delta_{ij} e^{2\pi i h_i} \ , \tag{5.36}$$

where $h_i$ is the conformal dimension of the corresponding primary, $\Phi_i$, of the 2D rational chiral algebra. Another important piece of data for us is the set of quantum dimensions

$$d_i = \frac{S_{0i}}{S_{00}} \ , \tag{5.37}$$

corresponding to the expectation value of a Wilson loop of type $i$ on $S^3$. Since our starting point is unitary, we have

$$d_i \geq 1 \ , \tag{5.38}$$

where $d_i = 1$ if and only if the corresponding anyon is abelian, i.e. if there exists $\bar{i}$ (which may or may not satisfy $i = \bar{i}$) such that

$$i \otimes \bar{i} = 1 \ . \tag{5.39}$$

Note that this fusion rule corresponds to the RCFT fusion $\phi_i \otimes \phi_{\bar{i}} = \phi_0$, where the $\phi_a$ are RCFT primaries ($\phi_0$ is the identity). The proof of this statement follows from the fact that $d_1 = 1$, $d_i = d_{\bar{i}}$, and the fact that the quantum dimensions satisfy the fusion rules of the theory [95, 135]

$$d_j d_k = \sum_k N_{jk}^{\ell} d_\ell \ , \tag{5.40}$$

where the integers $N_{jk}^{\ell} \geq 0$ are the fusion multiplicities. As a result, the $i$ anyon generates (part of) the abelian one-form symmetry of the theory (and $\bar{i}$ is $i$'s "inverse"). We call such an anyon an "abelian" anyon to distinguish it from the anyons, $a$, with $d_a > 1$, whose fusion rules are not those of a group ($a \times \bar{a}$ will involve at least two non-trivial fusion channels).

### Galois conjugation

Given an MTC, we may define various natural actions on it. One particularly important action is that of Galois conjugation. While the precise action of Galois conjugation at the level of the full MTC is subtle,[80] a Galois action at the level of the generalized quantum dimensions[81] is simpler to describe [128, 136].

The main point is that the quantum dimensions can be thought of as taking values in some "cyclotomic" field, $\mathbb{Q}(\xi)$, for $\xi = e^{\frac{2\pi i}{k}}$, which consists of appending $k$th roots of unity to the rational numbers, $\mathbb{Q}$.[82] The cyclotomic field admits the action of a Galois group, $G = \mathbb{Z}_k^\times$, consisting of the multiplicative units between 1 and $k$ (e.g., $\mathbb{Z}_4^\times = \{1, 3\}$). The action of $G$ is simple to describe: it leaves the base field (i.e., the

[80] One reason is that some of the data in the $F$ and $R$ matrices depends on certain gauge choices.

[81] These include not only the $d_i = \frac{S_{i0}}{S_{00}}$ but also the $\frac{S_{ij}}{S_{0j}}$ with $j \neq 0$.

[82] A similar story holds for the modular $S$ and $T$ data, although the cyclotomic field is, in general, different [128]. We will comment further on this fact below.

rational numbers) invariant and acts non-trivially on $\xi$ as

$$\xi \to \xi^p \ , \quad p \in \mathbb{Z}_k^\times \ , \tag{5.41}$$

for $p$ and $k$ co-prime. In general, Galois conjugation takes unitary theories to non-unitary ones (although there are exceptions). The most basic example being the Galois action that takes the Lee-Yang MTC to $(G_2)_1$, $(F_4)_1$, and the complex conjugate of Lee-Yang. We will return to this example shortly. Note that in the non-unitary conjugates of a unitary theory, the quantum dimension bound in (5.38) is typically violated.

Before proceeding, let us emphasize that the examples of Galois group we discuss here can be naturally related to one that acts on the full $S$ and $T$ matrices in an RCFT [128,137] by a surjective restriction (and similarly for the natural Galois action descending from the quantum group structure underlying the MTC).

### 5.4.2 Galois action, RG flows, and quantum dimensions

In this section, we study the action of a Galois group on some of the data underlying the $\tilde{D}_{p+1}$ theory. We start with the special cases of $p = 3$ and $p = 5$ before discussing the general case. As we will see, some additional interesting phenomena occur for $p = 3, 5$.

To that end, consider the case of $p = 3$. As discussed in section 5.3, the resulting $\tilde{D}_4$ theory is a theory with abelian fusion rules. Indeed, after anyon condensation in $SU(2)_4$, the resulting Chern-Simons theory has abelian anyons and $\mathbb{Z}_3$ fusion rules. From (5.34), we see that the resulting 4D IR theory is the trivial $(A_1, A_0)$ theory.[83] Later we will see that this phenomenon appears in other examples as well: when the UV theory consists of abelian anyons, the 4D Higgs branch theory is either trivial (after removing the decoupled hypermultiplet of spontaneous symmetry breaking) or free (at generic points). This statement is also consistent with the matching of quantum dimensions alluded to in the introduction

$$\langle W^i_{D,2} \rangle = d_{2_i} = \frac{S_{0,2_i}}{S_{0,0}} = 1 \ , \quad S^D = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix} \ . \tag{5.42}$$

In writing the $S$-matrix, we have taken the second (third) row / column to correspond to $2_1$ ($2_2$). These rows and columns correspond to the anyons that generate $\mathbb{Z}_3$. Indeed, as we explained in the previous subsection, anyons whose fusion rules are abelian have quantum dimension one. The IR theory is trivial (after considering the 2D theory related to the 4D theory we get by dropping the Goldstone multiplet) and so the only

[83] One can also see from (5.13) that the corresponding central charge with $p = 3$ and $N = 2$ vanishes (in the discussion below (5.14), this is because the IR chiral algebra is for the trivial $(2, 3)$ Virasoro minimal model).

IR field is the vacuum, $\phi_{(1,1)}$, with quantum dimension one.

Next let us consider the case of $p=5$, i.e., $\tilde{D}_6$. The corresponding modular $S$-matrix is[84]

$$S^D = \begin{pmatrix} \frac{1}{10}\left(5-\sqrt{5}\right) & \frac{1}{10}\left(5+\sqrt{5}\right) & \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{10}\left(5+\sqrt{5}\right) & \frac{1}{10}\left(5-\sqrt{5}\right) & -\frac{1}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & -\frac{1}{\sqrt{5}} & -\frac{1}{10}\left(5-\sqrt{5}\right) & \frac{1}{10}\left(5+\sqrt{5}\right) \\ \frac{1}{\sqrt{5}} & -\frac{1}{\sqrt{5}} & \frac{1}{10}\left(5+\sqrt{5}\right) & -\frac{1}{10}\left(5-\sqrt{5}\right) \end{pmatrix} . \tag{5.43}$$

Now, using the Verlinde formula

$$N_{\lambda\mu}^{\nu} = \sum_{\sigma} \frac{S_{\lambda,\sigma} S_{\mu,\sigma} S^*_{\sigma,\nu}}{S_{0_D,\sigma}} , \tag{5.44}$$

we find that

$$\Phi^i_{D,4} \times \Phi^i_{D,4} = \Phi_{D,0} + \Phi^i_{D,4} . \tag{5.45}$$

In particular, we see that $\left\{\Phi_{D,0}, \Phi^i_{D,4}\right\}$ are closed fusion subcategories (one two-element subcategory for each value of $i$; without loss of generality, we will drop $i$ from now on). Moreover, their fusion rules are the so-called "Fibonacci" fusion rules (e.g., see [138] for a review) shared by the Lee-Yang, conjugate Lee-Yang, $(G_2)_1$, and $(F_4)_1$ fusion categories. In our case, after normalizing the sub-$S$-matrix for $\{\Phi_{0_D}, \Phi_{4_D}\}$, we obtain

$$\begin{aligned} S &= \frac{1}{\sqrt{\xi^{-1}+3+\xi}} \begin{pmatrix} 1 & \xi^{-1}+1+\xi \\ \xi^{-1}+1+\xi & -1 \end{pmatrix} , \quad d_{\Phi_{D,0}} = 1 , \quad d_{\Phi^i_{D,4}} = \xi^{-1}+1+\xi , \\ T &= \mathrm{diag}(1,\xi^3) , \quad \xi = e^{\frac{2\pi i}{5}} . \end{aligned} \tag{5.46}$$

These are the $S$ and $T$ matrices for the $(F_4)_1$ MTC [117]. Using Galois conjugation as in (5.41) at the level of the $S$ and $T$ matrices, we can transform the above data into the data for Lee-Yang. More precisely, if we Galois conjugate by the element $2 \in \mathbb{Z}_5^\times$, we obtain[85]

$$d^{\overline{LY}}_{\phi_{(1,1)}} = 1 , \quad d^{\overline{LY}}_{\phi_{(1,2)}} = \xi^{-2}+1+\xi^2 , \quad T^{\overline{LY}} = \mathrm{diag}(1,\xi^6) , \tag{5.47}$$

[84] The modular $S$-matrix can be derived from the one for $\widehat{su(2)}_8$ as follows. First, note that the primaries of the $\tilde{D}_6$ chiral algebra are fixed in terms of the $\widehat{su(2)}_8$ primaries as in (5.20). This observation fixes the first three rows / columns in the modular $S$-matrix in terms of the entries in the $S$-matrix in (5.19). The remaining two rows and columns (i.e., for the two $\Phi_4^i$ primaries) can be fixed by demanding symmetry of the $S$-matrix, reality of the first row (and column), unitarity, and the $sl(2,\mathbb{Z})$ conditions $S^2=(ST)^3$ and $S^4=1$.

[85] Note that the Galois group studied in [129, 137] is $\mathbb{Z}_{60}^\times$. The reason for this difference is that the authors of these latter works consider the CFT-normalized $T$ matrix (i.e., with the $e^{-\frac{2\pi i c}{24}}$ prefactor). There is no inconsistency in using these two different groups since we have an appropriate surjective restriction $\mathbb{Z}_{60}^\times \to \mathbb{Z}_5^\times$.

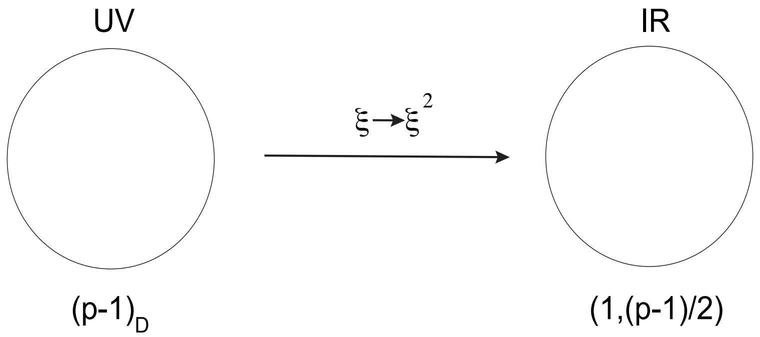



**Figure 7:** The anyonic imprint on the Higgs branch. The expectation value for the Wilson lines corresponding to the Dynkin label $(p-1)$ fields in the rational 2D theory related to the UV $(A_1, D_p)$ SCFT are mapped, via Galois conjugation, to the expectation value for the Wilson line corresponding to the lowest scaling dimension primary in the $(2, p)$ minimal model related to the IR $(A_1, A_{p-3})$ SCFT on the Higgs branch.

which is the complex conjugate of the Lee-Yang category. On the other hand, if we conjugate by the element $3 \in \mathbb{Z}_5^\times$, we obtain

$$d^{LY}_{\phi_{(1,1)}} = 1 \ , \quad d^{LY}_{\phi_{(1,2)}} = \xi^{-3} + 1 + \xi^3 \ , \quad T^{LY} = \mathrm{diag}(1, \xi^9) \ , \tag{5.48}$$

which is the Lee-Yang category. Note that both Lee-Yang and its complex conjugate have the same spectrum of quantum dimensions since

$$d^{LY}_{\phi_{(1,2)}} = \xi^{-3} + 1 + \xi^3 = \xi^{-2} + 1 + \xi^2 = d^{\overline{LY}}_{\phi_{(1,2)}} \ . \tag{5.49}$$

From this discussion, we see that the rational theory contains a sub-category that is Galois conjugate to the MTC for the IR chiral algebra in the flow discussed around (5.34) (the Lee-Yang or $(2,5)$ minimal model Virasoro algebra corresponding to $\chi[(A_1, A_2)]$). Therefore, the rational UV theory "knows" about the IR MTC.

More generally, one may ask if the MTC for the $\tilde{D}_{p+1}$ theory contains a closed fusion subcategory corresponding to the representations of the $(2,p)$ Virasoro algebra for $p \geq 7$. It turns out that for general $p \in \mathbb{Z}_{\mathrm{odd}}$, the $\tilde{D}_{p+1}$ MTC does not have a non-trivial closed subcategory. However, we can partly generalize what happens for $p = 5$ as follows. The vev of the Wilson line in the $\tilde{D}_{p+1}$ Chern-Simons theory that corresponds to the maximal spin representation (and therefore, via the correspondence discussed above, to the 4D Schur operators) is related, via Galois conjugation, to the vev of a Wilson line in the $(2,p)$ MTC corresponding to the 4D IR theory (see Fig. 7).[86]

[86]Note that we are not claiming the UV and IR MTCs are Galois conjugate. Indeed, the number of simple elements is different.

In other words

$$\langle W^i_{D,p-1}\rangle = \frac{1}{2\sin\left(\frac{\pi}{2p}\right)} = \frac{S^D_{0,(p-1)_i}}{S^D_{0,0}} \to_{2\in\mathbb{Z}_p^\times} \frac{S_{(1,1),(1,(p-1)/2)}}{S_{(1,1),(1,1)}} = \frac{(-1)^{\frac{p+1}{2}}}{2\cos\left(\frac{\pi}{p}\right)} = \langle W_{(1,(p-1)/2)}\rangle\ , \tag{5.50}$$

where "$\to_{2\in\mathbb{Z}_p^\times}$" denotes Galois conjugation by the element $2\in\mathbb{Z}_p^\times$ (since $p\in\mathbb{Z}_{\rm odd}$, this is always an element of the Galois group), and

$$S_{(r,s),(\rho,\sigma)} = \frac{2}{\sqrt{p}}(-1)^{s\rho+r\sigma}\sin\left(\frac{\pi p}{2}r\rho\right)\sin\left(\frac{2\pi}{p}s\sigma\right)\ . \tag{5.51}$$

is the $(2,p)$ minimal model $S$-matrix [95].[87] Note that the $\mathbb{Z}_p^\times$ Galois group we discuss here can be obtained from the appropriate surjective restriction of the $\mathbb{Z}_{2p}^\times$ Galois group (if $p-1=0$ mod 4) or $\mathbb{Z}_{4p}^\times$ Galois group (if $p-1=2$ mod 4) one finds by applying the discussion in [137] to the full $\tilde{D}_{p+1}$ RCFT modular data (a similar statement holds for the Galois group that naturally arises when considering the underlying quantum group).

For the interested reader, we give the proof of (5.50) in Appendix E. Here we mention a few observations before discussing some generalizations in the next section:

- The identification in (5.50) leads to some simple rules that one can easily verify for the theories in question. For example, if $\langle W^i_{D,p-1}\rangle\neq 1$, then both the UV and the IR theory have non-abelian anyons. The reason is that such a quantum dimension cannot equal one when raised to any power and so the corresponding Wilson line / anyon cannot satisfy group-like fusion (this statement holds even though the IR theory is non-unitary, and the quantum dimension bound in (5.38) is violated in the IR). Indeed, the $\tilde{D}_{p+1}$ and $(2,p)$ theories with $p>5$ have non-abelian anyons (in fact, any non-unitary MTC must have non-abelian anyons). When $\langle W^i_{D,p-1}\rangle = 1$, the UV theory has an abelian anyon, and the IR (after removing the decoupled hypermultiplet) must also have an abelian anyon in its MTC or be trivial. As we have seen, the only such case in our theories is the $p=3$ case, where the UV has $\mathbb{Z}_3$ abelian anyons and the IR is trivial (after considering the theory related to the 4D IR in which we have removed the Goldstone multiplet). In the next section, we will comment on some generalizations of these observations to other theories.

- The quantum dimension on the LHS of (5.50) is related to the field in the $\tilde{D}_{p+1}$ theory whose character reproduces the Schur index of the UV $(A_1,D_p)$ SCFT.

[87] Note that, as in the $p=5$ example, the Galois conjugate of the $(p-1)_i$ twists generally do not agree with the twist for $(1,(p-1)/2)$ in the IR MTC. On the other hand, conjugating by $3\in\mathbb{Z}_p^\times$ (when $p$ is not a multiple of 3) does yield an equality of the twists. However, for general $p$ not a multiple of three, we do not have a relation of quantum dimensions as in (5.50) if we choose $3\in\mathbb{Z}_p^\times$.

On the other hand, the quantum dimension on the RHS of (5.50) is related to the field whose character reproduces the Schur index of the IR $(A_1, A_{p-3})$ theory in the presence of an $\mathcal{N} = (2,2)$-preserving surface defect [49].

- It is interesting to note that in the MTCs that are related to our 4D $\mathcal{N} = 2$ SCFTs, all bosonically generated one-form symmetries (i.e., the corresponding generators have integer spin) have been gauged: the $\mathbb{Z}_2$ one-form symmetry in $SU(2)_{2(p-1)}$ has been gauged, and the corresponding bosons have condensed. In the $p = 3$ case we have a left-over one-form symmetry, $\mathbb{Z}_3$, after the $\mathbb{Z}_2$ gauging (note that the anyons generating the $\mathbb{Z}_3$ symmetry have spin 1/3). However, this symmetry has a 't Hooft anomaly—and hence cannot be gauged (e.g., see the recent discussion in [125]).

## 5.5 Connections with other theories

It would be interesting to understand how general the observations in the previous section are in the space of 4D $\mathcal{N} = 2$ SCFTs. As a modest first step, let us revisit the $D_2[SU(3)] = (A_2^3[-1], F)$ SCFT[88] we discussed in Chapter 4 and recounted briefly in the introduction. Recall from the introduction that the associated non-unitary chiral algebra is $\widehat{su(3)}_{-\frac{3}{2}}$ [30] and that the associated unitary RCFT is $\widehat{so(8)}_1$. The corresponding MTC is $Spin(8)_1$ (e.g., see the recent discussion in [118]).

As in the examples mentioned in the previous sections, the $Spin(8)_1$ TQFT has no one-form symmetries generated by bosons. Indeed, all the non-trivial lines are fermionic. One can gauge a $\mathbb{Z}_2$ one-form symmetry generated by one of the fermions and obtain the $SO(8)_1$ spin-TQFT.[89]

More generally, as explained in footnote 67, the results of Chapter 4 imply the following MTCs are associated with the $D_2[SU(2n+1)] = (A_{2n}^{2n+1}[1-2n], F)$ 4D $\mathcal{N} = 2$ theories

$$D_2[SU(2n+1)] \;\rightarrow\; \begin{cases} Spin(8)_1 \text{ MTC} \,, & \text{if } n(n+1) = 2 \text{ (mod4)} \\ D(\mathbb{Z}_2) \text{ (toric code) MTC} \,, & \text{if } n(n+1) = 0 \text{ (mod4)} \,. \end{cases} \tag{5.52}$$

The toric code MTC has two non-trivial bosons that can condense. However, this condensation leads to a trivial theory.[90] Therefore, we see that all the MTCs that are related to the doubly infinite classes of 4D $\mathcal{N} = 2$ SCFTs discussed in the present chapter do not allow for further non-trivial gauging of bosonic one-form symmetries. It

[88] We use the language of section 5.2 in writing $(A_2^3[-1], F)$.
[89] It might also be interesting to pursue ideas along the lines of [139].
[90] This statement follows, as in the related discussion around (5.19) for $SU(2)_{2(p-1)}$, from the modular $S$-matrix [117] of the toric code MTC; see also [124].

would be interesting to understand if this is a general feature of MTCs related to 4D theories in the way we have described.

As in the case of the $(A_1, D_3)$ theory, the MTCs described in (5.52) are abelian: they have $\mathbb{Z}_2 \times \mathbb{Z}_2$ fusion rules. Moreover, as for the $(A_1, D_3)$ theory, the Higgs branches of these theories at generic points are free: they consist of decoupled hypermultiplets (the would-be $A_{N-1}$ $W$-algebra minimal models in (5.14) do not exist, since $p < N$). Therefore, we see that, by again dropping decoupled hypermultiplets, UV and IR quantum dimensions can be related as in (5.50)[91]

$$\langle W_{[0,0,\cdots,1]}\rangle = \langle W_{[0,0,\cdots,1,0]}\rangle = 1 = \frac{S_{0,[0,\cdots,1]}}{S_{0,0}} = \frac{S_{0,[0,\cdots,1,0]}}{S_{0,0}} \quad \to_{1\in\mathbb{Z}_1^\times} \quad \frac{S_{00}}{S_{00}} = 1 = \langle W_0\rangle \ , \tag{5.53}$$

where the representations on the LHS correspond to the highest conformal weight primaries in the respective $\widehat{so(4n(n+1))}_1$ chiral RCFTs. As in the case of the $(A_1, D_3)$ theory, the quantum dimension on the RHS is for the trivial theory without an $\mathcal{N} = (2,2)$-preserving surface defect included.

There is an additional subtlety we should note for the $D_2[SU(2n+1)]$ theories with $n > 1$. In this case, we have non-generic flows to theories of the type $D_2[SU(2n'+1)]$ with $n' < n$ and decoupled hypermultiplets. As a result, we have interacting IR factors. However, as we have shown above, the related chiral RCFTs have only abelian anyons. Therefore, we again have a matching as in (5.53) if we also "rationalize" the IR theory. The fact that the IR chiral RCFTs have only abelian anyons is consistent with our Galois action described above: the relevant Galois groups for $Spin(8)_1$ and $D(\mathbb{Z}_2)$ are trivial.

[91] For $n = 1$, it is natural to include $\langle W_{[1,0,0,0]}\rangle$ since this line corresponds to the $\widehat{so(8)}_1$ primary with (co-highest) conformal weight.

# Chapter 6

# Flowing from 16 to 32 Supercharges

We now turn to exploring certain RG flows which start from a family of AD theories that include the $\mathcal{T}_{3,\frac{3}{2}}$ theory we first encountered in Chapter 3, and lead to supersymmetry enhancement from $\mathcal{N}=2$ to $\mathcal{N}=4$. In Sec. 6.2 we introduce in detail the class $\mathcal{S}$ construction of our UV starting points. We then specialize to the case of $\mathcal{T}_{3,\frac{3}{2}}$ and describe its various connections to free fields and $\mathcal{N}=4$ theories. We turn on certain mass deformations in the direct $S^1$ reduction of $\mathcal{T}_{3,\frac{3}{2}}$ and find that it triggers an RG flow to an $\mathcal{N}=8$ SYM theory in the IR. The corresponding RG flow in 4D is then argued to result in the aforementioned $\mathcal{N}=2$ to $\mathcal{N}=4$ supersymmetry enhancement. To find out more information about the $\mathcal{N}=4$ theory which sits at the IR endpoint of the flow, we study the Seiberg-Witten curve of $\mathcal{T}_{3,\frac{3}{2}}$ and discover a particular limit which leads to the SW curve of $\mathcal{N}=4$ $SU(2)$ gauge theory tuned to a weak coupling cusp. However, we cannot detect the exactly marginal deformation that is necessarily present in any $\mathcal{N}=4$ theory. We ponder about the implications of this result, including the possibility that the IR theory is in fact an exotic $\mathcal{N}=4$ theory. In Sec. 6.4, we extend our analysis to an infinite family of generalizations of $\mathcal{T}_{3,\frac{3}{2}}$.

This chapter is based on the paper [4].

## 6.1 Introduction

Emergent symmetries are ubiquitous in quantum field theory:[92] along renormalization group flows, couplings that break certain symmetries are sometimes renormalized to zero at long distance. The resulting infrared theory then has accidental symmetries

[92] Throughout this chapter we use "emergent" symmetries and "accidental" symmetries interchangeably.

that are not present in the ultraviolet theory.[93] [94]

Often, supersymmetry is one of these emergent symmetries. For example, in three dimensions, one may potentially find accidental $\mathcal{N} = 1$ SUSY in certain condensed matter systems [145, 146] (see also [147] for a discussion of emergent $\mathcal{N} = 2$ SUSY).

More generally, additional SUSY can emerge in RG flows that are already supersymmetric. Instances of this phenomenon in three dimensions include the $\mathcal{N} = 3 \to \mathcal{N} = 6$ (or $\mathcal{N} = 8$) enhancement in the ABJM flows starting from certain deformed super Yang-Mills theories in the UV [148] as well as $\mathcal{N} = 1 \to \mathcal{N} = 2$ enhancement studied in other contexts [149, 150] (see also [151] for a recent discussion in the context of 3D $\mathcal{N} = 2 \to \mathcal{N} = 4$). In four dimensions, enhancement from $\mathcal{N} = 1 \to \mathcal{N} = 2$ has also received considerable attention recently [59–61, 152–155].

In this chapter, we study SUSY enhancement along an infinite class of RG flows starting from strongly interacting 4D $\mathcal{N} = 2$ SCFTs labeled by integers $(n, k) \geq (2, 3)$[95] that do not have known Lagrangians[96] and ending at IR fixed points with thirty-two (Poincaré plus special) supercharges.[97] In particular, we provide evidence that these $\mathcal{N} = 2$ SCFTs flow, upon turning on "mass terms"[98] and compactifying the theories on an $S^1$ of radius $r$, to the 3D $\mathcal{N} = 8$ SCFTs one gets by turning on the gauge couplings of $u(n)$ 3D $\mathcal{N} = 8$ SYM for arbitrary $(n, k) \geq (2, 3)$. In the case of $(n, k) = (2, 3)$, we provide arguments that the $r \to \infty$ limit of the flow is to a 4D theory with $\mathcal{N} = 4$ SUSY.

While we believe it is likely that the $r \to \infty$ limit of these flows for any $(n, k) \geq (2, 3)$ has 4D $\mathcal{N} = 4$ SUSY (with a $3n$ complex dimensional moduli space) in the IR, we leave a detailed study of this question and an analysis of the resulting spectra to future work. One motivation for this chapter is simply to identify a space of theories in which SCFTs with $\mathcal{N} = 4$ SUSY in four dimensions may plausibly emerge somewhat more unconventionally. We hope these constructions will shed light on the space of possible $\mathcal{N} = 4$ theories (perhaps even on the question of whether these theories are necessarily of SYM type).

The plan of this chapter is as follows. In the next section we describe how our UV theories are engineered starting from the $A_{N-1}$ $(2, 0)$ theory. We then discuss the case

[93] In general, it is an interesting but difficult question to try to find constraints on the amount of accidental symmetry (e.g., see [140–143] for a discussion in the context of certain classes of RG flows).

[94] Here we have in mind symmetries that act on local operators. One may generalize the concept of emergent symmetry to include higher-form symmetries as well (e.g., see [144]).

[95] More precisely, as we will see below, these theories are specified by Young diagrams that are determined by $(n, k)$.

[96] These theories lack $\mathcal{N} = 2$ Lagrangians because they have $\mathcal{N} = 2$ chiral operators of non-integer scaling dimension. Moreover, they do not have known UV Lagrangians in the sense of [59–61, 152–155].

[97] Several examples of four-dimensional flows from $\mathcal{N} = 2 \to \mathcal{N} = 4$ were studied at the level of Coulomb branch geometries in [94].

[98] More accurately, these are deformations of the superpotential by dimension two holomorphic moment maps in the same $\mathcal{N} = 2$ multiplets as certain flavor symmetries.

of $(n, k) = (2, 3)$ and motivate certain expectations for the corresponding RG flow from the superconformal index discussion of [1]. We comment on the nature of the 4D IR fixed point that emerges in the $r \to \infty$ limit. Finally, we generalize our discussion to arbitrary $(n, k) \geq (2, 3)$.

## 6.2 The UV starting points

Our particular UV 4D $\mathcal{N} = 2$ SCFTs are obtained from certain twisted compactifications of the $A_{N-1}$ 6D $(2, 0)$ theory on a Riemann surface, $\mathcal{C} = \mathbb{CP}^1$. A co-dimension two defect intersects $\mathcal{C}$ at $z = \infty$ giving rise to an irregular puncture at this point [53] (see also [63, 156]). In our class of theories, $\mathcal{C}$ has no additional punctures.

One convenient way of studying certain aspects of the irregular puncture at $z = \infty$ and the resulting 4D theories is to first compactify the parent 6D theory on an $S^1$. We can then describe the irregular puncture in terms of the singular behavior of a twisted element of the vector multiplet of the corresponding $A_{N-1}$ 5D maximal SYM theory—the $\mathfrak{sl}(N, \mathbb{C})$-valued $(1, 0)$-form, $\Phi_z dz$ the Higgs field.[99] In particular, after performing a gauge transformation, we have (for the particular class of theories we are interested in)

$$\Phi_z = z^{\ell-2} T_{\ell-2} + z^{\ell-3} T_{\ell-3} + \cdots + T_0 + \frac{1}{z} T_{-1} + \cdots \ , \tag{6.1}$$

where the second set of ellipses contain non-singular terms in the limit $z \to \infty$, and the $T_i$ are traceless $N \times N$ matrices. In the above equation, $\ell > 1$ is an integer (the case $\ell = 1$ describes a regular singularity and is not relevant to our discussion below; the case $\ell \notin \mathbb{Z}$ is also not relevant).

Combined with a gauge field on $\mathcal{C}$, the configuration in (6.1) forms a solution to Hitchin's equations and describes the Higgs branch of the mirror of the $S^1$ reduction of our 4D theories of interest (the reduction of the 5D theory on $\mathcal{C}$). Therefore, it describes the Coulomb branch of the direct $S^1$ reduction and also, via the base of the corresponding fibration, the Coulomb branch of the 4D theory itself. The Seiberg-Witten curve of the 4D theory may be read off from the spectral curve (2.62)

$$\det(x - \Phi_z) = 0 \ . \tag{6.2}$$

In order for the description of the moduli space to not jump discontinuously as a function of the parameters residing in the $T_i$, a sufficient condition on the $T_i$ is that they are regular (*note:* $\mathcal{C}$'s puncture is still irregular!) semisimple (see [157] and references therein for a discussion in a closely related context). In particular, this statement means that the $T_i$ can be brought to the form of diagonal matrices with non-degenerate

[99] $\Phi_z$ is $\mathfrak{sl}(N, \mathbb{C})$-valued instead of $\mathfrak{su}(N, \mathbb{C})$-valued since it comes from $Y^1 + iY^2$ where $Y^1$ and $Y^2$ are two adjoint scalars in the 5D SYM. It is a $(1, 0)$-form because of the twist.

eigenvalues. These singularities give rise to 4D theories with Coulomb branch operators of non-integer scaling dimensions and generalize the theories described in [52, 54].[100]

The above class of theories, while very broad, is (modulo some caveats we will discuss) not closed under the natural SCFT operation of conformal gauging [75] or under the RG flow. In fact, these SCFTs form a part of a much broader but still relatively poorly understood class of theories called the "type III" theories [53] (these theories are expected to exhibit various interesting phenomena; e.g., see [67, 158, 159] or Chapter 3).

To define the type III SCFTs, we relax the condition of regularity of the $T_i$. In this case, the requirement of smoothness away from the origin of the moduli space implies that [157]

$$L_{-1} \subseteq L_0 \subseteq \cdots \subseteq L_{\ell-2} \ , \tag{6.3}$$

where the $L_a$ are the Levi subalgebras associated with the $T_i$.[101] This restriction can be conveniently described in terms of certain Young diagrams [53]

$$T_i \ \leftrightarrow \ Y_i = [n_{i,1}, n_{i,2}, \cdots, n_{i,k_i}] \ , \quad n_{i,a} \geq n_{i,a+1} \in \mathbb{Z}_{>0} \ , \quad \sum_{a=1}^{k_i} n_{i,a} = N \ , \tag{6.4}$$

where the columns of height $n_{i,a}$ represent the eigenvalue degeneracies of the $T_i$. The condition (6.3) amounts to the statement that Young diagram $i$ and Young diagram $i-1$ are related by taking some number of columns (possibly zero) in diagram $i$ and decomposing each of them into columns in diagram $i-1$.

In this picture, the $T_{-1}$ matrix has a special status: it contains mass parameters (or, equivalently, vevs for the corresponding background vector multiplets) of the theory. By $\mathcal{N}=2$ SUSY, such mass parameters correspond to elements of the Cartan subalgebra of the $\mathcal{N}=2$ flavor symmetry group. In particular, we see that the rank of the flavor symmetry group, $G$, satisfies

$$\mathrm{rank}(G) \geq k_{-1} - 1 \ , \tag{6.5}$$

where the inequality is saturated for cases in which all symmetries are visible in the Hitchin system description (see the next section for an example with hidden symmetries).

As we will discuss below, one particular piece of progress in understanding type III theories relevant to us in this chapter is the first computation of the superconformal

[100] Just as in the case of regular singularities, irregular singularities may be enriched by the presence of certain co-dimension one symmetry defects. Such a construction can lead to 4D SCFTs if there is also a regular singularity present [68].

[101] In particular, $L_a$ is defined as the centralizer (in $A_{N-1}$) of the $T_i$ with $a \leq i \leq \ell-2$. Note that the conditions in (6.3) are necessary but not sufficient to have a sensible SCFT.

index (and determination of the associated chiral algebra in the sense of [27]) in the non-regular case we described in Chapter 3.

In the remainder of this work, the particular theories we will be interested in have type III singularities of the form

$$Y_{0,1} = [n, \cdots, n] \ , \quad Y_{-1} = [n, \cdots, n, n-1, 1] \ , \tag{6.6}$$

where $n \geq 2$, there are $k_{0,1} = k \geq 3$ columns in $Y_{0,1}$, and there are $k_{-1} = k + 1 \geq 4$ columns in $Y_{-1}$ (so that $N = nk$). We discuss the case of $(n, k) = (2, 3)$ in the next section.

## 6.3 The $(n, k) = (2, 3)$ case

In this section, we specialize to the UV $\mathcal{N} = 2$ SCFT given by the Young diagrams

$$Y_1 = Y_0 = [2, 2, 2] \ , \quad Y_{-1} = [2, 2, 1, 1] \ . \tag{6.7}$$

This theory was originally described implicitly in [53]. However, the construction in [75] makes it clear that, although such a type III non-regular theory might seem exotic, it actually arises quite naturally when one uses more traditional SCFTs as building blocks. Indeed, as we saw in Chapter 3 the setup in [75] starts by taking two copies of the isolated $(A_1, D_4)$ SCFT, adding nine hypermultiplets, and conformally gauging a diagonal $su(3)$ flavor symmetry.[102] Then, as one dials the $su(3)$ coupling to infinity, a dual weakly coupled description emerges with a diagonal $su(2)$ of an $(A_1, D_4)$ theory and the $\mathcal{T}_{3,\frac{3}{2}}$ theory gauged. The $\mathcal{T}_{3,\frac{3}{2}}$ theory is another name for the SCFT with $Y_{0,1} = [2, 2, 2]$ and $Y_{-1} = [2, 2, 1, 1]$.

The $\mathcal{T}_{3,\frac{3}{2}}$ theory has $su(2)^2 \times su(3)$ flavor symmetry (of which a diagonal $su(2) \subset su(2)^2$ is gauged in the above duality), although only an $su(2) \times su(3)$ symmetry is visible in (6.7) (according to our analysis in section 3.2, this theory splits into an interacting piece, $\mathcal{T}_X$, with $su(2) \times su(3)$ flavor symmetry, and a free hypermultiplet with $su(2)$ flavor symmetry). More precisely, we see from (6.7) that $k_{-1} = 4$, and so the visible flavor symmetry has rank three.

One remarkable feature of the duality described in [75] is that, even though the theory in question is constructed from various strongly interacting non-Lagrangian building blocks (the $(A_1, D_4)$ and $\mathcal{T}_{3,\frac{3}{2}}$ SCFTs), each of these building blocks has certain observables that are closely related to the corresponding observables in free theories,

[102] Note that the resulting theory has $Y_{-1,0,1} = [2, 2, 1, 1]$ and is non-regular type III even though the various isolated SCFT building blocks are not: the hypermultiplet is described by $Y_{-1,0,1} = [1, 1]$, while $(A_1, D_4)$ is described by $Y_{-1,0,1} = [1, 1, 1]$. As alluded to above, this discussion shows (modulo potential dualities involving theories with one irregular singularity and a regular one) that the theories described by regular semisimple $T_i$ are not closed under conformal gauging.

as we saw in Chapter 3. In the case of $(A_1, D_4)$ SCFT (and its generalizations), we explored this connection in Chapter 4.

In the case of the $\mathcal{T}_{3,\frac{3}{2}}$ theory, the connection with free fields can be seen by examining its Schur index. After removing a decoupled free hypermultiplet to obtain the $\mathcal{T}_X$ SCFT discussed above, we have the following Schur index (3.29)

$$\mathcal{I} = \sum_{\lambda=0}^{\infty} q^{\frac{3}{2}\lambda} \text{P.E.} \left[ \frac{2q^2}{1-q} + 2q - 2q^{1+\lambda} \right] \text{ch}_{R_\lambda}^{su(2)}(q,w) \text{ch}_{R_{\lambda,\lambda}}^{su(3)}(q,z_1,z_2) \ , \tag{6.8}$$

where $\lambda$ is an integer, $q$ is a superconformal fugacity, and $w, z_{1,2}$ are flavor fugacities for $su(2)$ and $su(3)$ respectively. In (6.8), $ch_{R_\lambda}^{su(2)}$ and $ch_{R_\lambda}^{su(2)}$ are characters for modules of $\widehat{su(2)}_{-2}$ and $\widehat{su(3)}_{-3}$ affine Kac-Moody algebras at the crtitical level with primaries transforming with Dynkin labels $\lambda$ and $(\lambda, \lambda)$ of $su(2)$ and $su(3)$ respectively.

The formula in (6.8) is closely related to the index for 8 free half-hypermultiplets (the so-called $T_2$ theory [62])

$$\mathcal{I}_{T_2} = \sum_{\lambda=0}^{\infty} q^{\frac{\lambda}{2}} \text{ P.E.} \left[ \frac{2q^2}{1-q} + 2q - 2q^{1+\lambda} \right] \text{ch}_{R_\lambda}^{su(2)}(q,x) \text{ch}_{R_\lambda}^{su(2)}(q,y) \text{ch}_{R_\lambda}^{su(2)}(q,z) \ , \tag{6.9}$$

where $x, y, z$ are fugacities for the $su(2)^3 \subset sp(4)$ flavor symmetry (the particular rewriting of the $T_2$ index above was suggested in [82]). Indeed, in both cases we sum over a "diagonal" set of representations (of $su(2)^3$ in the $T_2$ case and of $su(2) \times su(3)$ in the $\mathcal{T}_X$ case), and the structure constants (the plethystic exponential factors) are identical.

The $T_2$ theory has a natural connection with $su(2)$ $\mathcal{N} = 4$ SYM. Indeed, by diagonally gauging an $su(2) \times su(2)$ factor we are left with $su(2)$ $\mathcal{N} = 4$ SYM and a decoupled free hypermultiplet. Note that the remaining $\mathcal{N} = 2$ $su(2)$ flavor symmetry becomes part of the $su(4)_R$ symmetry of the $\mathcal{N} = 4$ theory.[103] The deformation that connects $T_2$ to $\mathcal{N} = 4$ SYM is exactly marginal (although if we just want to get $\mathcal{N} = 4$, then we should also turn on a mass parameter for the hypermultiplet or else add a decoupled $u(1)$ $\mathcal{N} = 2$ vector multiplet).

The $\mathcal{T}_{3,\frac{3}{2}}$ SCFT also has a connection to $\mathcal{N} = 4$. For example, as in the case of $su(2)$ $\mathcal{N} = 4$ SYM, the $su(2) \subset su(4)_R$ $\mathcal{N} = 2$ flavor symmetry of the interacting piece (the $\mathcal{T}_X \subset \mathcal{T}_{3,\frac{3}{2}}$ theory) has a global Witten anomaly.[104] More generally, it follows

[103]Technically this is a diagonal flavor symmetry that acts both on the SYM theory and the decoupled hyper. Note that the $su(2)$ symmetry of the $\mathcal{N} = 4$ factor has a Witten anomaly [78]: it has 3 doublets charged under it (the corresponding holomorphic moment maps are $\sum_a Q^a Q^a, \sum_a \tilde{Q}^a \tilde{Q}^a, \sum_a \tilde{Q}^a Q^a$). This anomaly translates into the fact that, at generic points on the moduli space, we have a massless $u(1)$ $\mathcal{N} = 4$ theory: the singlet hypermultiplet is a doublet of $su(2)$ and therefore also gives rise to a Witten anomaly.

[104]In the $su(2)$ $\mathcal{N} = 4$ case, this statement follows from the fact that the adjoint hypermultiplet transforms as three doublets of the $su(2) \subset su(4)_R$ symmetry. By similar reasoning, there is a non-vanishing Witten anomaly for this symmetry in $su(2r)$ $\mathcal{N} = 4$ theories.

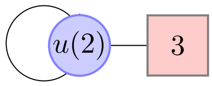



**Figure 8:** The quiver corresponding to the $S^1$ reduction of the $\mathcal{T}_{3,\frac{3}{2}}$ SCFT [1]. Here the closed loop attached to the gauge node denotes an adjoint hypermultiplet of $u(2)$. This adjoint breaks up into a $\mathbf{3}+\mathbf{1}$ of $su(2) \subset u(2)$, with the singlet corresponding to the free decoupled hyper in $\mathcal{T}_{3,\frac{3}{2}} = \mathcal{T}_X \oplus$ hyper [1].

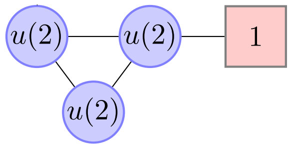



**Figure 9:** The quiver corresponding to the mirror of the $S^1$ reduction of the $\mathcal{T}_{3,\frac{3}{2}}$ theory [1,53]. The Young diagrams describing the $\mathcal{T}_{3,\frac{3}{2}}$ theory are $Y_{1,0} = [2,2,2]$, $Y_{-1} = [2,2,1,1]$ [53].

from anomaly matching that any $\mathcal{N}=4$ theory (Lagrangian or not) with a rank one Coulomb branch (by which we mean that the low energy theory consists of a massless $U(1)$ $\mathcal{N}=4$ vector multiplet at generic points along the three-real-dimensional moduli space) must have a non-vanishing Witten anomaly for the $su(2) \subset su(4)_R$ $\mathcal{N}=2$ symmetry.[105]

Another connection between the $\mathcal{T}_{3,\frac{3}{2}}$ SCFT and $\mathcal{N}=4$ can be found by, instead of introducing dynamical gauge fields (for $su(2)\times su(2)$) as in the $T_2$ case, introducing vevs for background gauge fields (i.e., mass terms) for the $su(3)$ symmetry. This statement is most obvious by first considering the $S^1$ reduction of the $\mathcal{T}_{3,\frac{3}{2}}$ theory. At the level of the index (6.8), this reduction is implemented by taking $q \to 1$ (which corresponds to taking the radius of the $S^1 \subset S^1 \times S^3$ factor in the index to zero) and throwing away a flavor-independent divergent prefactor that encodes certain anomalies of the 4D theory (see [30,37,66,85,86]). Performing this procedure, we showed in Appendix C that (6.8) reduces to the $S^3$ partition function of the 3D theory in Fig. 8. This result confirms the rules conjectured in [53], which produce the mirror quiver gauge theory in Fig. 9 (e.g., see [160]).

From Fig. 8, it is clear that if we turn on any superpotential mass term for the fundamental flavors we will flow to an $\mathcal{N}=8$ SCFT that is the IR endpoint of the usual $\mathcal{N}=8$ $u(2)$ SYM flow.[106] This theory is then the same as the dimensional

[105]This statement generalizes for odd rank $\mathcal{N}=4$ theories (again without appealing to the existence of a Lagrangian).

[106]It is also clear that the Witten anomaly of the 4D $\mathcal{T}_X$ theory is reflected in the fact that there are three doublets of the flavor $su(2)$ arising from the adjoint hypermultiplet of $su(2) \subset u(2)$.

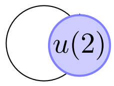



**Figure 10:** The quiver describing the endpoint of the flow initiated by turning on generic $su(3)$ mass parameters in Fig. 8. We find a $u(2)$ $\mathcal{N}=8$ theory (the $u(1)$ piece becomes a direct sum of a twisted hypermultiplet and a conventional hypermultiplet).

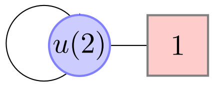



**Figure 11:** The quiver corresponding to the IR endpoint of the RG flow from Fig. 8 after turning on masses for two fundamental flavors (these are non-generic $su(3)$ mass parameters in (6.10)). This theory has accidental $\mathcal{N}=8$ supersymmetry in the IR as in Fig. 10 [161].

reduction of the $u(2)$ 4D $\mathcal{N}=4$ theory.

To see that we end up with 3D $\mathcal{N}=8$ for any value of the superpotential mass terms, note that these mass terms are valued in the adjoint of $su(3)$ and can be parameterized as follows

$$m = \text{diag}(m_1, m_2, -m_1 - m_2) \ . \tag{6.10}$$

Therefore, turning on generic $m_{1,2}$ results in giving masses to all the fundamental flavors, and we are left with the $\mathcal{N}=8$ quiver in Fig. 10. On the other hand, if we choose $m_1 \neq 0$ with $m_2 = 0$, $m_1 = 0$ with $m_2 \neq 0$, or $m_{1,2} \neq 0$ with $m_1 + m_2 = 0$, we give mass to two out of the three fundamental flavors and obtain the quiver in Fig. 11. However, as is well-known (e.g., see [161]), this theory flows to the $\mathcal{N}=8$ quiver of Fig. 10 in the IR.

Combining the procedure of putting the theory on a circle with turning on $su(3)$ mass terms gives us our desired RG flow from sixteen to thirty-two supercharges (see Fig. 12 with $(n,k)=(2,3)$). Indeed, this procedure is unambiguous since the 4D $su(3)$ holomorphic moment maps get mapped to gauge-invariant bilinears of the 3D theory

$$\mu_1 \rightarrow_{r\to 0} Q_1\tilde{Q}^1 - Q_3\tilde{Q}^3 \ , \quad \mu_2 \rightarrow_{r\to 0} Q_2\tilde{Q}^2 - Q_3\tilde{Q}^3 \ , \tag{6.11}$$

where $r$ is the $S^1$ radius, and $Q$, $\tilde{Q}$ are fundamental flavors of $u(2)$. In these expressions, gauge indices have been contracted, and the remaining indices are $su(3)$ flavor indices. Moreover, since there are no non-perturbative $\mathcal{N}=4$-preserving deformations we can contemplate that arise from putting the theory on a circle,[107] and since our mass

[107]This situation is unlike the one considered in [162] for 4D $\mathcal{N}=1$ theories. Note that in our case, flavor symmetries are all non-anomalous in both 4D and 3D due to the larger amount of SUSY.

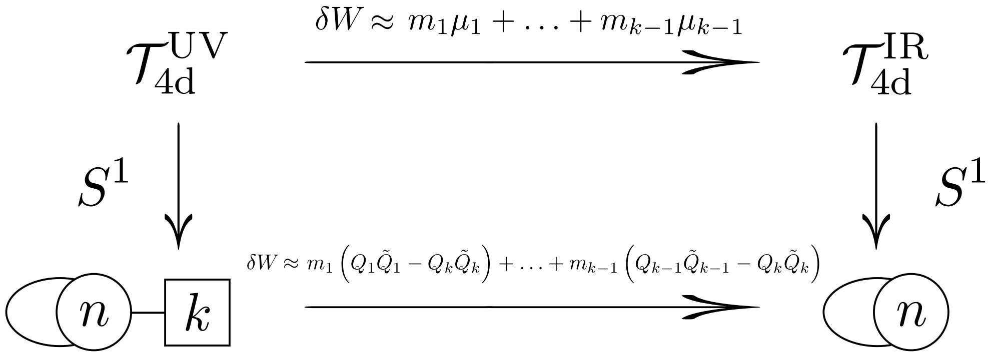



**Figure 12:** The RG flows described in this chapter, with $\mathcal{T}_{4d}^{UV}$ the non-Lagrangian 4D $\mathcal{N} = 2$ SCFT described around (6.6). Horizontal arrows indicate superpotential deformations by holomorphic moment maps / mass terms of $su(k)$. Vertical arrows indicate $S^1$ reductions from 4D to 3D. All arrows preserve eight Poincaré supercharges. As described in the text, we expect this diagram to commute.

deformation does not induce Chern-Simons terms in 3D, we expect the limit of reducing the theory on a circle and turning on mass terms to commute.

### 6.3.1 The $r \to \infty$ limit and (exotic?) 4D $\mathcal{N} = 4$

Two natural questions arise from the above discussion:

- Is the IR of the $r \to \infty$ limit of the above RG flow (i.e., $\mathcal{T}_{4d}^{IR}$) a 4D $\mathcal{N} = 4$ SCFT?
- If $\mathcal{T}_{4d}^{IR}$ has 4D $\mathcal{N} = 4$ SUSY, is this theory $u(2)$ SYM?

Note that the presence of a 3D Lagrangian does not immediately shed light on these questions since, in principle, it is possible that the $\mathcal{N} = 8$ SUSY is accidental in 3D. Moreover, the existence of a 3D Lagrangian does not obviously imply a 4D Lagrangian. Indeed, the $\mathcal{T}_{3,\frac{3}{2}}$ theory does not have a Lagrangian even though the $S^1$ reduction does (as in Fig. 8).

One way to explore these questions is to construct the Seiberg-Witten curve for the $\mathcal{T}_{3,\frac{3}{2}}$ theory[108] and define a scaling limit that produces the Seiberg-Witten curve of the IR theory, $\mathcal{T}_{4d}^{IR}$ (e.g., see [57] for a successful recent application of this technique).

The Seiberg-Witten curve corresponding to an SCFT describes the Coulomb branch that one obtains by deforming the SCFT by relevant or marginal prepotential couplings, mass parameters (i.e., background vector multiplets), and expectation values of $\mathcal{N} = 2$ chiral operators. In general it is not clear whether a particular marginal or relevant parameter of the UV SCFT must necessarily appear in the curve, since the curve is an

[108] By (6.2), the Seiberg-Witten curve for this theory is guaranteed to exist. In general, it is not clear whether a given $\mathcal{N} = 2$ SCFT must have such a curve.

effective description of the theory.[109] However, all parameters appearing in the curve are of the type just described.

To obtain the curve in the case of the $\mathcal{T}_{3,\frac{3}{2}}$ theory, we start by writing the Higgs field as in (6.1)

$$\begin{aligned}\Phi_z &= z\,\mathrm{diag}(a_1,a_1,a_2,a_2,-a_1-a_2,-a_1-a_2)+\mathrm{diag}(b_1,b_1,b_2,b_2,-b_1-b_2,-b_1-b_2)\\ &+ \frac{1}{z}\,\mathrm{diag}(m_1,m_1,m_2,m_2,-m_1-m_2+m_3,-m_1-m_2-m_3)\\ &+ \frac{1}{z^2}\,\mathrm{diag}(c_1,c_2,c_3,c_4,c_5,-c_1-c_2-c_3-c_4-c_5)+\mathcal{O}(z^{-3})\ ,\end{aligned} \tag{6.12}$$

where the degeneracies of the eigenvalues in each singular term correspond to the Young diagrams in (6.7) (the non-singular pieces, starting with the $c_i$, describe vevs of $\mathcal{N}=2$ chiral operators). In principle, since we are interested in studying the RG flow along the top arrow in Fig. 12, we may turn off the $su(2)$ mass parameter. At the level of (6.12), this manoeuvre corresponds to setting $m_3=0$ so that

$$\begin{aligned}\Phi_z &= z\,\mathrm{diag}(a_1,a_1,a_2,a_2,-a_1-a_2,-a_1-a_2)+\mathrm{diag}(b_1,b_1,b_2,b_2,-b_1-b_2,-b_1-b_2)\\ &+ \frac{1}{z}\,\mathrm{diag}(m_1,m_1,m_2,m_2,-m_1-m_2,-m_1-m_2)\\ &+ \frac{1}{z^2}\,\mathrm{diag}(c_1,c_2,c_3,c_4,c_5,-c_1-c_2-c_3-c_4-c_5)+\mathcal{O}(z^{-3})\ ,\end{aligned} \tag{6.13}$$

Indeed, we then see that the terms in (6.13) are subject to a natural action of the $S_3$ Weyl group of $su(3)$, which acts via permutation of the degenerate two-by-two blocks (in particular, the curve we get from (6.2) will be invariant under this action). When we study the Seiberg-Witten curve of the UV theory we will keep track of the $su(2)$ mass parameter as well.

To write the curve for the $\mathcal{T}_{3,\frac{3}{2}}$ SCFT, it is convenient to first shift $x$ and $z$ by constants[110] so that the $\mathcal{O}(z^0)$ matrix in (6.13) is of the form $\mathrm{diag}(0,0,0,0,b,b)$. Now, plugging this result into (6.2) yields

$$\begin{aligned}&u_2+\Big((x-a_1z)(x-a_2z)(x+(a_1+a_2)z)+\frac{M_1}{2}(x-a_1z)+\frac{M_2}{2}(x-a_2z)\\ &\quad-b(x-a_1z)(x-a_2z)\Big)^2+M_3^2(x-a_1z)(x-a_2z)\\ &\quad+u_1\Big(-b(a_1-a_2)(x-a_1z)(x-a_2z)-(x-a_1z)^2(x-a_2z)(a_1+2a_2)\\ &\quad+(x-a_1z)(x-a_2z)^2(2a_1+a_2)+\frac{a_1-a_2}{2}(M_1(x-a_1z)+M_2(x-a_2z))\Big)=0\ ,\end{aligned} \tag{6.14}$$

[109] For example, in the case of the $\mathcal{T}_{3,\frac{3}{2}}$ theory, there are actually two independent $su(2)$ mass parameters (since we have flavor symmetry $su(2)^2\times su(3)$), but, as discussed in [1], only one appears in the curve coming from (6.2). Note that this additional mass parameter might become visible through an alternate construction of the curve that does not go through the particular Hitchin system we described above.

[110] Note that these shifts affect the 1-form only by exact terms, and BPS masses are unchanged.

where

$$\begin{aligned}
M_1 =& -2(a_1+2a_2)m_2 \ , \quad M_2 = -2(2a_1+a_2)m_1 \ , \\
M_3^2 =& -(2a_1+a_2)(a_1+2a_2)m_3^2 \ , \quad u_1 = -(2a_1+a_2)(c_1+c_2)+2bm_1 \ , \\
u_2 =& (a_1-a_2)^2((2a_1+a_2)c_1-bm_1)((2a_1+a_2)c_2-bm_1) \\
&+ (a_1-a_2)(2a_1+a_2)(a_1+2a_2)m_1m_3^2 \ . 
\end{aligned} \tag{6.15}$$

In the above equations, $u_1$ is the vev of the $\mathcal{N}=2$ chiral ring generator of dimension 3/2, while $u_2$ is the vev of the $\mathcal{N}=2$ chiral ring generator of dimension 3. The parameter $b$ is the relevant coupling of dimension 1/2. The dimensionless parameters, $a_i$, are not physical since they are absorbed by changing coordinates, $x$ and $z$. Note also that, when the $su(2)$ mass parameter is turned off, we have $m_3=0$ and therefore $M_3=0$. The above curve transforms homogenously (with the couplings and vevs acting as spurions) under the $u(1)_R$ scaling of the UV SCFT.

To study the RG flow described by the top arrow in Fig. 12, we would like to turn on some RG scale, $m$, in (6.14) and take $m\to\infty$.[111] We can make contact with a curve resembling that of the $su(2)$ $\mathcal{N}=4$ theory (we expect to have an additional $\mathcal{N}=4$ $u(1)$ decoupled) if we set

$$\begin{aligned}
&u_1=0 \ , \quad M_1=m \ , \quad M_2=0 \ , \quad M_3=0 \ , \quad b=qm^{\frac{1}{2}} \ , \\
&x-a_1z=2m^{-\frac{1}{2}}X \ , \quad x-a_2z=m^{\frac{1}{2}}Z \ , \quad u_2=-Um \ .
\end{aligned} \tag{6.16}$$

Here, $X$ and $Z$ act as good coordinates describing the curve of $\mathcal{T}_{4d}^{IR}$ and do not scale with $m$ (they have scaling dimension one and zero respectively) while $U$ is a dimension two vev. We interpret some of the terms that are turned off as decoupling or becoming irrelevant along the RG flow (although, as discussed below, additional quantities decouple).[112] Keeping only the leading terms in (6.14) as $m\to\infty$ and solving for $X$, we obtain

$$X^2 = \frac{U}{\left(\frac{2(2a_1+a_2)}{a_1-a_2}Z^2-2qZ+1\right)^2} \ . \tag{6.17}$$

This equation is the $su(2)$ $\mathcal{N}=4$ curve[113] tuned to a cusp (i.e., a weak gauge coupling limit). Indeed, although there is an apparently dimension zero parameter, $q$, arising in (6.16) (reflecting the fact that the dimension half coupling in the Hitchin system

[111] Note that this method is a rather indirect way of studying the RG flow: we try to carve out the Coulomb branch of $\mathcal{T}_{4d}^{IR}$ as a subspace of the Coulomb branch of $\mathcal{T}_{4d}^{UV}$ rather than considering the flow starting from the UV SCFT and then deforming by $\delta W\sim m_1\mu_1+m_2\mu_2$ with zero vevs (the 3D picture of the RG flow suggests that we should remain at the origin of the 4D Coulomb branch).

[112] As we will see, the fact that $M_3\to 0$ in (6.16) is simply interpreted as the fact that our curve will be tuned to a cusp.

[113] By (6.16), the 1-form is (modulo exact terms) also the 1-form for $su(2)$ $\mathcal{N}=4$ (up to a constant we can tune).

forms a dimensionless combination with the square-root of the mass parameter), this naively marginal parameter is irrelevant in the IR description given above.

Note that we may also exchange $a_1 \leftrightarrow a_2$ and $M_1 \leftrightarrow M_2$ and obtain a similar limit of the curve. Finally, we may also construct a closely related limit of the curve by taking the linear combination $(2a_1 + a_2)M_1 + (a_1 + 2a_2)M_2$ to vanish.[114]

We have not been able to find a more general non-trivial scaling limit than the one described above. In particular, we are not able to see the putative $\mathcal{T}_{4d}^{IR}$ marginal deformation (away from the cusp) in the Seiberg-Witten description we have found. Note that if $\mathcal{T}_{4d}^{IR}$ has $\mathcal{N} = 4$ SUSY, it necessarily possesses an exactly marginal deformation residing in the stress-energy tensor multiplet (this statement follows from $\mathcal{N} = 4$ SUSY and is not related to the existence of an $\mathcal{N} = 4$ Lagrangian). However, there may be several reasons for the absence of this marginal direction in the Coulomb branch effective action:

- A radical option is that $\mathcal{T}_{4d}^{IR}$ is an exotic 4D $\mathcal{N} = 4$ theory without a Lagrangian description. In a Lagrangian theory, we expect that W-boson masses will vary as a function of the marginal gauge coupling (this statement follows from the Higgs mechanism). These changes in mass are reflected in the periods of the curve, since the W-bosons are BPS particles. On the other hand, as far as we are aware, there is no argument that the most general exactly marginal parameter in an $\mathcal{N} = 2$ or $\mathcal{N} = 4$ SCFT must appear in the IR effective action captured by the Seiberg-Witten description. If the IR effective action is indeed given by (6.17), it means that the exactly marginal parameter of the UV SCFT becomes irrelevant in the IR (as opposed to being related to a marginal coupling in the IR). In this case, varying the exactly marginal parameter may have a more profound effect on the non-BPS sector.

- A less radical option is that the exactly marginal parameter of $\mathcal{T}_{4d}^{IR}$ is a standard gauge coupling, but it is hidden in the flow from $\mathcal{T}_{3,\frac{3}{2}}$. This option is not implausible since the existence of a conformal manifold is accidental in this case. If this possibility is realized, then perhaps the marginal coupling becomes visible by choosing a different UV starting point than $\mathcal{T}_{4d}^{UV}$.

- The most conservative option is simply that there is a more general scaling limit that describes the curve of $\mathcal{T}_{4d}^{IR}$ for all values of the exactly marginal parameter. In this case, $\mathcal{T}_{4d}^{IR}$ may again be a standard $\mathcal{N} = 4$ Lagrangian theory.

We hope to conduct a more detailed study of these options using additional techniques.[115] In the next section we set this goal aside for now and present infinitely many generalizations of the above discussion.

[114] Said more invariantly, when we take the scaling limit in (6.16), Tr $T_{-1}^2 \to \infty$ and Tr $T_{-1}^3 \to 0$.

[115] The F-theory realization of $\mathcal{T}_X$ in terms of D3 branes probing an $A_2$ singularity in [71] seems to

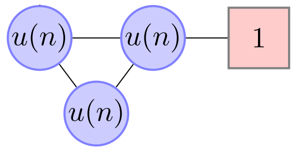



**Figure 13:** The quiver corresponding to the mirror of the $S^1$ reduction of the type III AD theory with $Y_{1,0} = [n, n, n]$, $Y_{-1} = [n, n, n-1, 1]$ [53].

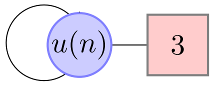



**Figure 14:** The quiver corresponding to the $S^1$ reduction of the type III AD theory with $Y_{1,0} = [n, n, n]$, $Y_{-1} = [n, n, n-1, 1]$. The closed loop attached to the gauge node denotes an adjoint hypermultiplet of $u(n)$.

## 6.4 Generalizations

It is rather straightforward to generalize the above discussion to other values of $n$ and $k$. For example, we can take any $n \geq 2$. $\mathcal{T}_{4d}^{UV}$ is now described by the following Young diagrams, which generalize (6.7)

$$Y_1 = Y_0 = [n, n, n] \ , \quad Y_{-1} = [n, n, n-1, 1] \ . \tag{6.18}$$

Applying the discussion in [53], one can easily check that this theory has rank $n$ with $\mathcal{N} = 2$ chiral ring generators of scaling dimensions

$$\Delta = \left\{ \frac{3}{2} \ , \ 3 \ , \ \frac{9}{2} \ , \ \cdots \ , \ \frac{3n}{2} \right\} \ . \tag{6.19}$$

Note that, as in $\mathcal{N} = 4$, the scaling dimensions of chiral operators are integer multiples of the dimension of the lowest dimensional chiral operator (although here, unlike in $\mathcal{N} = 4$, the scaling dimension of the lowest dimensional chiral operator is half-integer).[116]

In this case, the 3D mirror quiver generalizing Fig. 9 is given in Fig. 13 following the rules in [53]. The mirror of this quiver (i.e., the direct $S^1$ reduction) is the $u(n)$ theory with an adjoint hypermultiplet and three fundamental flavors as in Fig. 14 (e.g., see the discussion in [160]).

We may then reproduce the discussion for $n = 2$ for general $n \geq 2$ by turning on

suggest one of the first two options is being realized. This type of construction may also shed light on the more general $\mathcal{T}_{4d}^{IR}$ theories considered in the next section.

[116] In fact, the scaling dimensions of the operators in (6.19) correspond to those of $u(n)$ $\mathcal{N} = 4$ SYM up to an overall multiplication by $3/2$.

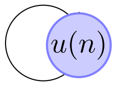



**Figure 15:** The result of turning on generic $su(3)$ masses in the quiver in Fig. 14.

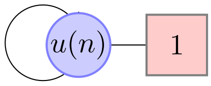



**Figure 16:** The result of turning on masses for two out of the three flavors in Fig. 15. Quantum mechanically, this remaining fundamental flavor also gets a mass [161].

masses for the three fundamental flavors in the $S^1$ reduction. For generic masses, we end up with the quiver in Fig. 15. For non-generic $su(3)$ masses, we end up with the quiver in Fig. 16, which, by the discussion in [161], flows to the 3D $\mathcal{N} = 8$ quiver in Fig. 15. By combining the procedure of $S^1$ reduction with turning on masses, we again, as in the more detailed discussion of the $n = 2$ case, get the commuting RG diagram of Fig. 12 with accidental enhancement to thirty-two (Poincaré plus special) supercharges in the IR. We again suspect (but have not proven) that the $r \to \infty$ limit of this flow has $\mathcal{N} = 4$ SUSY.

Finally, note that we can have an even more general UV starting point given by

$$Y_{1,0} = [n, n, \cdots, n] \ , \quad Y_{-1} = [n, \cdots, n, n-1, 1] \ , \tag{6.20}$$

where, as in (6.6), $n \geq 2$, there are $k \geq 3$ columns in $Y_{0,1}$, and there are $k + 1 \geq 4$ columns in $Y_{-1}$ (so that $N = nk$, where we obtain our theory from the $A_{N-1}$ $(2, 0)$ theory). Here the mirror looks as in Fig. 13, but now there is a $k$-sided polygon of $u(n)$ nodes with one node coupled to a fundamental flavor. The direct reduction of the theory is given in Fig. 17. Just as in the previous cases, we may give masses to these $k$ fundamental flavors and flow to a theory with thirty-two (Poincaré plus special) supercharges, thus obtaining the RG diagram in Fig. 12.

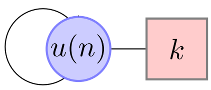



**Figure 17:** The quiver corresponding to the $S^1$ reduction of the type III AD theory with Young diagrams described in (6.20). The closed loop attached to the gauge node denotes an adjoint hypermultiplet of $u(n)$.

# Chapter 7

# Conclusions and Outlook

The wider theme of this thesis was the investigation of strongly coupled QFTs using modern nonperturbative techinques. More precisely, we explored various powerful tools such as the superconformal index, the chiral algebra correspondence, or the class $\mathcal{S}$ construction in the context of 4D $\mathcal{N} = 2$ SCFTs called Argyres-Douglas theories. These techniques allowed us to uncover many interesting properties of these rather mysterious non-Lagrangian theories, including their duality structure and renormalization group flows. In the remainder of this chapter, we review our main results, present some open questions and comment on recent developements.

Using very little data, in Chapter 3 we found the Schur index and chiral algebra of the exotic isolated irreducible SCFT, $\mathcal{T}_X$,[117] that emerges in the simplest AD generalization of Argyres-Seiberg duality. $\mathcal{T}_X$ was later identified in [71] as the rank-two $SU(3)$ instanton SCFT, and its chiral algebra was given a free field realization. We also saw that this theory has a remarkable resemblance to its cousin $T_N$ theories. This connection has recently been explained in [70], where $\mathcal{T}_X$ and $(A_1, D_4)$ (plus a hypermultiplet) was given a surprising class $\mathcal{S}$ construction as a trinion theory based on the twisted $A_2$ $(2, 0)$ theory. This construction utilized only regular punctures, contradicting the common lore that irregular punctures are necessary to generate non-integer Coulomb branch dimensions.[118] This class $\mathcal{S}$ construction explains our index formula written in terms of AKM characters in (3.29) simply as the TQFT-expression for the index of the twisted $A_2$ trinion theory. Moreover, as a result of the class $\mathcal{S}$ construction of $\mathcal{T}_X$ and $(A_1, D_4)$, our duality finds a satisfying place within the geometric framework of class $\mathcal{S}$ S-duality.

Our results in Chapter 3 raise many open questions. In particular the $\mathcal{T}_X$ chiral algebra has only bosonic operators. It would be interesting to know if this is part

[117] Note that this chiral algebra lies outside the classes of AD chiral algebras considered in the literature before (e.g., see [19, 30, 32, 33, 38, 39]).

[118] As the authors of [70] note however, Coulomb branch dimensions of class $\mathcal{S}$ SCFTs based on twisted $A_{2n}$ theories seem to be restricted to half-integers.

of some larger pattern for isolated $1 < \mathcal{N} < 3$ SCFTs. We also saw that $\mathcal{T}_X$ has $SU(3) \times SU(2) \times U(1)$ flavor symmetry (when viewed as an $\mathcal{N} = 1$ theory). We are not aware of another way to find this symmetry group in string or field theory from a minimality condition (recall that in our case, this symmetry emerges from requiring that we study the minimal generalization of Argyres-Seiberg duality to $\mathcal{N} = 2$ SCFTs with non-integer chiral primaries). Can the minimality we are discussing be made more precise so that one can find this SCFT using the conformal bootstrap (perhaps, in light of (3.43) and (3.28), it will be useful to study the $\langle J^I W^a_K J^L W^b_M \rangle$ four-point function)? What if we gauge the flavor symmetry–can this SCFT act as a hidden sector for beyond the standard model physics (since the $U(1)$ is not asymptotically free, this gauged theory can, at best, be part of an effective field theory)?

It would also be interesting to find a manifestation of the 4D Witten anomaly for the (inconsistent) SCFT in Fig. 6 in the corresponding 2D chiral algebra (as discussed in Sec. 3.8). It is also interesting to observe that the expression in (3.29) makes it rather trivial to write down simple formulae for the indices of conformal manifolds built out of $\mathcal{T}_X$ theories (as in the case of the $T_N$ theories). For typical conformal manifolds built out of AD theories (e.g., as in the case of the $(A_N, A_M)$ conformal manifolds studied in [66]), this procedure is considerably more complicated.

In Chapter 4 we followed up on our observation made in Chapter 3 that there seems to be some kind of a connection between the $(A_1, D_4)$ Argyres-Douglas theory and free fields. In particular, we have seen that the simple non-unitary 4D Lagrangian (4.27) allows us (through manipulations in two dimensions) to exactly compute the unrefined Schur indices for the $D_2[SU(2N+1)]$ SCFTs, including the $(A_1, D_4) \equiv D_2[SU(3)]$ theory. We were also able to compute other (linear combinations of) characters of the associated chiral algebras via the, to our knowledge, novel mathematical identities in (4.25). Based on known relations between chiral algebras in 2D and 3D QFT, it is reasonable to expect that aspects of the physics of the non-vacuum modules of the chiral algebras are captured by (worldvolumes of) 4D objects that have non-trivial braiding statistics as in [163]. Indeed, there is considerable evidence that this intuition holds [49, 164], and it would be interesting to study surface and line defects in our setup. In particular, the Lagrangian in (4.27) seems to compute the Schur indices of the $D_2[SU(2N+1)]$ SCFTs in the presence of certain surface defects[119], while we presumably need to introduce defects in our non-unitary theory in order to compute—directly in 4D—the other Schur indices of the $D_2[SU(2N+1)]$ theories.

As another future direction, we may hope to find information about new observables that are closely related to the chiral algebra as in [40, 165, 166]. Moreover, since we have a Lagrangian description of certain Schur observables, it is tempting to see what

[119] Or, more precisely, it computes linear combinations of the usual Schur indices and the Schur indices in the presence of certain surface defects.

(if anything) the corresponding correlation functions / OPE coefficients compute in the original strongly interacting theory. Even more simply, it would be interesting to understand if it is possible to map flavor symmetries between our two sets of theories.

We also expect that our procedure of starting with a unitary 4D theory, mapping to 2D using [27], conjugating / permuting the characters, reinterpreting the characters as objects in a unitary 2D theory, and then lifting to a non-unitary theory in 4D could generalize (with certain modifications) to other $\mathcal{N} = 2$ theories. While we do not expect that the non-unitary 4D theories will in general have a completely free description in terms of hypermultiplets, gauge fields and perhaps the constructions in [13] could play a role (possibly when the original 4D theory has a conformal manifold [19, 66, 104, 158, 159]). Indeed, we expect non-unitary Lagrangians to be a more diverse and flexible group of objects than their unitary counterparts, and so we anticipate them to describe "more" theories.

Still, we should point out that our non-unitary 4D theories described in Chapter 4 have an avatar of 2D unitarity: a modified notion of reflection positivity exists in our theories. Related 2D constructions have played a role in non-unitary extensions of Zamolodchikov's $c$-theorem [167]. Such structures, involving "hidden" unitarity, may also shed more light on the question of which non-unitary theories in 2D are able to encode the unitary 4D physics in the original construction of [27].

It would also be interesting to understand any relation between our construction and the Lagrangians appearing in [59–61, 152, 154]. While our Lagrangians govern only a particular sector of the theories we study (and perhaps only a particular set of observables in such a sector), they are considerably simpler than the "full" Lagrangians in these latter works.[120] Also, our approach is different: we sacrifice unitarity instead of the $\mathcal{N} = 2$ superconformal algebra. More generally, it would be interesting to find connections between our discussion and other effective Lagrangian descriptions of sectors of QFTs (e.g., as in [168]).

Finally, we mention that the $D_2[SU(2N + 1)]$ series of theories have recently been given a class $\mathcal{S}$ interpretation in [70] as trinion theories based on the twisted $A_{2N}$ $(2, 0)$ theories. It would be nice to see if there are any connections between this construction and our results in Chapter 4.

In Chapter 5 we investigated the natural RG flows between the AD theories $(A_1, D_p)$ and $(A_1, A_{p-3})$, that we get by closing the regular puncture involved in the class $\mathcal{S}$ construction of $(A_1, D_p)$. Motivated by a partially flavored character relation between the logarithmic chiral algebras of $(A_1, D_p)$ and a series of rational chiral algebras, we found that certain pieces of the modular data underlying the chiral algebras of the IR $(A_1, A_{p-3})$ theories are related to those of the UV $(A_1, D_p)$ theories, via the action of

[120] Although, at present, there are no known "full" Lagrangians for the $D_2[SU(2N+1)]$ theories with $N > 1$.

Galois conjugation.

Here we mention some additional observations, comments, and open questions. Another way to find a unitary interpretation of the $\chi[D_2[SU(3)]] = \widehat{su(3)}_{-\frac{3}{2}}$ characters discussed around (5.1) is as follows. Consider the chiral $\widehat{su(3)}_3$ CFT. Three of the ten primaries of this theory (transforming under $su(3)$ representations $[0,0]$, $[3,0]$, and $[0,3]$) are related to the abelian lines that generate the $\mathbb{Z}_3$ one-form symmetry of the $SU(3)_3$ MTC. Gauging this one-form symmetry projects out the lines in representations $[0,1], [1,2], [2,0], [0,2], [2,1], [1,0]$. The remaining $[1,1]$ representation is a fixed point under $\mathbb{Z}_3$ fusion, and so we add two more copies of it. This object then gives rise to the three unitary dimension $1/2$ chiral primaries in the associated chiral RCFT whose characters match the $\widehat{su(3)}_{-\frac{3}{2}}$ vacuum character.

This approach is reminiscent of the $\mathbb{Z}_2$ gauging in the case of $\widehat{su(2)}_{2(p-1)}$ discussed at length in Chapter 5 and also in [45]. As in the $\widehat{su(2)}$ case, it potentially gives us a canonical way to relate the unitary and non-unitary theories when we turn on (discrete) flavor fugacities. This example, combined with those in the rest of this chapter, suggest a link between the physics of the $(A^N_{N-1}[p-N], F)$ theories, the admissible characters of their associated chiral algebras, $\widehat{su(N)}_{N\frac{1-p}{p}}$, and the $\mathbb{Z}_{N(p-1)}$ gaugings of $\widehat{su(N)}_{N(p-1)}$. The relation is already somewhat more elaborate in the case of $N = 2n + 1 \geq 5$ and $p = 2$, since the anyons related to the one-form symmetry correspond to 2D RCFT chiral primaries of conformal dimension larger than 1. For general $p$ and $N$ one must also take into account the fact that some of the admissible characters of the logarithmic theories have negative coefficients.[121] Perhaps these can be related to rational theories after turning on some flavor fugacities (or, more generally, fugacities for generators corresponding to a unitary $W$-algebra). Clearly it would be interesting to understand this point better

In our examples, we "rationalized" UV chiral CFTs constructed via [27] by associating rational theories with them. On the other hand, the IR was already rational, though non-unitary, at the $(A_1, A_{p-3})$ endpoints (since it was a $(2, p)$ minimal model). More generally, to allow for an anyonic imprint on the Higgs branch as in (5.50) and Fig. 7, we have to "rationalize" the IR theory as well. Indeed, we saw an example of this phenomenon in the $D_2[SU(2n + 1)] \to D_2[SU(2n' + 1)]$ flows. It would be interesting to understand this process more generally.

Our most non-trivial correspondence (i.e., the one with a non-trivial Galois action) was between UV chiral RCFTs and IR chiral algebras that are $C_2$-cofinite. In physics language, this means that we are studying IR theories on the Higgs branch that have no Higgs branches themselves [43] (e.g., the $(A_1, A_{p-3})$ theories with $p \in \mathbb{Z}_{\rm odd}$ do not

[121] This statement can be easily seen by considering the linear modular differential equations satisfied by the Schur index (e.g., see [43] for an introduction in the context of the 4D/2D correspondence of [27]). For other interesting recent work on LMDEs and their implications for 2D CFT, see [46, 169, 170].

have Higgs branches). The authors of [43] and their collaborators have embarked on a program to classify 4D $\mathcal{N} = 2$ SCFTs using these $C_2$-cofinite theories as basic building blocks. It would be interesting if our work sheds light on this program.

We did not pursue qDS reduction on the RCFT side. Clearly this is interesting to do. Perhaps the recent notion of Galois conjugation at the level of RCFT characters [129] will prove useful to make contact between the UV and IR. The LMDE-based discussion in [46] may also play a role.

In Chapter 6 we studied an infinite set of RG flows that start from 4D $\mathcal{N} = 2$ SCFTs that lack a Lagrangian description and end up, after turning on generalized mass terms, flowing to theories that have thirty-two (Poincaré plus special) supercharges. We are able to demonstrate this fact compellingly when we also compactify these theories on a circle (and we have the flow diagram in Fig. 12).

We also gave some preliminary, but far from conclusive, arguments that these theories flow to 4D $\mathcal{N} = 4$ SCFTs (at least for $(n, k) = (2, 3)$) when we take the radius of the circle to infinity. One important matching quantity was the Witten anomaly for the $\mathcal{T}_X \subset \mathcal{T}_{3,\frac{3}{2}}$ SCFT.

In Chapter 3, we wondered how to construct such Witten anomalous theories directly in terms of punctured compactifications of the $(2, 0)$ theory. Recently, there has been progress on this topic [68,171]. Moreover, the authors of [68] find an $\mathcal{N} = 4$ theory starting directly from an irregular singularity (and a regular singularity, both in the presence of a co-dimension one defect). It would be interesting to see if their theory is related to $\mathcal{T}^{IR}_{4d}$ in the case of $(n, k) = (2, 3)$.

We have also seen that the scaling limit we chose does not reproduce the standard $\mathcal{N} = 4$ curve, since the IR description seems tuned to a cusp. As discussed in section 6.3.1, this result may have various causes ranging from the existence of an exotic $\mathcal{N} = 4$ non-Lagrangian theory to the presence of a hidden marginal direction or to the existence of a more general scaling limit that describes the curve of $\mathcal{T}^{IR}_{4d}$. It would be interesting to find out which of these options is realized.[122] The F-theory construction of $\mathcal{T}_X$ in [71] suggests that one of the latter two options take place, at least in the case of $(n, k) = (2, 3)$, but this type of construction might also shed light on the $(n, k) > (2, 3)$ theories.

Our construction in Chapter 6 has also been extended in [175] to SCFTs obtained from marginal gaugings of $D_2[SU(2N + 1)]$ theories, suggesting that this type of SUSY enhancement from 16 to 32 supercharges might be more prevalent among class $\mathcal{S}$ theories with 3D mirrors than previously thought.

[122] It would also be interesting to understand if our RG flows shed any light on the question of classifying $su(N)$ $\mathcal{N} = 4$ SYM theories [72, 172, 173] and if we can understand some of our theories from a holographic point of view (along the lines of [174]).

# Appendix A

# Proof of the XYY formula

In this appendix we review the fact that the conjectured XYY formula for the Schur index of the $(A_1, D_4)$ theory [31] reproduced in (3.13) can be proven using Theorem 5.5 of [32] (in fact, this result follows directly from (11) of [79]).[123]

To that end, we start with the XYY formula

$$\begin{aligned}\mathcal{I}_{(A_1,D_4)}(q,a,b) &= \text{P.E.}\left[\frac{q}{1-q^2}\,\chi^{SU(3)}_{\text{Adj}}(a,b)\right]\\ &= \text{P.E.}\left[\frac{q}{1-q^2}\left(2+\frac{1}{a^2b}+\frac{1}{ab^2}+\frac{a}{b}+\frac{b}{a}+a^2b+ab^2\right)\right]\ . \end{aligned}\tag{A.1}$$

Expanding the plethystic exponentials, we obtain

$$\text{P.E.}\left[\frac{a}{1-b}\right] = \prod_{i=0}^{\infty}\frac{1}{1-ab^i}\ , \tag{A.2}$$

and we can then rewrite (A.1) as

$$\begin{aligned}\text{P.E.}\left[\frac{q}{1-q^2}\,\chi^{SU(3)}_{\text{Adj}}(a,b)\right] = \prod_{n=0}^{\infty}&\frac{1}{(1-q^{2n+1})^2}\frac{1}{(1-\frac{1}{a^2b}q^{2n+1})}\frac{1}{(1-\frac{1}{ab^2}q^{2n+1})}\frac{1}{(1-\frac{a}{b}q^{2n+1})}\times\\ &\times\frac{1}{(1-\frac{b}{a}q^{2n+1})}\frac{1}{(1-a^2bq^{2n+1})}\frac{1}{(1-ab^2q^{2n+1})}\ .\end{aligned}\tag{A.3}$$

It is then straightforward to show that (A.3) becomes the Schur index of the $(A_1, D_4)$ SCFT given by Theorem 5.5 of [32] (setting $p = 2$ and with the $q^{1/3}$ prefactor stripped

[123] Note that the authors of [79] also demonstrate more general conjectures [31] for theories closely related to the $(A_1, D_4)$ SCFT.

off)

$$\begin{aligned}\mathcal{I}_{(A_1,D_4)}(q,x,y) &= \prod_{n=0}^{\infty} \frac{\left(1-y^2q^{2(n+1)}\right)\left(1-q^{2(n+1)}\right)^2\left(1-y^{-2}q^{2(n+1)}\right)}{\left(1-y^2q^{n+1}\right)\left(1-q^{n+1}\right)^2\left(1-y^{-2}q^{n+1}\right)\left(1-xyq^{2\left(n+\frac{1}{2}\right)}\right)}\times \\ &\times \frac{1}{\left(1-x^{-1}yq^{2\left(n+\frac{1}{2}\right)}\right)\left(1-xy^{-1}q^{2\left(n+\frac{1}{2}\right)}\right)\left(1-x^{-1}y^{-1}q^{2\left(n+\frac{1}{2}\right)}\right)} \\ &= \prod_{n=0}^{\infty} \frac{1}{\left(1-q^{2n+1}\right)^2\left(1-y^{\pm 2}q^{2n+1}\right)\left(1-x^{\pm}y^{\pm}q^{2n+1}\right)}\ ,\end{aligned} \tag{A.4}$$

under the fugacity map

$$a = y\,x^{1/3} \qquad b = y^{-1}\,x^{1/3}\ . \tag{A.5}$$

The relation in (A.5) corresponds to the decomposition of the $SU(3)$ fugacities into fugacities of $SU(2)\times U(1)$. Before concluding, note that, as in (3.16), the "$\pm$" superscripts in (A.4) are understood as a product over each sign, e.g.

$$\frac{1}{1-y^{\pm 2}q^{2n+1}} \equiv \frac{1}{1-y^{2}q^{2n+1}}\,\frac{1}{1-y^{-2}q^{2n+1}}\ . \tag{A.6}$$

# Appendix B

# Details of the Inversion Formula

In this appendix we find an integral expression for the superconformal index of the $\mathcal{T}_{3,\frac{3}{2}}$ theory in the Schur limit by employing the inversion theorem proved in [80]. Our use of the inversion theorem is similar to its use in the case of the $E_6$ SCFT by the authors of [81], but there are some technical differences here since our $SU(2)$ duality frame in Fig. 3 has, in addition to the $\mathcal{T}_{3,\frac{3}{2}}$ theory, a strongly interacting $(A_1, D_4)$ SCFT instead of a pair of hypermultiplets as in the $E_6$ case. Nonetheless, we will argue that, using the results reviewed in Appendix A and an argument about analytic properties of the index, we can invert the gauge integral of the index in the $SU(2)$ duality frame.

In order to find the index in the two duality frames we need the index of the basic building blocks in Figs. 2 and 3. To that end, we reproduce here the single letter Schur index of the $\mathcal{N} = 2$ vector multiplet (transforming in the adjoint of the gauge group) and half-hypermultiplet (transforming in representation $R$ of the combined gauge and flavor groups) (2.42) dressed with characters for their respective gauge and flavor representations

$$\begin{aligned} \mathcal{I}_{\text{vect}}(q, \mathbf{x}) &= -\frac{2q}{1-q}\chi_{\text{adj}}(\mathbf{x}) \ , \\ \mathcal{I}_{\frac{1}{2}H}(q, \mathbf{x}, \mathbf{z}) &= \frac{\sqrt{q}}{1-q}\,\chi_R(\mathbf{x}, \mathbf{z}) \ . \end{aligned} \tag{B.1}$$

We can "glue" these indices along with the index of the $(A_1, D_4)$ SCFT given in (3.13) by integrating their product over the Haar measure of the diagonal subgroup we are gauging just like we did for Lagrangian theories in (2.32). We start with the $SU(3)$ side of the duality where we are gauging the diagonal part of the $SU(3)$ flavor symmetries of the two $(A_1, D_4)$ theories along with 3 fundamental hypermultiplets as in Fig. 2. The latter degrees of freedom supply the $U(3)$ symmetry, which is decomposed as

$U(3) = SU(3)_z \otimes U(1)_s$. The index on this side of the duality is then given by

$$\mathcal{I}_{SU(3)}(q,s,z_1,z_2) = \frac{(q;q)^4}{6} \oint_{\mathbb{T}^2} \prod_{k=1}^{2} \frac{dx_k}{2\pi i x_k} \prod_{i\neq j}(x_i - x_j)\left(q\frac{x_i}{x_j};q\right)^2 \times$$
$$\times \text{ P.E.}\left[\frac{q}{1-q^2}\,\chi^{SU(3)}_{\text{Adj}}(x_1,x_2)\right]^2 \prod_{i,j}\left(\sqrt{q}\left(\frac{z_j s^{1/3}}{x_i}\right)^{\pm};q\right)^{-1}, \tag{B.2}$$

where $\mathbb{T}$ is the positively oriented unit circle, $\prod_{k=1}^{2}\frac{dx_k}{2\pi i x_k}\frac{1}{3!}\prod_{i\neq j}(x_i-x_j)$ is the Haar measure of $SU(3)$, and the $x_i$ $(i=1,2,3)$ satisfy the constraint $\prod_{i=1}^{3} x_i = 1$. We can rewrite (B.2) slightly using elementary computations described in Appendix A

$$\text{P.E.}\left[\frac{q}{1-q^2}\,\chi^{SU(3)}_{\text{adj}}(x_1,x_2)\right] = (q;q^2)^{-2}\prod_{i\neq j}\left(q\frac{x_i}{x_j};q^2\right)^{-1}, \quad x_3 = x_1^{-1}x_2^{-1}\,. \tag{B.3}$$

Substituting (B.3) into (B.2) and performing some simplifications yields the following explicit formula

$$\mathcal{I}_{SU(3)}(q,s,z_1,z_2) = \frac{(q^2;q^2)^4}{6}\oint_{\mathbb{T}^2}\prod_{i=1}^{2}\frac{dx_i}{2\pi i x_i}\prod_{i\neq j}(x_i-x_j)\left(q^2\frac{x_i}{x_j};q^2\right)^2\prod_{i,j}\left(\sqrt{q}\left(\frac{z_j s^{1/3}}{x_i}\right)^{\pm};q\right)^{-1}. \tag{B.4}$$

Since the index is invariant under duality transformations, (B.4) has to equal the index on the $SU(2)$ side of the duality where we are gauging the diagonal $SU(2)_e$ of the $(A_1,D_4)$ and $\mathcal{T}_{3,\frac{3}{2}}$ theories as in Fig. 3. We can write the index in this duality frame as

$$\mathcal{I}_{SU(2)}(q,s,z_1,z_2) = \frac{(q;q)^2}{2}\oint_{\mathbb{T}}\frac{de}{2\pi i e}(e^{\pm 2}q;q)^2(1-e^{\pm 2})\times$$
$$\times\text{P.E.}\left[\frac{q}{1-q^2}\,\chi^{SU(3)}_{\text{adj}}\left(es^{\frac{1}{3}}, e^{-1}s^{\frac{1}{3}}, s^{-\frac{2}{3}}\right)\right]\mathcal{I}_{\mathcal{T}_{3,\frac{3}{2}}}(q,e,z_1,z_2)\,, \tag{B.5}$$

where $\frac{de}{2\pi i e}\frac{1}{2}(e-e^{-1})(e^{-1}-e)$ is the Haar measure of $SU(2)$. Rewriting the plethystic exponential as in (B.3) and performing some simplifications leads to

$$\mathcal{I}_{SU(2)}(q,s,z_1,z_2) = \frac{(q^2;q^2)^2}{2}\oint_{\mathbb{T}}\frac{de}{2\pi i e}\frac{(e^{\pm 2}q;q)(e^{\pm 2};q^2)\mathcal{I}_{\mathcal{T}_{3,\frac{3}{2}}}(q,e,z_1,z_2)}{(qs^{\pm}e^{\pm};q^2)}\,. \tag{B.6}$$

Finally, to make contact with the inversion theorem, we replace $q \to \sqrt{q}$

$$\mathcal{I}_{SU(2)}(q,s,z_1,z_2)|_{q\to\sqrt{q}} = \frac{(q;q)^2}{2}\oint_{\mathbb{T}}\frac{de}{2\pi i e}\frac{(e^{\pm 2};q)}{(\sqrt{q}s^{\pm}e^{\pm};q)}(e^{\pm 2}\sqrt{q};\sqrt{q})\mathcal{I}_{\mathcal{T}_{3,\frac{3}{2}}}(q,e,z_1,z_2)|_{q\to\sqrt{q}}\,. \tag{B.7}$$

Now we will explain how to use the inversion theorem in order to extract $\mathcal{I}_{\mathcal{T}_{3,\frac{3}{2}}}$ from this equation. Extracting $\mathcal{I}_{\mathcal{T}_{3,\frac{3}{2}}}$ is highly non-trivial since it is not at all obvious why (B.7) preserves all the information about this quantity.

## B.1 Inversion Theorem

This section closely follows Appendix B of [81]. The input to the inversion theorem of [80] is the following type of contour integral

$$\hat{f}(w) = \kappa \oint_{C_w} \frac{ds}{2\pi i s} \delta(s, w; T^{-1}, p, q) f(s) \ , \tag{B.8}$$

where $\kappa = \frac{1}{2}(p;p)(q;q)$, $w$ is on the unit circle, and the integral kernel is defined as

$$\delta(s, w; T, p, q) \equiv \frac{\Gamma(T s^{\pm 1} w^{\pm 1}; p, q)}{\Gamma(T^2; p, q)\Gamma(s^{\pm 2}; p, q)} \ . \tag{B.9}$$

In (B.9), $T$ is a function of $p, q, t \in \mathbb{C}$ satisfying

$$\max\left(|p|, |q|\right) < |T|^2 < 1 \ , \tag{B.10}$$

$\Gamma(z; p, q)$ is defined as

$$\Gamma(z; p, q) \equiv \prod_{j,k \geq 0} \frac{1 - z^{-1} p^{j+1} q^{k+1}}{1 - z p^j q^k} \ , \tag{B.11}$$

and $f(s) \equiv f(s, p, q, t)$ is a function that is holomorphic in the annulus

$$\mathbb{A} = \{|T| - \varepsilon < |s| < |T|^{-1} + \varepsilon\} \ , \tag{B.12}$$

for small but finite $\varepsilon > 0$ and also satisfies

$$f(s) = f(s^{-1}) \ . \tag{B.13}$$

The contour $C_w = C_w^{-1}$ lies in the annulus $\mathbb{A}$ with the points $T^{-1}w^{\pm}$ in its interior (and therefore the points $Tw^{\pm}$ in its exterior). If these conditions are all satisfied, then the inversion theorem states that $f$ can be recovered from the contour integral

$$f(s) = \kappa \oint_{\mathbb{T}} \frac{de}{2\pi i e} \delta(e, s; T, p, q) \hat{f}(e) \ . \tag{B.14}$$

As first applied to the index in [81], this inversion theorem is used as follows. First, one finds a representation of the conformal manifold index that is of the form of the RHS of (B.14). In particular, $\hat{f}(e)$ should contain the index of the isolated SCFT (the $E_6$ theory in [81] or the $\mathcal{T}_{3,\frac{3}{2}}$ SCFT in the case at hand) we wish to determine. One then makes an analytic assumption that $\hat{f}(e)$ can be written as in (B.8) for some function $f(s)$ satisfying (B.13) while being analytic in the annulus, $\mathbb{A}$. Then, the inversion theorem implies that $f(s)$ is the index of the conformal manifold. However, in general, one is not guaranteed that the analytic assumption described above holds.[124]

As a result, to apply this theorem in our case, we first need to choose $\hat{f}(e)$ in (B.14) so that (B.14) coincides with (B.7). To that end, using

$$\Gamma(z;p,q) = \text{P.E.}\left[\frac{z - pq/z}{(1-p)(1-q)}\right] , \tag{B.15}$$

and (A.2) one finds that the "delta function" in (B.14) satisfies (for our choice of $T$ discussed below)

$$\delta(e,s;T,p,q) = \frac{(T^2;q)(e^{\pm 2};q)}{(Te^{\pm}s^{\pm};q)}\tilde{\delta}(e;T,p,q) , \tag{B.16}$$

where $\tilde{\delta}(e;T,p,q)$ contains $p$-dependent terms. By comparing (B.7) with (B.14), one can see that if we choose $T=\sqrt{q}$ and

$$\hat{f}(e) = (e^{\pm 2}\sqrt{q};\sqrt{q}) \times (e^{\pm 2}p;p)^{-1} \times \mathcal{I}_{\mathcal{T}_{3,\frac{3}{2}}}(q,e,z_1,z_2)|_{q\to\sqrt{q}} , \tag{B.17}$$

the two expressions coincide.

However, there is an additional wrinkle in our application of the inversion theorem relative to the $E_6$ case in [81]. Indeed, under the analytic assumption described in the paragraph below (B.14), we have

$$\begin{aligned}(w^{\pm 2}\sqrt{q};\sqrt{q}) \times (w^{\pm 2}p;p)^{-1} \quad &\times \quad \mathcal{I}_{\mathcal{T}_{3,\frac{3}{2}}}(q,w,z_1,z_2)|_{q\to\sqrt{q}} = \frac{(q;q)(p;p)}{2} \times \oint_{C_w}\frac{ds}{2\pi i s}\frac{(\frac{1}{q};q)(s^{\pm 2};q)}{(\frac{1}{\sqrt{q}}s^{\pm}w^{\pm};q)}\times \\ &\times \quad \tilde{\delta}(s,w;\frac{1}{\sqrt{q}},p,q) \times \mathcal{I}_{SU(3)}(q,s,z_1,z_2)|_{q\to\sqrt{q}} ,\end{aligned} \tag{B.18}$$

where, as in (B.16), we have separated $\delta$ into a $p$-independent part and a $p$-dependent part, $\tilde{\delta}$. While the $p$-dependence in (B.18) can be cancelled so that

$$\begin{aligned}(w^{\pm 2}\sqrt{q};\sqrt{q}) \quad &\times \quad \mathcal{I}_{\mathcal{T}_{3,\frac{3}{2}}}(q,w,z_1,z_2)|_{q\to\sqrt{q}} = \frac{(q;q)}{2} \times \oint_{C_w}\frac{ds}{2\pi i s}\frac{(\frac{1}{q};q)(s^{\pm 2};q)}{(\frac{1}{\sqrt{q}}s^{\pm}w^{\pm};q)}\times \\ &\times \quad \mathcal{I}_{SU(3)}(q,s,z_1,z_2)|_{q\to\sqrt{q}} ,\end{aligned} \tag{B.19}$$

the condition (B.10) fails for $T=\sqrt{q}$, and $\delta(s,w;\frac{1}{\sqrt{q}};p,q) = 0$ (since $(\frac{1}{q};q) = 0$).

[124] Therefore, the authors of [81] performed many non-trivial consistency checks of this procedure in the $E_6$ case. Our results in the main text can be viewed as highly non-trivial consistency checks of this procedure for the $\mathcal{T}_{3,\frac{3}{2}}$ SCFT.

Therefore, the RHS of (B.19) vanishes.[125]

To get a more sensible answer, we can consider taking $T = \sqrt{q}(1+\varepsilon')$ for $\varepsilon' \ll 1$. In this case, we have

$$\left(\frac{1}{q};q\right) \to \left(\frac{1-2\varepsilon'}{q};q\right) \neq 0 \ , \tag{B.20}$$

and the expression on the RHS of (B.19) is non-vanishing since it becomes

$$\frac{(q;q)}{2} \times \oint_{C_w} \frac{ds}{2\pi i s} \frac{(\frac{1-2\varepsilon'}{q};q)(s^{\pm 2};q)}{(\frac{1-\varepsilon'}{\sqrt{q}}s^{\pm}w^{\pm};q)} \times \mathcal{I}_{SU(3)}(q,s,z_1,z_2)|_{q\to\sqrt{q}} \ . \tag{B.21}$$

In particular, note that the double poles at $s = T^{-1}w^{\pm 1}$ and $s = qT^{-1}w^{\pm 1}$ in (B.19) are resolved into eight single poles in (B.21) with one of each pair still taken to be in the integration contour (for a total of four) and a factor of $\varepsilon'^{-1}$ from the residues that cancels the factor of $\varepsilon'$ arising from (B.20) (all other contributions will be parametrically smaller in $\varepsilon'$). Taking the $\varepsilon' \to 0$ limit then gives us a prescription for computing the Schur index with

$$\begin{aligned}(w^{\pm 2}\sqrt{q};\sqrt{q}) \times \mathcal{I}_{\mathcal{T}_{3,\frac{3}{2}}}(q,w,z_1,z_2)|_{q\to\sqrt{q}} &= \lim_{\varepsilon'\to 0} \frac{(q;q)}{2} \times \oint_{C_w} \frac{ds}{2\pi i s} \frac{(\frac{1-2\varepsilon'}{q};q)(s^{\pm 2};q)}{(\frac{1-\varepsilon'}{\sqrt{q}}s^{\pm}w^{\pm};q)} \times \\ &\times \ \mathcal{I}_{SU(3)}(q,s,z_1,z_2)|_{q\to\sqrt{q}} \ .\end{aligned} \tag{B.22}$$

The contour integration around an infinite number of poles thus reduces to the residues of just four poles whose contribution gives us the simple expression

$$\mathcal{I}_{\mathcal{T}_{3,\frac{3}{2}}}(q,w,z_1,z_2) = \frac{1}{(w^{\pm 2}q;q)}\left[\frac{1}{1-w^2}\mathcal{I}_{SU(3)}(q,wq,z_1,z_2) + \frac{w^2}{w^2-1}\mathcal{I}_{SU(3)}(q,\frac{q}{w},z_1,z_2)\right] \ . \tag{B.23}$$

We can justify the above discussion a posteriori by noting that the non-trivial checks in the main text strongly suggest that (B.22) is a consistent prescription. While a similar procedure works for the Schur index of the $E_6$ SCFT discussed in [81], our case at hand is somewhat more special. Indeed, we used the fact that the $(A_1, D_4)$ SCFT has a Schur index whose $s$ dependence (after taking $q \to \sqrt{q}$) in (B.7) is the same as for $\delta(e, s; \sqrt{q}, p, q)$. On the other hand, when we take $T \to \sqrt{q}(1+\varepsilon')$, we do not necessarily expect that the $(A_1, D_4)$ SCFT has a limit of the index whose $s$ dependence matches the $s$ dependence in $\delta(e, s; \sqrt{q}(1+\varepsilon'), p, q)$ to all orders in $\varepsilon'$. However, the $\mathcal{O}(\epsilon')$ resolution of the double poles into single poles described above should correspond to a shift in the fugacities of the index so that previously degenerate contributions from sets of operators are no longer degenerate (this statement is quite natural since generic single letter contributions to the index will be shifted at $\mathcal{O}(\epsilon')$ if we identify $T$ with

[125] A similar situation occurs in the $E_6$ example of [81] if one first takes the Schur limit and then performs the integration.

a fugacity) and that higher-order differences with respect to $\delta(e, s; \sqrt{q}(1 + \varepsilon'), p, q)$ do not affect the validity of our computation in the limit of small $\varepsilon'$.

# Appendix C

# $q \to 1$ and $S^3$ partition function

The superconformal index can alternatively be viewed as a partition function on $S^3 \times S^1$. Moreover, the fugacity $q = e^{-\beta}$ introduced in the main text controls the relative radii of the $S^3$ and $S^1$ factors. In particular, in the $\beta \to 0$ limit, the $S^1$ factor shrinks relative to the $S^3$ factor and, up to divergent terms, we expect the index to reduce to the $S^3$ partition function, $Z_{S^3}$.

In the limit of $\beta \to 0$, our expression for the $\mathcal{T}_X$ index in (3.29) can be described by the rules in [88]. In particular, the sum over $\lambda$ is replaced by an integral on $m$, where

$$\lambda = -\frac{2\pi m}{\beta} \ , \tag{C.1}$$

and the group fugacities are $w = e^{-i\beta\zeta}$, $z_i = e^{-i\beta\zeta_i}$. We drop group fugacity independent factors in (3.29) and only work to leading order in $\beta$. The $\beta \to 0$ limit of the remaining quantities are given by the following dictionary [88]

$$\begin{aligned}
\text{P.E.}[-2q^{\lambda+1}] \quad &\to \quad (1 - e^{2\pi m})^2 \ , \\
\dim_q R_\lambda^{SU(2)} \quad &\to \quad \sinh(\pi m) \ , \\
\dim_q R_{\lambda,\lambda}^{SU(3)} \quad &\to \quad \sinh(2\pi m) \sinh^2(\pi m) \ , \\
\text{P.E.}\left[\frac{q}{1-q}\chi_{adj}\right] \quad &\to \quad \prod_{j<k} \frac{(\zeta_j - \zeta_k)}{\sinh \pi(\zeta_j - \zeta_k)} \ ,
\end{aligned}$$

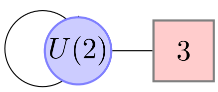



**Figure 18:** The quiver diagram describing the $S^1$ reduction of the $\mathcal{T}_{3,\frac{3}{2}}$ theory (it is mirror to the mirror in Fig. 4). The closed loop beginning and ending at the $U(2)$ node denotes an adjoint hypermultiplet of $U(2)$.

$$\begin{aligned} \chi^{SU(2)}_{R_\lambda}(w) &\to \frac{\sin(2\pi m\zeta)}{\zeta} , \\ \chi^{SU(3)}_{R_{\lambda,\lambda}}(z_1,z_2,z_3) &\to \frac{\sin\pi m(\zeta_1-\zeta_2)\sin\pi m(2\zeta_1+\zeta_2)\sin\pi m(2\zeta_2+\zeta_1)}{(\zeta_1-\zeta_2)(2\zeta_1+\zeta_2)(2\zeta_2+\zeta_1)} . \end{aligned} \tag{C.2}$$

Using (C.2) and replacing the sum over $\lambda$ with an integral over $m$, the $\beta \to 0$ limit of (3.29) becomes

$$\int_{-\infty}^{\infty}\frac{dm}{\sinh 2\pi m\sinh\pi m}\frac{\sin\pi m(\zeta_1-\zeta_2)\sin\pi m(2\zeta_1+\zeta_2)\sin\pi m(2\zeta_2+\zeta_1)}{\sinh\pi(\zeta_1-\zeta_2)\sinh\pi(2\zeta_1+\zeta_2)\sinh\pi(2\zeta_2+\zeta_1)}\frac{\sin 2\pi m\zeta}{\sinh 2\pi\zeta} . \tag{C.3}$$

One can integrate (C.3) by turning it into a contour integral and using the residue theorem. The result is the following

$$\begin{aligned} &\frac{1}{32}\operatorname{sech}\pi\zeta\,(2\operatorname{csch}\pi(\zeta_1-\zeta_2)\operatorname{csch}\pi(\zeta_1+2\zeta_2)\operatorname{sech}\pi(\zeta-2\zeta_1-\zeta_2)\operatorname{sech}\pi(\zeta+2\zeta_1+\zeta_2) \\ &\quad -\operatorname{csch}\pi(2\zeta_1+\zeta_2)\operatorname{csch}\pi(\zeta_1+2\zeta_2)\operatorname{sech}\pi(\zeta+\zeta_1-\zeta_2)\operatorname{sech}\pi(\zeta-\zeta_1+\zeta_2) \\ &\quad -\operatorname{csch}\pi\zeta\operatorname{csch}\pi\,(\zeta_1-\zeta_2)\,\operatorname{csch}\pi\,(\zeta_1+2\zeta_2) \\ &\qquad \times((2\zeta+3\zeta_1+5\zeta_2)\operatorname{sech}\pi\,(\zeta-\zeta_1-2\zeta_2)\,\operatorname{sech}\pi\,(\zeta+\zeta_1-\zeta_2) \\ &\qquad -(4\zeta+3\zeta_1+5\zeta_2)\,\operatorname{sech}\pi\,(\zeta-\zeta_1+\zeta_2)\,\operatorname{sech}\pi\,(\zeta+\zeta_1+2\zeta_2)) \\ &\quad -\frac{1}{2}\operatorname{csch}\pi\zeta\operatorname{csch}\pi\,(2\zeta_1+\zeta_2)\,\operatorname{csch}\pi\,(\zeta_1+2\zeta_2) \\ &\qquad \times((4\zeta+\zeta_1+\zeta_2)\operatorname{sech}\pi\,(\zeta-\zeta_1-2\zeta_2)\operatorname{sech}\pi\,(\zeta-2\zeta_1-\zeta_2) \\ &\qquad -(2\zeta+\zeta_1+\zeta_2)\operatorname{sech}\pi\,(\zeta+2\zeta_1+\zeta_2)\operatorname{sech}\pi\,(\zeta+\zeta_1+2\zeta_2)) \\ &+(\zeta_1\leftrightarrow\zeta_2) . \end{aligned} \tag{C.4}$$

This answer can then be compared with the partition function of the $S^1$ reduction of $\mathcal{T}_X$ or of the mirror theory in Fig. 4. The direct $S^1$ reduction of $\mathcal{T}_{3,\frac{3}{2}}$ is described by an $\mathcal{N}=4$ $U(2)$ gauge theory whose Lagrangian quiver is illustrated in Fig. 18 [160]. Once we decouple the contribution of the $SU(2)$ gauge singlet part of the adjoint hypermultiplet, $\frac{1}{\cosh\pi m'}$, which is the 3D descendant of the decoupled hyper of $\mathcal{T}_{3,\frac{3}{2}}$ we can write down the partition function of the 3D reduction of $\mathcal{T}_X$ [176] [177]

$$\begin{aligned} Z^{\text{quiver}}_{S^3} &= \frac{1}{2}\int_{-\infty}^{\infty}dx_1dx_2\frac{\sinh^2(\pi(x_1-x_2))e^{2\pi i\eta(x_1+x_2)}}{\cosh\pi(x_1-x_2-m')\cosh\pi(x_2-x_1-m')} \\ &\times\frac{1}{\cosh\pi m'\cosh\pi(x_1-m_1)\cosh\pi(x_2-m_1)\cosh\pi(x_1-m_2)} \\ &\times\frac{1}{\cosh\pi(x_2-m_2)\cosh\pi(x_1+m_1+m_2)\cosh\pi(x_2+m_1+m_2)} . \end{aligned} \tag{C.5}$$

This integral can be evaluated similary to (C.3) with the same result (up to an unimportant overall constant and after using the map $\zeta\to m'$, $\zeta_i\to m_i$) as in (C.4) (again, a similar statement holds for the partition function of the mirror in Fig. 4, which involves six integrations and for which one should use the fugacity map in (D.4)).

# Appendix D

# The Hall-Littlewood index of $\mathcal{T}_X$

In this appendix, we derive the HL index in (3.42). In the language of [22], the HL operators are a subset of the Schur operators, we encountered them earlier in 3.7 when discussing the HL limit of the index to which they contribute.

When a 4D $\mathcal{N}=2$ theory is put on a circle, we can often compute the HL limit of the index from the 3D $\mathcal{N}=4$ Higgs branch Hilbert series provided the compactification is sufficiently well-behaved. Equivalently, mirror symmetry allows us to compute the HL limit of the 4D theory from the Coulomb branch Hilbert series of the mirror theory.

Indeed, we can try to compute $\mathcal{I}_{HL}^{\mathcal{T}_X}$ by first computing $\mathcal{I}_{HL}^{\mathcal{T}_{3,\frac{3}{2}}}$ from the 3D mirror gauge theory that follows from the rules reproduced in Fig. 4 and described in [53].[126] Using the results in [178], we can write this index as follows

$$\mathcal{I}_{HL}^{\mathcal{T}_{3,\frac{3}{2}}}(t) = \frac{1}{(1-t)^3} \sum_{a_1, a_{A,i}, a_{B,i} \in \Gamma^*_{\hat{G}}/\mathcal{W}_{\hat{G}}} \zeta_A^{a_{A,1}+a_{A,2}} \zeta_B^{a_{B,1}+a_{B,2}} \zeta_C^{a_{C,1}+a_{C,2}} \cdot P(a_{A,i}, a_{B,i}, a_{C,i}) \cdot t^{\Delta} \,, \tag{D.1}$$

where the arguments of $P$ denote integral GNO flux (restricted to a Weyl chamber of the weight lattice of the GNO dual gauge group as described in [178]), $\zeta_{A,B,C}$ are fugacities for the $U(1)^3$ topological symmetry, $\Delta$ is a monopole scaling dimension for operators charged under the GNO flux, and

$$\begin{aligned}
P(a_{A,1}=a_{A,1}, a_{B,1}=a_{B,2}, a_{C,1}=a_{C,2}) &= \frac{1}{(1-t^2)^3} \,, \\
P(a_{A,1}>a_{A,1}, a_{B,1}=a_{B,2}, a_{C,1}=a_{C,2}) &= P(a_{A,1}=a_{A,1}, a_{B,1}>a_{B,2}, a_{C,1}=a_{C,2}) = \\
P(a_{A,1}=a_{A,1}, a_{B,1}=a_{B,2}, a_{C,1}>a_{C,2}) &= \frac{1}{(1-t)(1-t^2)^2} \,, \\
P(a_{A,1}>a_{A,1}, a_{B,1}>a_{B,2}, a_{C,1}=a_{C,2}) &= P(a_{A,1}>a_{A,1}, a_{B,1}=a_{B,2}, a_{C,1}>a_{C,2}) = \\
P(a_{A,1}=a_{A,1}, a_{B,1}>a_{B,2}, a_{C,1}>a_{C,2}) &= \frac{1}{(1-t)^2(1-t^2)} \,,
\end{aligned}$$

[126] Note that we found substantial evidence in favor of this proposed quiver in the main body of the text.

$$P(a_{A,1} > a_{A,1}, a_{B,1} > a_{B,2}, a_{C,1} > a_{C,2}) = \frac{1}{(1-t)^3} . \tag{D.2}$$

The monopole scaling dimension in (D.1) is given by [75]

$$\begin{aligned}\Delta &= \frac{1}{2}\Big(|a_{A,1}| + |a_{A,2}|\Big) + \frac{1}{2}\Big(|a_{A,1} - a_{B,1}| + |a_{A,2} - a_{B,1}| + |a_{A,1} - a_{B,2}| + |a_{A,2} - a_{B,2}| \\ &+ |a_{A,1} - a_{C,1}| + |a_{A,2} - a_{C,1}| + |a_{A,1} - a_{C,2}| + |a_{A,2} - a_{C,2}| + |a_{B,1} - a_{C,1}| \\ &+ |a_{B,2} - a_{C,1}| + |a_{B,1} - a_{C,2}| + |a_{B,2} - a_{C,2}|\Big) - \Big(|a_{A,1} - a_{A,2}| + |a_{B,1} - a_{B,2}| \\ &+ |a_{C,1} - a_{C,2}|\Big) . \end{aligned} \tag{D.3}$$

After identifying fugacities according to

$$\zeta_A = w z_1^{-2} z_2^{-1} , \quad \zeta_B = z_1 z_2^2 , \quad \zeta_C = z_1 z_2^{-1} , \tag{D.4}$$

we can then expand the HL index in $t$ to find

$$\begin{aligned}\mathcal{I}_{HL}^{\mathcal{T}_{3,\frac{3}{2}}}(t) &= 1 + \chi_1 t^{\frac{1}{2}} + (2\chi_2 + \chi_{1,1})t + (\chi_1 + 2\chi_3 + 2\chi_1\chi_{1,1})t^{\frac{3}{2}} + (2 + \chi_2 + 3\chi_4 + \\ &+ 2\chi_{1,1} + 3\chi_2\chi_{1,1} + \chi_{2,2})t^2 + (3\chi_5 + \chi_3(2 + 4\chi_{1,1}) + \chi_1(2 + \chi_{1,1} + \\ &+ \chi_{3,0} + \chi_{0,3} + 2\chi_{2,2}))t^{\frac{5}{2}} + \mathcal{O}(t^3) \end{aligned} \tag{D.5}$$

We immediately see a free hypermultiplet at $\mathcal{O}(t^{\frac{1}{2}})$ as expected from our discussion in the main text. Stripping off this free hypermultiplet, we get the putative HL index of the $\mathcal{T}_X$ theory

$$\begin{aligned}\mathcal{I}_{HL}^{\mathcal{T}_X}(t, w, z_1, z_2) &= 1 + (\chi_2 + \chi_{1,1})t + \chi_1\chi_{1,1}t^{\frac{3}{2}} + (1 + \chi_4 + \chi_{1,1} + \chi_2\chi_{1,1} + \chi_{2,2})t^2 + \\ &+ (\chi_1\chi_{1,1} + \chi_3\chi_{1,1} + \chi_1\chi_{3,0} + \chi_1\chi_{0,3} + \chi_1\chi_{2,2})t^{\frac{5}{2}} + \mathcal{O}(t^3) , \end{aligned} \tag{D.6}$$

described around (3.42).

# Appendix E

# Proof of (5.50)

In this appendix, we will prove (5.50). For ease of reference, we reproduce it below

$$\langle W^i_{D,p-1}\rangle = \frac{1}{2\sin\left(\frac{\pi}{2p}\right)} = \frac{S^D_{0,(p-1)_i}}{S^D_{0,0}} \rightarrow_{2\in\mathbb{Z}_p^\times} \frac{S_{(1,1),(1,(p-1)/2)}}{S_{(1,1),(1,1)}} = \frac{(-1)^{\frac{p+1}{2}}}{2\cos\left(\frac{\pi}{p}\right)} = \langle W_{(1,(p-1)/2)}\rangle\ . \tag{E.1}$$

To obtain these elements we will use the S-transformation properties of the $\widehat{su(2)}_{2(p-1)}$ primaries given by the S-matrix in (5.19) which we reproduce below

$$S_{l_1,l_2} = \frac{1}{\sqrt{p}}\sin\left[\frac{(l_1+1)(l_2+1)\pi}{2p}\right]\ . \tag{E.2}$$

As discussed in (5.20), primaries of the condensed $\tilde{D}_{p+1}$ theory take the following form in terms of primaries of $\widehat{su(2)}_{2(p-1)}$

$$\Phi_{D,\ell} = \Phi_\ell \oplus \Phi_{2(p-1)-\ell}\ ,\quad \ell\in\{0,2,4,\cdots,p-3\}\ ,\quad \Phi^i_{D,p-1} = \Phi^i_{p-1}\ , \tag{E.3}$$

where $i = 1,2$. To calculate elements of the first row of the $\tilde{D}_{p+1}$ S-matrix we need to write the S-transformation of the condensed vacuum in terms of condensed fields using (E.2)

$$\begin{aligned}(S^D\chi_D)_0 &= \sum_{\ell=0,\ell\in\mathbb{Z}_{\text{even}}}^{p-3} S^D_{0,\ell}\chi_{D,\ell} + \sum_{i=1}^{2} S^D_{0,(p-1)_i}\chi_{D,(p-1)_i} = \sum_{\ell=0}^{2(p-1)} (S_{0,\ell}+S_{2(p-1),\ell})\chi_\ell^{\widehat{su(2)}_{2(p-1)}}\\ &= \sum_{\ell=0}^{2(p-1)} \frac{1}{\sqrt{p}}\sin\left(\frac{\ell+1}{2p}\pi\right)\chi_\ell^{\widehat{su(2)}_{2(p-1)}} + \sum_{\ell=0}^{2(p-1)} \frac{1}{\sqrt{p}}\sin\left(\frac{(2p-1)(\ell+1)}{2p}\pi\right)\chi_\ell^{\widehat{su(2)}_{2(p-1)}}\\ &= \sum_{\ell=0}^{p-1} \frac{2}{\sqrt{p}}\sin\left(\frac{2\ell+1}{2p}\pi\right)\chi_{2\ell}^{\widehat{su(2)}_{2(p-1)}}\ .\end{aligned} \tag{E.4}$$

In going to the last equality, we have used the relation $\sin\left(\frac{(2p-1)(\ell+1)}{2p}\pi\right) = (-1)^\ell\sin\left(\frac{\ell+1}{2p}\pi\right)$.

Now, we can solve for the first $(p-1)/2$ entries of the first row of the $S^D$ matrix

$$S^D_{0,\ell} = \frac{2}{\sqrt{p}} \sin\left(\frac{2\ell+1}{2p}\pi\right) \ . \tag{E.5}$$

The last two entries of the first row are also constrained to obey

$$S^D_{0,(p-1)_1} + S^D_{0,(p-1)_2} = \frac{2}{\sqrt{p}} \ , \quad S^D_{0,(p-1)_1} \in \mathbb{R} \ , \tag{E.6}$$

where the reality of these entries is required by the reality of the quantum dimensions. Unitarity of the $S$-matrix requires the first row to have unit norm and so

$$S^D_{0,(p-1)_1} = S^D_{0,(p-1)_2} = \frac{1}{\sqrt{p}} \ . \tag{E.7}$$

In particular, we see that the quantum dimension in the UV theory is indeed

$$\frac{S^D_{0,(p-1)_i}}{S^D_{0,0}} = \frac{1}{2\sin\left(\frac{\pi}{2p}\right)} \ , \tag{E.8}$$

as claimed in (E.1).

Now let us study the quantum dimension of $\phi_{(1,(p-1)/2)}$. This quantity is easily computed from the $(2,p)$ S-matrix

$$S_{(r,s),(\rho,\sigma)} = \frac{2}{\sqrt{p}}(-1)^{s\rho+r\sigma} \sin\left(\frac{\pi p}{2} r\rho\right) \sin\left(\frac{2\pi}{p} s\sigma\right) \ . \tag{E.9}$$

Indeed, we find

$$\frac{S_{(1,1),(1,(p-1)/2)}}{S_{(1,1),(1,1)}} = \frac{(-1)^{\frac{p+1}{2}}}{2\cos\left(\frac{\pi}{p}\right)} \ , \tag{E.10}$$

as claimed in (E.1).

Now we would like to discuss the Galois action that relates the two quantum dimensions. First, we claim that the Galois group acting on the quantum dimensions (and also the $T$ matrices) can be taken to be $G = \mathbb{Z}_p^\times$ (see the main text for a discussion of the reduction to $G$ from the larger groups one finds using the methods of [137] and also from the underlying quantum groups). For the $T$ matrices (defined with the normalization in (5.36)), this statement follows from (5.17) since $\ell \in \mathbb{Z}_{\rm even}$ and so the $\theta_\ell$ are $p^{\rm th}$ roots of unity (a similar statement holds on the $(2,p)$ minimal model side). At the level of the quantum dimensions, it is sufficient to show that $\sin\left(\frac{2\ell+1}{2p}\pi\right)$ can be written in the field $\mathbb{Q}(\xi)$, where $\xi = e^{\frac{2\pi i}{p}}$.

To see this statement is correct, note that since $p$ is odd, we have either $p+2\ell+1 =$

$4n_\ell$ or $p+2\ell+1 = 4n_\ell + 2$ for $n_\ell \in \mathbb{Z}$. In either case, we have

$$\sin\left(\frac{2\ell+1}{2p}\pi\right) = \frac{1}{2}\left(e^{\frac{i\pi}{2}\left(\frac{2\ell+1}{p}-1\right)} + e^{-\frac{i\pi}{2}\left(\frac{2\ell+1}{p}-1\right)}\right) . \tag{E.11}$$

Let us now suppose $p+2\ell+1 = 4n_\ell$. We then have

$$\begin{aligned} \sin\left(\frac{2\ell+1}{2p}\pi\right) &= -\frac{1}{2}\left(e^{\frac{\pi i}{2}\left(\frac{2\ell+1}{p}+1\right)} + e^{-\frac{\pi i}{2}\left(\frac{2\ell+1}{p}+1\right)}\right) = -\frac{1}{2}\left(e^{\frac{2\pi i n_\ell}{p}} + e^{-\frac{2\pi i n_\ell}{p}}\right) \\ &= \frac{(-1)^{\frac{p-1}{2}}}{2}\left(\xi^{n_\ell} + \xi^{-n_\ell}\right) \in \mathbb{Q}(\xi) , \end{aligned} \tag{E.12}$$

as desired. Similarly, for $p+2\ell+1 = 4n_\ell + 2$, we have

$$\sin\left(\frac{2\ell+1}{2p}\pi\right) = \frac{1}{2}\left(e^{\frac{2\pi i(\ell-n_\ell)}{p}} + e^{-\frac{2\pi i(\ell-n_\ell)}{p}}\right) = \frac{(-1)^{\frac{p-1}{2}}}{2}\left(\xi^{n_\ell-\ell} + \xi^{\ell-n_\ell}\right) \in \mathbb{Q}(\xi) , \tag{E.13}$$

which completes our proof of the claim that $G = \mathbb{Z}_p^\times$.

Let us now apply the Galois action $2 \in G$ to the unitary quantum dimension. We have from the previous two equations that

$$\frac{1}{2\sin\frac{\pi}{2p}} = \frac{(-1)^{\frac{p-1}{2}}}{\xi^n + \xi^{-n}} , \quad n = \left\lfloor \frac{p+1}{4} \right\rfloor . \tag{E.14}$$

Now, applying the Galois action yields

$$\frac{1}{2\sin\frac{\pi}{2p}} = \frac{(-1)^{\frac{p-1}{2}}}{\xi^n + \xi^{-n}} \longrightarrow \frac{(-1)^{\frac{p-1}{2}}}{\xi^{2n} + \xi^{-2n}} = \frac{(-1)^{\frac{p-1}{2}}}{2\cos\left(\frac{4\pi n}{p}\right)} = \frac{(-1)^{\frac{p+1}{2}}}{2\cos\left(\frac{\pi}{p}\right)} , \tag{E.15}$$

where in the last equality we used the relation $\cos\left(\frac{4\pi n}{p}\right) = -\cos\left(\pi\frac{4n-p}{p}\right) = -\cos\frac{\pi}{p}$ for $p = 4n \pm 1$. This completes the proof of our assertion in (E.1) / (5.50).

# Bibliography


[1] M. Buican, Z. Laczko, and T. Nishinaka, *$\mathcal{N}$ = 2 S-duality revisited*, *JHEP* **09** (2017) 087, [arXiv:1706.0379].

[2] M. Buican and Z. Laczko, *Nonunitary Lagrangians and unitary non-Lagrangian conformal field theories*, *Phys. Rev. Lett.* **120** (2018), no. 8 081601, [arXiv:1711.0994].

[3] M. Buican and Z. Laczko, *Rationalizing CFTs and Anyonic Imprints on Higgs Branches*, *JHEP* **03** (2019) 025, [arXiv:1901.0759].

[4] M. Buican, Z. Laczko, and T. Nishinaka, *Flowing from 16 to 32 Supercharges*, *JHEP* **10** (2018) 175, [arXiv:1807.0278].

[5] R. Rattazzi, V. S. Rychkov, E. Tonni, and A. Vichi, *Bounding scalar operator dimensions in 4D CFT*, *JHEP* **12** (2008) 031, [arXiv:0807.0004].

[6] I. García-Etxebarria and D. Regalado, $\mathcal{N} = 3$ *four dimensional field theories*, *JHEP* **03** (2016) 083, [arXiv:1512.0643].

[7] O. Aharony and M. Evtikhiev, *On four dimensional N = 3 superconformal theories*, *JHEP* **04** (2016) 040, [arXiv:1512.0352].

[8] Y. Tachikawa, *N=2 supersymmetric dynamics for pedestrians*, *Lecture Notes in Physics* (2015).

[9] P. C. Argyres, M. R. Plesser, and N. Seiberg, *The Moduli space of vacua of N=2 SUSY QCD and duality in N=1 SUSY QCD*, *Nucl. Phys.* **B471** (1996) 159–194, [hep-th/9603042].

[10] N. Seiberg and E. Witten, *Electric-magnetic duality, monopole condensation, and confinement in n=2 supersymmetric yang-mills theory*, *Nuclear Physics B* **426** (Sep, 1994) 19–52.

[11] N. Seiberg and E. Witten, *Monopoles, duality and chiral symmetry breaking in N=2 supersymmetric QCD*, *Nucl. Phys.* **B431** (1994) 484–550, [hep-th/9408099].

[12] M. Bertolini, *Lectures on supersymmetry*, 2019.

[13] R. Dijkgraaf, B. Heidenreich, P. Jefferson, and C. Vafa, *Negative Branes, Supergroups and the Signature of Spacetime*, *JHEP* **02** (2018) 050, [arXiv:1603.0566].

[14] E. Witten, *Solutions of four-dimensional field theories via m-theory*, *Nuclear Physics B* **500** (Sep, 1997) 3–42.

[15] W. Lerche, *Introduction to seiberg-witten theory and its stringy origin*, *Nuclear Physics B - Proceedings Supplements* **55** (May, 1997) 83–117.

[16] Y. Nakayama, *Scale invariance vs conformal invariance*, *Phys. Rept.* **569** (2015) 1–93, [arXiv:1302.0884].

[17] P. C. Argyres and N. Seiberg, *S-duality in N=2 supersymmetric gauge theories*, *JHEP* **12** (2007) 088, [arXiv:0711.0054].

[18] D. Green, Z. Komargodski, N. Seiberg, Y. Tachikawa, and B. Wecht, *Exactly Marginal Deformations and Global Symmetries*, *JHEP* **06** (2010) 106, [arXiv:1005.3546].

[19] M. Buican and T. Nishinaka, *Conformal Manifolds in Four Dimensions and Chiral Algebras*, *J. Phys.* **A49** (2016), no. 46 465401, [arXiv:1603.0088].

[20] M. Buican, T. Nishinaka, and C. Papageorgakis, *Constraints on chiral operators in $\mathcal{N} = 2$ SCFTs*, *JHEP* **12** (2014) 095, [arXiv:1407.2835].

[21] M. Buican, *Minimal Distances Between SCFTs*, *JHEP* **01** (2014) 155, [arXiv:1311.1276].

[22] F. A. Dolan and H. Osborn, *On short and semi-short representations for four-dimensional superconformal symmetry*, *Annals Phys.* **307** (2003) 41–89, [hep-th/0209056].

[23] J. Kinney, J. Maldacena, S. Minwalla, and S. Raju, *An index for 4 dimensional super conformal theories*, *Communications in Mathematical Physics* **275** (Jun, 2007) 209–254.

[24] L. Rastelli and S. S. Razamat, *The supersymmetric index in four dimensions*, *J. Phys.* **A50** (2017), no. 44 443013, [arXiv:1608.0296].

[25] E. Witten, *Constraints on Supersymmetry Breaking*, *Nucl. Phys.* **B202** (1982) 253.

[26] A. Gadde, L. Rastelli, S. S. Razamat, and W. Yan, *Gauge Theories and Macdonald Polynomials*, *Commun. Math. Phys.* **319** (2013) 147–193, [arXiv:1110.3740].

[27] C. Beem, M. Lemos, P. Liendo, W. Peelaers, L. Rastelli, and B. C. van Rees, *Infinite Chiral Symmetry in Four Dimensions*, *Commun. Math. Phys.* **336** (2015), no. 3 1359–1433, [arXiv:1312.5344].

[28] S. Benvenuti, B. Feng, A. Hanany, and Y.-H. He, *Counting BPS Operators in Gauge Theories: Quivers, Syzygies and Plethystics*, *JHEP* **11** (2007) 050, [hep-th/0608050].

[29] A. Manenti, *Differential operators for superconformal correlation functions*, arXiv:1910.1286.

[30] M. Buican and T. Nishinaka, *On the superconformal index of Argyres-Douglas theories*, *J. Phys.* **A49** (2016), no. 1 015401, [arXiv:1505.0588].

[31] D. Xie, W. Yan, and S.-T. Yau, *Chiral algebra of Argyres-Douglas theory from M5 brane*, arXiv:1604.0215.

[32] T. Creutzig, *W-algebras for Argyres-Douglas theories*, arXiv:1701.0592.

[33] J. Song, D. Xie, and W. Yan, *Vertex operator algebras of Argyres-Douglas theories from M5-branes*, arXiv:1706.0160.

[34] C. Beem, C. Meneghelli, and L. Rastelli, *Free Field Realizations from the Higgs Branch*, *JHEP* **09** (2019) 058, [arXiv:1903.0762].

[35] D. Xie and W. Yan, *4d $\mathcal{N} = 2$ SCFTs and lisse W-algebras*, arXiv:1910.0228.

[36] C. Beem, *Flavor symmetries and unitarity bounds in $\mathcal{N} = 2$ SCFTs*, arXiv:1812.0609.

[37] M. Buican and T. Nishinaka, *Argyres–Douglas theories, $S^1$ reductions, and topological symmetries*, *J. Phys.* **A49** (2016), no. 4 045401, [arXiv:1505.0620].

[38] C. Cordova and S.-H. Shao, *Schur Indices, BPS Particles, and Argyres-Douglas Theories*, *JHEP* **01** (2016) 040, [arXiv:1506.0026].

[39] S. Cecotti, J. Song, C. Vafa, and W. Yan, *Superconformal Index, BPS Monodromy and Chiral Algebras*, arXiv:1511.0151.

[40] L. Fredrickson, D. Pei, W. Yan, and K. Ye, *Argyres-Douglas Theories, Chiral Algebras and Wild Hitchin Characters*, arXiv:1701.0878.

[41] J. Song, *Macdonald index and chiral algebra*, *Journal of High Energy Physics* **2017** (Aug, 2017).

[42] D. Xie and W. Yan, *Schur sector of Argyres-Douglas theory and W-algebra*, arXiv:1904.0909.

[43] C. Beem and L. Rastelli, *Vertex operator algebras, Higgs branches, and modular differential equations*, *JHEP* **08** (2018) 114, [arXiv:1707.0767].

[44] S. D. Mathur, S. Mukhi, and A. Sen, *On the classification of rational conformal field theories*, *Physics Letters B* **213** (1988), no. 3 303 – 308.

[45] S. Mukhi and S. Panda, *Fractional-level current algebras and the classification of characters*, *Nuclear Physics B* **338** (1990), no. 1 263 – 282.

[46] A. R. Chandra and S. Mukhi, *Towards a Classification of Two-Character Rational Conformal Field Theories*, arXiv:1810.0947.

[47] D. Gaiotto, L. Rastelli, and S. S. Razamat, *Bootstrapping the superconformal index with surface defects*, *JHEP* **01** (2013) 022, [arXiv:1207.3577].

[48] C. Cordova, D. Gaiotto, and S.-H. Shao, *Surface Defect Indices and 2d-4d BPS States*, *JHEP* **12** (2017) 078, [arXiv:1703.0252].

[49] C. Córdova, D. Gaiotto, and S.-H. Shao, *Surface defects and chiral algebras*, *Journal of High Energy Physics* **2017** (May, 2017).

[50] T. Nishinaka, S. Sasa, and R.-D. Zhu, *On the Correspondence between Surface Operators in Argyres-Douglas Theories and Modules of Chiral Algebra*, arXiv:1811.1177.

[51] L. Bianchi and M. Lemos, *Superconformal surfaces in four dimensions*, arXiv:1911.0508.

[52] P. C. Argyres and M. R. Douglas, *New phenomena in SU(3) supersymmetric gauge theory*, *Nucl. Phys.* **B448** (1995) 93–126, [hep-th/9505062].

[53] D. Xie, *General Argyres-Douglas Theory*, *JHEP* **01** (2013) 100, [arXiv:1204.2270].

[54] P. C. Argyres, M. R. Plesser, N. Seiberg, and E. Witten, *New N=2 superconformal field theories in four-dimensions*, *Nucl. Phys.* **B461** (1996) 71–84, [hep-th/9511154].

[55] T. Eguchi, K. Hori, K. Ito, and S.-K. Yang, *Study of N=2 superconformal field theories in four-dimensions*, *Nucl. Phys.* **B471** (1996) 430–444, [hep-th/9603002].

[56] T. Eguchi and K. Hori, *N=2 superconformal field theories in four-dimensions and A-D-E classification*, in *The mathematical beauty of physics: A memorial volume for Claude Itzykson. Proceedings, Conference, Saclay, France, June 5-7, 1996*, pp. 67–82, 1996. hep-th/9607125.

[57] D. Gaiotto, N. Seiberg, and Y. Tachikawa, *Comments on scaling limits of 4d N=2 theories*, *JHEP* **01** (2011) 078, [arXiv:1011.4568].

[58] S. Cecotti, A. Neitzke, and C. Vafa, *R-Twisting and 4d/2d Correspondences*, arXiv:1006.3435.

[59] K. Maruyoshi and J. Song, *Enhancement of Supersymmetry via Renormalization Group Flow and the Superconformal Index*, *Phys. Rev. Lett.* **118** (2017), no. 15 151602, [arXiv:1606.0563].

[60] K. Maruyoshi and J. Song, *$\mathcal{N} = 1$ deformations and RG flows of $\mathcal{N} = 2$ SCFTs*, *JHEP* **02** (2017) 075, [arXiv:1607.0428].

[61] P. Agarwal, K. Maruyoshi, and J. Song, *$\mathcal{N}$ =1 Deformations and RG flows of $\mathcal{N}$ =2 SCFTs, part II: non-principal deformations*, *JHEP* **12** (2016) 103, [arXiv:1610.0531].

[62] D. Gaiotto, *N=2 dualities*, *JHEP* **08** (2012) 034, [arXiv:0904.2715].

[63] D. Gaiotto, G. W. Moore, and A. Neitzke, *Wall-crossing, Hitchin Systems, and the WKB Approximation*, arXiv:0907.3987.

[64] A. Gadde, E. Pomoni, L. Rastelli, and S. S. Razamat, *S-duality and 2d topological qft*, *Journal of High Energy Physics* **2010** (Mar, 2010).

[65] A. Gadde, L. Rastelli, S. S. Razamat, and W. Yan, *The 4d Superconformal Index from q-deformed 2d Yang-Mills*, *Phys. Rev. Lett.* **106** (2011) 241602, [arXiv:1104.3850].

[66] M. Buican and T. Nishinaka, *On Irregular Singularity Wave Functions and Superconformal Indices*, arXiv:1705.0717.

[67] Y. Wang and D. Xie, *Classification of Argyres-Douglas theories from M5 branes*, *Phys. Rev.* **D94** (2016), no. 6 065012, [arXiv:1509.0084].

[68] Y. Wang and D. Xie, *Codimension-two defects and Argyres-Douglas theories from outer-automorphism twist in 6d* $(2,0)$ *theories*, arXiv:1805.0883.

[69] A. Kapustin, *Solution of $n = 2$ gauge theories via compactification to three dimensions*, *Nuclear Physics B* **534** (Nov, 1998) 531–545.

[70] C. Beem and W. Peelaers, *Argyres-Douglas Theories in Class S Without Irregularity*, arXiv:2005.1228.

[71] C. Beem, C. Meneghelli, W. Peelaers, and L. Rastelli, *VOAs and rank-two instanton SCFTs*, arXiv:1907.0862.

[72] O. Aharony, N. Seiberg, and Y. Tachikawa, *Reading between the lines of four-dimensional gauge theories*, *JHEP* **08** (2013) 115, [arXiv:1305.0318].

[73] J. A. Minahan and D. Nemeschansky, *An N=2 superconformal fixed point with E(6) global symmetry*, *Nucl. Phys.* **B482** (1996) 142–152, [hep-th/9608047].

[74] K. Papadodimas, *Topological Anti-Topological Fusion in Four-Dimensional Superconformal Field Theories*, *JHEP* **08** (2010) 118, [arXiv:0910.4963].

[75] M. Buican, S. Giacomelli, T. Nishinaka, and C. Papageorgakis, *Argyres-Douglas Theories and S-Duality*, *JHEP* **02** (2015) 185, [arXiv:1411.6026].

[76] D. Gaiotto and E. Witten, *S-Duality of Boundary Conditions In N=4 Super Yang-Mills Theory*, *Adv. Theor. Math. Phys.* **13** (2009), no. 3 721–896, [arXiv:0807.3720].

[77] K. A. Intriligator and N. Seiberg, *Mirror symmetry in three-dimensional gauge theories*, *Phys. Lett.* **B387** (1996) 513–519, [hep-th/9607207].

[78] E. Witten, *An SU(2) Anomaly*, *Phys. Lett.* **B117** (1982) 324–328.

[79] V. G. Kac and M. Wakimoto, *A remark on boundary level admissible representations*, *Comptes Rendus Mathematique* **355** (2017), no. 2 128–132.

[80] V. P. Spiridonov and S. O. Warnaar, *Inversions of integral operators and elliptic beta integrals on root systems*, *Advances in Mathematics* **207** (2006), no. 1 91–132.

[81] A. Gadde, L. Rastelli, S. S. Razamat, and W. Yan, *The Superconformal Index of the $E_6$ SCFT*, *JHEP* **08** (2010) 107, [arXiv:1003.4244].

[82] M. Lemos and W. Peelaers, *Chiral Algebras for Trinion Theories*, *JHEP* **02** (2015) 113, [arXiv:1411.3252].

[83] C. Beem, M. Lemos, P. Liendo, L. Rastelli, and B. C. van Rees, *The $\mathcal{N} = 2$ superconformal bootstrap*, *JHEP* **03** (2016) 183, [arXiv:1412.7541].

[84] M. Buican and T. Nishinaka, *Argyres-Douglas Theories, the Macdonald Index, and an RG Inequality*, *JHEP* **02** (2016) 159, [arXiv:1509.0540].

[85] L. Di Pietro and Z. Komargodski, *Cardy formulae for SUSY theories in $d = 4$ and $d = 6$*, *JHEP* **12** (2014) 031, [arXiv:1407.6061].

[86] A. Arabi Ardehali, *High-temperature asymptotics of supersymmetric partition functions*, *JHEP* **07** (2016) 025, [arXiv:1512.0337].

[87] L. Di Pietro and M. Honda, *Cardy Formula for 4d SUSY Theories and Localization*, *JHEP* **04** (2017) 055, [arXiv:1611.0038].

[88] T. Nishioka, Y. Tachikawa, and M. Yamazaki, *3d Partition Function as Overlap of Wavefunctions*, *JHEP* **08** (2011) 003, [arXiv:1105.4390].

[89] D. Xie and P. Zhao, *Central charges and RG flow of strongly-coupled N=2 theory*, *JHEP* **03** (2013) 006, [arXiv:1301.0210].

[90] S. Benvenuti and S. Giacomelli, *Compactification of dualities with decoupled operators and* 3d *mirror symmetry*, arXiv:1706.0222.

[91] P. Liendo, I. Ramirez, and J. Seo, *Stress-tensor OPE in $\mathcal{N} = 2$ superconformal theories*, *JHEP* **02** (2016) 019, [arXiv:1509.0003].

[92] I. A. Ramírez, *Mixed OPEs in $\mathcal{N} = 2$ superconformal theories*, *JHEP* **05** (2016) 043, [arXiv:1602.0726].

[93] P. C. Argyres and J. R. Wittig, *Infinite coupling duals of N=2 gauge theories and new rank 1 superconformal field theories*, *JHEP* **01** (2008) 074, [arXiv:0712.2028].

[94] P. Argyres, M. Lotito, Y. Lu, and M. Martone, *Geometric constraints on the space of N=2 SCFTs III: enhanced Coulomb branches and central charges*, arXiv:1609.0440.

[95] P. Di Francesco, P. Mathieu, and D. Senechal, *Conformal Field Theory*. Graduate Texts in Contemporary Physics. Springer-Verlag, New York, 1997.

[96] L. Rastelli, "Harvard university seminar." November 2014.

[97] D. Anninos, T. Hartman, and A. Strominger, *Higher Spin Realization of the dS/CFT Correspondence*, *Class. Quant. Grav.* **34** (2017), no. 1 015009, [arXiv:1108.5735].

[98] T. Hertog, G. Tartaglino-Mazzucchelli, T. Van Riet, and G. Venken, *Supersymmetric ds/cft*, *Journal of High Energy Physics* **2018** (Feb, 2018).

[99] S. Cecotti and M. Del Zotto, *Infinitely many $\mathcal{N}$ = 2 scft with ade flavor symmetry*, *Journal of High Energy Physics* **2013** (Jan, 2013).

[100] S. Cecotti, M. Del Zotto, and S. Giacomelli, *More on the $\mathcal{N} = 2$ superconformal systems of type d p (g)*, *Journal of High Energy Physics* **2013** (Apr, 2013).

[101] C. Cordova, G. B. De Luca, and A. Tomasiello, *AdS8 Solutions in Type II Supergravity*, arXiv:1811.0698.

[102] C. Beem, L. Rastelli, and B. C. van Rees, *$\mathcal{W}$ symmetry in six dimensions*, *JHEP* **05** (2015) 017, [arXiv:1404.1079].

[103] T. Nishinaka and Y. Tachikawa, *On 4d rank-one $\mathcal{N} = 3$ superconformal field theories*, *JHEP* **09** (2016) 116, [arXiv:1602.0150].

[104] J. Choi and T. Nishinaka, *On the chiral algebra of Argyres-Douglas theories and S-duality*, *JHEP* **04** (2018) 004, [arXiv:1711.0794].

[105] T. Creutzig, *Logarithmic W-algebras and Argyres-Douglas theories at higher rank*, *JHEP* **11** (2018) 188, [arXiv:1809.0172].

[106] F. Bonetti, C. Meneghelli, and L. Rastelli, *VOAs labelled by complex reflection groups and* 4d *SCFTs*, arXiv:1810.0361.

[107] T. Arakawa, *Chiral algebras of class $\mathcal{S}$ and Moore-Tachikawa symplectic varieties*, arXiv:1811.0157.

[108] P. Bouwknegt and K. Schoutens, *W symmetry in conformal field theory*, *Phys. Rept.* **223** (1993) 183–276, [hep-th/9210010].

[109] G. W. Moore and N. Seiberg, *LECTURES ON RCFT*, in *1989 Banff NATO ASI: Physics, Geometry and Topology Banff, Canada, August 14-25, 1989*, pp. 1–129, 1989. [,1(1989)].

[110] P. Etingof, S. Gelaki, D. Nikshych, and V. Ostrik, *Tensor categories*, vol. 205. American Mathematical Soc., 2016.

[111] B. Bakalov and A. A. Kirillov, *Lectures on tensor categories and modular functors*, vol. 21. American Mathematical Soc., 2001.

[112] M. R. Gaberdiel, *Fusion rules and logarithmic representations of a WZW model at fractional level*, *Nucl. Phys.* **B618** (2001) 407–436, [hep-th/0105046].

[113] T. Creutzig and D. Ridout, *Logarithmic Conformal Field Theory: Beyond an Introduction*, *J. Phys.* **A46** (2013) 4006, [arXiv:1303.0847].

[114] D. Ridout and S. Wood, *The Verlinde formula in logarithmic CFT*, *J. Phys. Conf. Ser.* **597** (2015), no. 1 012065, [arXiv:1409.0670].

[115] D. Ridout and S. Wood, *Relaxed singular vectors, Jack symmetric functions and fractional level* $\widehat{\mathfrak{sl}}(2)$ *models*, *Nucl. Phys.* **B894** (2015) 621–664, [arXiv:1501.0731].

[116] D. Gaiotto, A. Kapustin, N. Seiberg, and B. Willett, *Generalized Global Symmetries*, *JHEP* **02** (2015) 172, [arXiv:1412.5148].

[117] E. Rowell, R. Stong, and Z. Wang, *On classification of modular tensor categories*, *Communications in Mathematical Physics* **292** (2009), no. 2 343–389.

[118] N. Seiberg and E. Witten, *Gapped Boundary Phases of Topological Insulators via Weak Coupling*, *PTEP* **2016** (2016), no. 12 12C101, [arXiv:1602.0425].

[119] O. Perse, *Vertex operator algebras associated to certain admissible modules for affine lie algebras of type a*, *Glasnik matematicki* **43** (2008), no. 1 41–57.

[120] M. Dedushenko, S. Gukov, H. Nakajima, D. Pei, and K. Ye, *3d TQFTs from Argyres-Douglas theories*, arXiv:1809.0463.

[121] A. Cappelli, C. Itzykson, and J. B. Zuber, *Modular Invariant Partition Functions in Two-Dimensions*, *Nucl. Phys.* **B280** (1987) 445–465.

[122] A. Cappelli, C. Itzykson, and J. B. Zuber, *The ADE Classification of Minimal and A1(1) Conformal Invariant Theories*, *Commun. Math. Phys.* **113** (1987) 1.

[123] F. Bais and J. Slingerland, *Condensate-induced transitions between topologically ordered phases*, *Physical Review B* **79** (2009), no. 4 045316.

[124] T. Neupert, H. He, C. von Keyserlingk, G. Sierra, and B. A. Bernevig, *Boson condensation in topologically ordered quantum liquids*, *Physical Review B* **93** (2016), no. 11 115103.

[125] P.-S. Hsin, H. T. Lam, and N. Seiberg, *Comments on One-Form Global Symmetries and Their Gauging in 3d and 4d*, arXiv:1812.0471.

[126] C. Beem, W. Peelaers, L. Rastelli, and B. C. van Rees, *Chiral algebras of class S*, *JHEP* **05** (2015) 020, [arXiv:1408.6522].

[127] J. De Boer and J. Goeree, *Markov traces and II(1) factors in conformal field theory*, *Commun. Math. Phys.* **139** (1991) 267–304.

[128] A. Coste and T. Gannon, *Remarks on Galois symmetry in rational conformal field theories*, *Phys. Lett.* **B323** (1994) 316–321.

[129] J. A. Harvey and Y. Wu, *Hecke Relations in Rational Conformal Field Theory*, *JHEP* **09** (2018) 032, [arXiv:1804.0686].

[130] J. Song, *Superconformal indices of generalized Argyres-Douglas theories from 2d TQFT*, *JHEP* **02** (2016) 045, [arXiv:1509.0673].

[131] D. Gepner and E. Witten, *String Theory on Group Manifolds*, *Nucl. Phys.* **B278** (1986) 493–549.

[132] M. Buican and A. Gromov, *Anyonic Chains, Topological Defects, and Conformal Field Theory*, *Commun. Math. Phys.* **356** (2017), no. 3 1017–1056, [arXiv:1701.0280].

[133] E. Witten, *Quantum Field Theory and the Jones Polynomial*, *Commun. Math. Phys.* **121** (1989) 351–399. [,233(1988)].

[134] V. G. Kac and M. Wakimoto, *Modular invariant representations of infinite dimensional Lie algebras and superalgebras*, *Proc. Nat. Acad. Sci.* **85** (1988) 4956–5960.

[135] A. Kitaev, *Anyons in an exactly solved model and beyond*, *Annals of Physics* **321** (2006), no. 1 2–111.

[136] J. de Boer and T. Tjin, *The Relation between quantum W algebras and Lie algebras*, *Commun. Math. Phys.* **160** (1994) 317–332, [hep-th/9302006]. [,317(1993)].

[137] P. Bantay, *The Kernel of the modular representation and the Galois action in RCFT*, *Commun. Math. Phys.* **233** (2003) 423–438, [math/0102149].

[138] Z. Wang, *Topological quantum computation*. No. 112. American Mathematical Soc., 2010.

[139] D. Aasen, E. Lake, and K. Walker, *Fermion condensation and super pivotal categories*, arXiv:1709.0194.

[140] S. Abel, M. Buican, and Z. Komargodski, *Mapping Anomalous Currents in Supersymmetric Dualities*, *Phys. Rev.* **D84** (2011) 045005, [arXiv:1105.2885].

[141] M. Buican, *A Conjectured Bound on Accidental Symmetries*, *Phys. Rev.* **D85** (2012) 025020, [arXiv:1109.3279].

[142] M. Buican, *Non-Perturbative Constraints on Light Sparticles from Properties of the RG Flow*, *JHEP* **10** (2014) 026, [arXiv:1206.3033].

[143] T. C. Collins, D. Xie, and S.-T. Yau, *K stability and stability of chiral ring*, arXiv:1606.0926.

[144] C. Córdova, T. T. Dumitrescu, and K. Intriligator, *Exploring 2-Group Global Symmetries*, arXiv:1802.0479.

[145] L. Balents, M. P. A. Fisher, and C. Nayak, *Nodal Liquid Theory of the Pseudo-Gap Phase of High-T(c) Superconductors*, *Int. J. Mod. Phys.* **B12** (1998) 1033, [cond-mat/9803086].

[146] T. Grover, D. N. Sheng, and A. Vishwanath, *Emergent Space-Time Supersymmetry at the Boundary of a Topological Phase*, *Science* **344** (2014), no. 6181 280–283, [arXiv:1301.7449].

[147] S.-S. Lee, *Emergence of supersymmetry at a critical point of a lattice model*, *Phys. Rev.* **B76** (2007) 075103, [cond-mat/0611658].

[148] O. Aharony, O. Bergman, D. L. Jafferis, and J. Maldacena, *N=6 superconformal Chern-Simons-matter theories, M2-branes and their gravity duals*, *JHEP* **10** (2008) 091, [arXiv:0806.1218].

[149] D. Gaiotto, Z. Komargodski, and J. Wu, *Curious Aspects of Three-Dimensional* $\mathcal{N} = 1$ *SCFTs*, arXiv:1804.0201.

[150] F. Benini and S. Benvenuti, $N = 1$ *QED in 2+1 dimensions: Dualities and enhanced symmetries*, arXiv:1804.0570.

[151] D. Gang and M. Yamazaki, *An appetizer for supersymmetry enhancement*, arXiv:1806.0771.

[152] A. Gadde, S. S. Razamat, and B. Willett, *"Lagrangian" for a Non-Lagrangian Field Theory with* $\mathcal{N} = 2$ *Supersymmetry*, *Phys. Rev. Lett.* **115** (2015), no. 17 171604, [arXiv:1505.0583].

[153] S. Benvenuti and S. Giacomelli, *Lagrangians for generalized Argyres-Douglas theories*, *JHEP* **10** (2017) 106, [arXiv:1707.0511].

[154] S. Giacomelli, *RG flows with supersymmetry enhancement and geometric engineering*, arXiv:1710.0646.

[155] P. Agarwal, A. Sciarappa, and J. Song, $\mathcal{N}$ *=1 Lagrangians for generalized Argyres-Douglas theories*, *JHEP* **10** (2017) 211, [arXiv:1707.0475].

[156] G. Bonelli, K. Maruyoshi, and A. Tanzini, *Wild Quiver Gauge Theories*, *JHEP* **02** (2012) 031, [arXiv:1112.1691].

[157] E. Witten, *Gauge theory and wild ramification*, arXiv:0710.0631.

[158] D. Xie and S.-T. Yau, *Argyres-Douglas matter and N=2 dualities*, arXiv:1701.0112.

[159] D. Xie and K. Ye, *Argyres-Douglas matter and S-duality: Part II*, *JHEP* **03** (2018) 186, [arXiv:1711.0668].

[160] S. Cremonesi, G. Ferlito, A. Hanany, and N. Mekareeya, *Coulomb Branch and The Moduli Space of Instantons*, *JHEP* **12** (2014) 103, [arXiv:1408.6835].

[161] A. Kapustin, B. Willett, and I. Yaakov, *Nonperturbative Tests of Three-Dimensional Dualities*, *JHEP* **10** (2010) 013, [arXiv:1003.5694].

[162] O. Aharony, S. S. Razamat, N. Seiberg, and B. Willett, *3d dualities from 4d dualities*, *JHEP* **07** (2013) 149, [arXiv:1305.3924].

[163] C. Wang and M. Levin, *Braiding statistics of loop excitations in three dimensions*, *Physical Review Letters* **113** (Aug, 2014).

[164] Y. Pan and W. Peelaers, *Chiral algebras, localization and surface defects*, *Journal of High Energy Physics* **2018** (Feb, 2018).

[165] M. Fluder and J. Song, *Four-dimensional lens space index from two-dimensional chiral algebra*, *Journal of High Energy Physics* **2018** (Jul, 2018).

[166] Y. Imamura, *Orbifold schur index and ir formula*, *Progress of Theoretical and Experimental Physics* **2018** (Apr, 2018).

[167] O. A. Castro-Alvaredo, B. Doyon, and F. Ravanini, *Irreversibility of the renormalization group flow in non-unitary quantum field theory*, *Journal of Physics A: Mathematical and Theoretical* **50** (Sep, 2017) 424002.

[168] S. Hellerman and S. Maeda, *On the large r-charge expansion in $\mathcal{N} = 2$ superconformal field theories*, *Journal of High Energy Physics* **2017** (Dec, 2017).

[169] J.-B. Bae, S. Lee, and J. Song, *Modular Constraints on Superconformal Field Theories*, arXiv:1811.0097.

[170] J.-B. Bae, K. Lee, and S. Lee, *Monster Anatomy*, arXiv:1811.1226.

[171] Y. Tachikawa, Y. Wang, and G. Zafrir, *Comments on the twisted punctures of $A_{even}$ class S theory*, arXiv:1804.0914.

[172] P. C. Argyres and M. Martone, *4d $\mathcal{N}$ =2 theories with disconnected gauge groups*, *JHEP* **03** (2017) 145, [arXiv:1611.0860].

[173] I. García-Etxebarria, *New $\mathcal{N} = 4$ theories in four dimensions (talk at strings 2017)*, .

[174] H. Lin, O. Lunin, and J. M. Maldacena, *Bubbling AdS space and 1/2 BPS geometries*, *JHEP* **10** (2004) 025, [hep-th/0409174].

[175] M. Buican, L. Li, and T. Nishinaka, *Peculiar Index Relations, 2D TQFT, and Universality of SUSY Enhancement*, *JHEP* **01** (2020) 187, [arXiv:1907.0157].

[176] N. Hama, K. Hosomichi, and S. Lee, *Notes on SUSY Gauge Theories on Three-Sphere*, *JHEP* **03** (2011) 127, [arXiv:1012.3512].

[177] S. Benvenuti and S. Pasquetti, *3D-partition functions on the sphere: exact evaluation and mirror symmetry*, *JHEP* **05** (2012) 099, [arXiv:1105.2551].

[178] S. Cremonesi, A. Hanany, and A. Zaffaroni, *Monopole operators and Hilbert series of Coulomb branches of* 3d $\mathcal{N} = 4$ *gauge theories*, *JHEP* **01** (2014) 005, [arXiv:1309.2657].