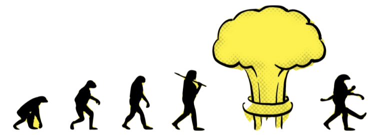

Nuclear Physics and Astrophysics

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These notes are evolving, so please let me know of any typos, factual errors etc. They will be updated weekly on QM+ (and may include updates to early parts we have already covered).

Note that material in purple 'Digression' boxes is not examinable.

Updated 16:29, on 05/12/2019.

Contents

1 Basic Nuclear Properties

A nucleus sits at the heart of an atom, containing almost all of the mass in a fraction – $\sim 10^{-15}$ – of the volume. In terms of the nuclear radius, it is about the same scale as a person is to London.

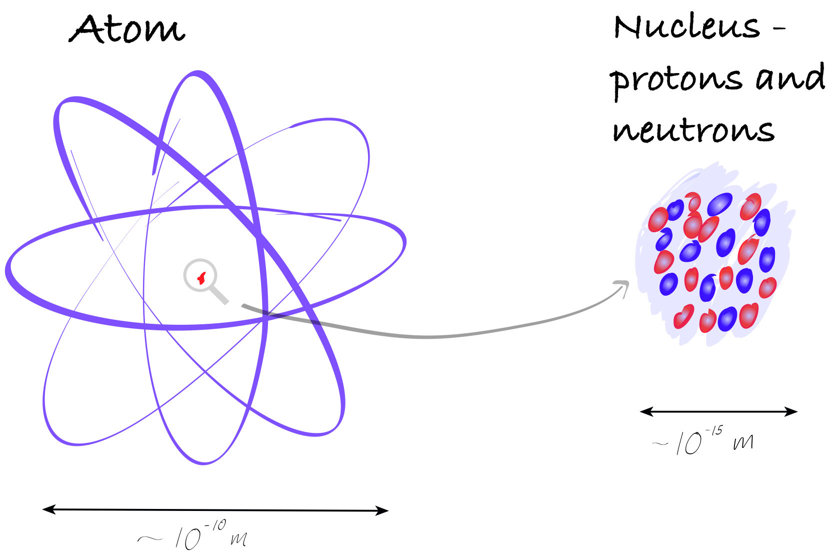


Figure 1: An atom consists of a tiny nucleus surrounded by electron shells. The nucleus itself is made of a collection of neutrons and protons, held together by the **residual strong force**, or **nuclear force**.

The nucleus is made of charged **protons** and neutral **neutrons**. This is held together by the residual **strong force** which overcomes the Coulomb repulsion from the charges protons.

The charge on a proton is $+e$ (where $e = 1.60217662 \times 10^{-19}$ Coulombs is the magnitude of the charge on an electron), so a neutral atom with Z protons must have Z electrons. Consequently, the number of protons determines **chemical** properties of an **atom** (even though chemistry is mainly about the interaction of electrons shared between atoms). This is why the periodic table is arranged by the number of protons an element has (see Fig. 2).

Say we have a nuclide called 'X' with Z protons. This has N neutrons, but with $N \neq Z$ in general. The number Z is called the **atomic number**. The **mass number** is $A = Z + N$, which is the total number of particles in a nuclide X. The nuclide is denoted in full detail as:

$$^{A}_{Z}\mathrm{X}_{N} \, . \tag{1}$$

This notation is actually quite over the top, because we know that $N = A - Z$ we can write:

$$^{A}_{Z}\mathrm{X}_{N} \longrightarrow {}^{A}_{Z}\mathrm{X} \, , \tag{2}$$

because from A and Z we can figure out N. Furthermore, X is *defined* by the number of protons,

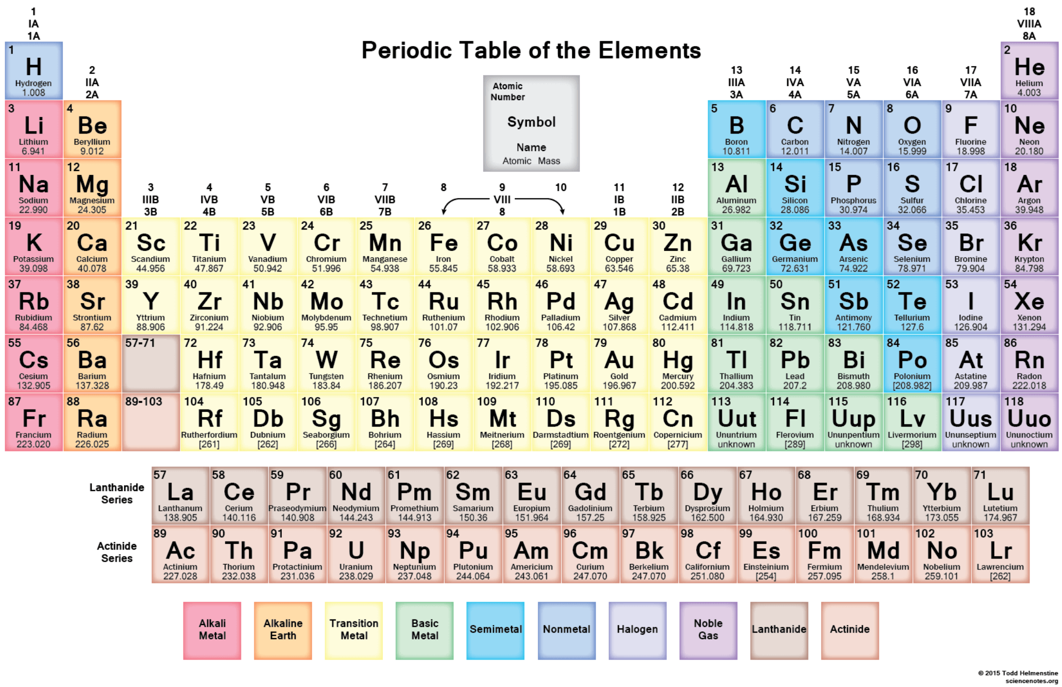


Figure 2: The periodic table orders the elements by atomic number – which is the number of protons in the nucleus. The atomic mass given is close to – but not equal to – the mass number A.

or Z, so we can also write much more compactly:

$$^{A}_{Z}\mathrm{X} \longrightarrow {}^{A}\mathrm{X}\,. \tag{3}$$

From this and the periodic table we can work out N. Typically a variety of notations are used depending on the context, including $_Z\mathrm{X}_N$ where you can read off Z and N directly.

Example:
A nucleus with $Z = 6$ is called carbon, so $^{12}_{6}\mathrm{C}_6$ is just $^{12}\mathrm{C}$, and $^{14}_{6}\mathrm{C}_8$ is just $^{14}\mathrm{C}$. These are examples of **isotopes**. Isotopes have the same amount of protons (Z), but a different about of neutrons (N), and behave the same chemically.

We can also have $^{14}_{7}\mathrm{N}_7$ ($^{14}\mathrm{N}$) and $^{14}_{8}\mathrm{O}_6$ ($^{14}\mathrm{O}$). These are examples of **isotones**. Isotones have the same amount of neutrons (N), but a different number of protons (Z).

Alternatively, say you are given the chemical symbol Te (Tellurium). Look up the periodic table to find it has 52 protons, so can be written $_{52}\mathrm{Te}$. There are a number of different isotopes known – these can be found from a **table of nuclides**.

The family of all (known and possible) nuclides are often visualised on a **table of nuclides**, which is just a Cartesian plot with N and Z making up the axes, and each nuclide is just a point on this plot – see Fig. 3. (We will always use the convention that N forms the horizontal axis – watch out for the opposite convention!) 'Live' versions of this from which you can find nuclear data about each nuclide can be found at

https://www-nds.iaea.org/relnsd/vcharthtml/VChartHTML.html
http://www.nndc.bnl.gov/nudat2/
http://atom.kaeri.re.kr/nuchart/
http://people.physics.anu.edu.au/~ecs103/chart/

Look these up and play around with all the things you can do – each of these sites has essentially the same data about each nuclide (most of which won't make sense yet), but with some neat ways to visualise it.

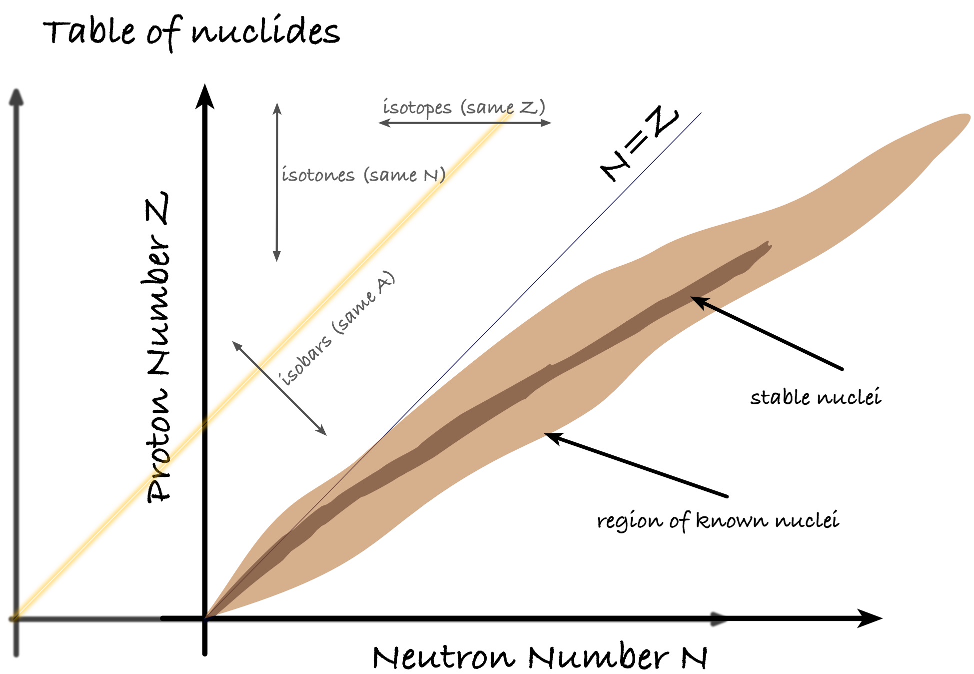


Figure 3: Sketch of the Table of Nuclides, sometimes known as a **Karlsruhe Nuclide Chart**.

The periodic table is sort of a projection onto the vertical axis on figure 3. Moving horizontally traverses different **isotopes** (same number of Z, so the same element), and vertically different **isotones** (same N, different elements). Along lines at the angle shown at 135° are **isobars**, which are nuclides with the same mass number A.

Only some of the nuclides are stable which form a **region of stability** on this diagram. Others tend to be unstable and will **decay**, either towards this region along isobars (β-**decay**), or along it parallel to the line $N = Z$ (α-**decay**) – much more on this later! The range of lifetimes for this to happen can be $\ll 10^{-23}$ s to longer than the age of the universe. Outside of the shaded region, isotopes have not been discovered or made, or cannot be formed.

1.1 Length Scales, Units and Dimensions

We could just use SI units for everything, but given the nucleus is so small and required very high energies to probe, a set of units and conventions has naturally arisen.

Length :

Scales involved are $\sim 10^{-15}$ m which is a **femto-metre**, denoted as 1 fm. It is often called a **Fermi** (after Enrico Fermi).

Area :

1 **barn** is equal to 100 fm^2. This is approximately the cross-section of ^{238}U.

Time Scales :

Nuclei decay on a very large range of times scales, from $\ll 10^{-23}$ s to $>$ Myr. Therefore we just use seconds.

Energy Scales :

For atomic physics a natural unit is the electron-volt (eV), where $1\,\text{eV} = 1.602 \times 10^{-19}$ J. This is a sensible unit of measurement because to ionize H (knock its electron off), it requires 13.6 eV (i.e., it's a number where we don't have huge powers of 10 to carry around).

For nuclides however, $\text{MeV} = 1 \times 10^6$ eV is a more typical value to use. For example, to split $^2_1H_1$ into a proton and a neutron (which you can think of as knocking off a neutron to leave ^1H) will require 2.2 MeV of energy.

Nuclear Mass and Energy Scales :

The mass of a proton is $\sim 1.67 \times 10^{-27}$ kg which is a bit of an inconvenient number, so we set it to near unity using the **unified atomic mass unit** (u). This is defined from carbon:

One **atom** of ^{12}C has mass 12 u exactly.

Note that this includes the electrons of the atom, and is not just the nucleus itself!

In these units, 1 neutron has a mass of 1.008 664 915 88 u and 1 proton has a mass of 1.007 276 466 879 u. We also have that $1\,\text{u} = 1.660\,539\,040 \times 10^{-27}$ kg.

Mass has an energy scale as well using $E = mc^2$, where c is the speed of light. Therefore

$$1\,\text{u} = 931.494\,095\,4\,\text{MeV}/\text{c}^2 \quad \Leftrightarrow \quad c^2 = 931.494\,095\,4\,\text{MeV}/1\,\text{u}\,. \tag{4}$$

Thus, this is approximately the rest mass energy of a nucleon.

Note:
$M(^{12}C) < 6\,\text{m}_\text{p} + 6\,\text{m}_\text{n}$. This is due to a mass defect accounting for **binding energy**.

2 Nuclear Properties and Models

In this section we shall build two very different approximate models of the nucleus: the semi-classical **liquid drop model** described by the **semi-empirical mass formula** which predicts a nuclei mass; and **the shell model**, which is a quantum approximation which tells us about why some models are particularly stable, and predicts the spin and parity of the models. First we need to consider some basic properties of nuclei and the nuclear force.

2.1 Nuclear Radius and Distribution of Nucleons

How should we begin to think about a nucleus? Assuming it's roughly spherical, we can ask: What is the nuclear radius? And, how does the charge and density of the nucleus change with radius? How are protons and neutrons distributed throughout the nucleus?

Something holds a nucleus together, which for sake of argument we shall refer to as the **nuclear force**. The nucleus has a net positive charge of Ze meaning that the protons are also pushing each other apart, so we need to take into account the **Coulomb** as well as the Nuclear forces. Presumably the nucleus has not collapsed to a point so something else is pushing nucleons apart at very short distances. Taking these into account, we can sketch the **nuclear potential** as felt by a proton or a neutron. The potential must look something like Fig. 4.

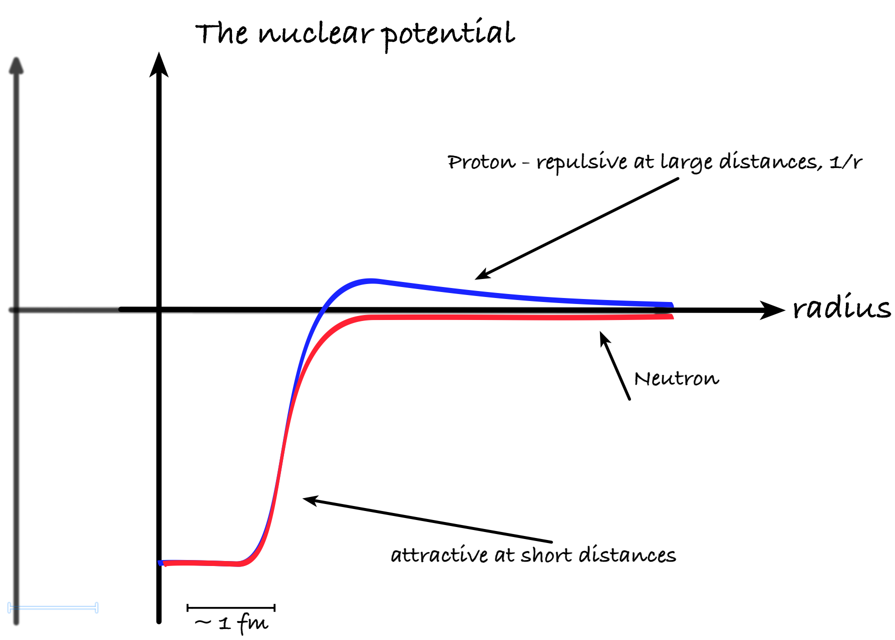


Figure 4: A sketch of the Nuclear Potential. At large distances the potential for protons becomes positive owing to the Coulomb repulsion between the proton (charge $+e$) and the nucleus (charge $+Ze$).

Digression: '$E = mc^2$' versus '$E = pc$'
The proper formula is $E^2 = m^2c^4 + p^2c^2$ or we sometimes see $E = \gamma mc^2$, where:

- m is rest mass
- p is total momentum $= \gamma mv$
- γ is Lorentz factor $= \dfrac{1}{\sqrt{1-\frac{v^2}{c^2}}}$

So,

$$E^2 = \gamma^2 m^2 c^4 = \frac{m^2c^4}{1-\frac{v^2}{c^2}} = m^2c^4\left(\frac{1-\frac{v^2}{c^2}+\frac{v^2}{c^2}}{1-\frac{v^2}{c^2}}\right) = m^2c^4\left(1+\gamma^2\frac{v^2}{c^2}\right) = m^2c^4 + p^2c^2\,. \tag{6}$$

For **low** velocities, $\gamma^2 \simeq 1 + \frac{v^2}{c^2}$,

$$E^2 = m^2c^4\left(1+\frac{v^2}{c^2}+\frac{v^4}{c^4}+...\right) \longrightarrow E = mc^2 + \frac{1}{2}mv^2 + O\left(\frac{v^4}{c^4}\right)\,, \tag{7}$$

where we used the binomial expansion on taking the square root: $(1+x)^n \approx 1+nx$ for $x \ll 1$.
For **relativistic** velocities, $p \gg mc$, use $E \simeq pc$. For e^- above $mc^2 = 0.5\,\text{MeV} \ll 100\,\text{MeV}$.

Note:
Interpreting potential diagrams ...

Potential energy plots are common in physics because they help visualise how a system will behave. There are two things to remember: for a fixed energy the kinetic energy will be the total energy minus the potential energy. So, the kinetic energy will be the potential curve 'upside down'. Also, the force is minus the derivative of the potential. This means you can think of how the system will react by just thinking of a ball rolling on the potential curve itself.

We can probe the shape of the shape and internal structure of the nucleus using **scattering** – basically firing high energy particles at a nucleus (really a distribution of nucleons in some material), and seeing in which directions they fly off. To investigate the charge distribution we use electrons which do not couple (feel) to the nuclear force but will be scattered by the positively charged protons. To find the mass distribution we need to use particles which couple to the nuclear force such as protons, neutrons or alpha particles.

What energy levels would we need to probe nuclear scales?

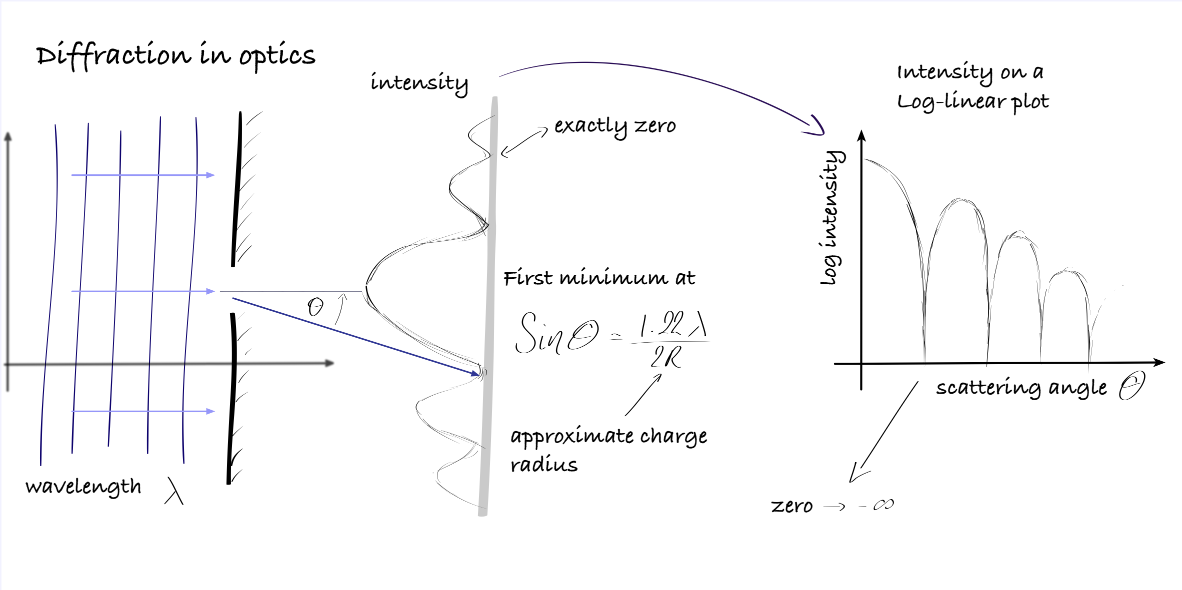
We know that $E = pc$ (relativistic), and a particle of de Broglie wavelength λ has momentum $p = h/\lambda$, so if $\lambda \sim 1\,\text{fm}$ then:

$$E = \frac{hc}{\lambda} \simeq \frac{(4.1\times10^{-15}\,\text{eV s})\times(3\times10^{8}\,\text{m s}^{-1})}{10^{-15}\,\text{m}} \sim 10^9\,\text{eV} \sim 1\,\text{GeV}\,. \tag{5}$$

This also shows that higher energies are needed to probe smaller scales.

We can probe the shape of the nucleus by studying the angular distribution of scattered e^- or other particles such as α-particles. From the scattering we see diffraction effects, but the intensity

Digression: Diffraction in optics.
In optics a common example is to look at light incident on a thin slit (of width $2R$) to see diffraction effects. For a plane wave we have something like this:



The diffraction pattern on the screen has an intensity as a function of scattering angle θ given by $I \sim \mathrm{sinc}^2\left(\frac{2\pi R}{\lambda}\sin\theta\right)$. The fact this this has distinct zeros is characteristic of the sharp edges of the slit – in reality the intensity will be much smaller here but not *exactly* zero owing to the fact the slit is not *exactly* sharp at the edge.
To see this on a graph we often use a log scale on the vertical axis (and plot the absolute value of the function) – in this way we can see very large and very small things on the same plot. Where the function goes *exactly* to zero, this becomes a sharp spike as the function heads off to $-\infty$.

does not go to zero exactly anywhere, see Fig. 5. This shows that the nucleus is not a sharp disk, but is actually a more of a 3D blob.

How does this relate to the charge distribution? We will now investigate the key aspects of this figure, and why it tells us the nucleus is more like a blob than a point or a solid ball.

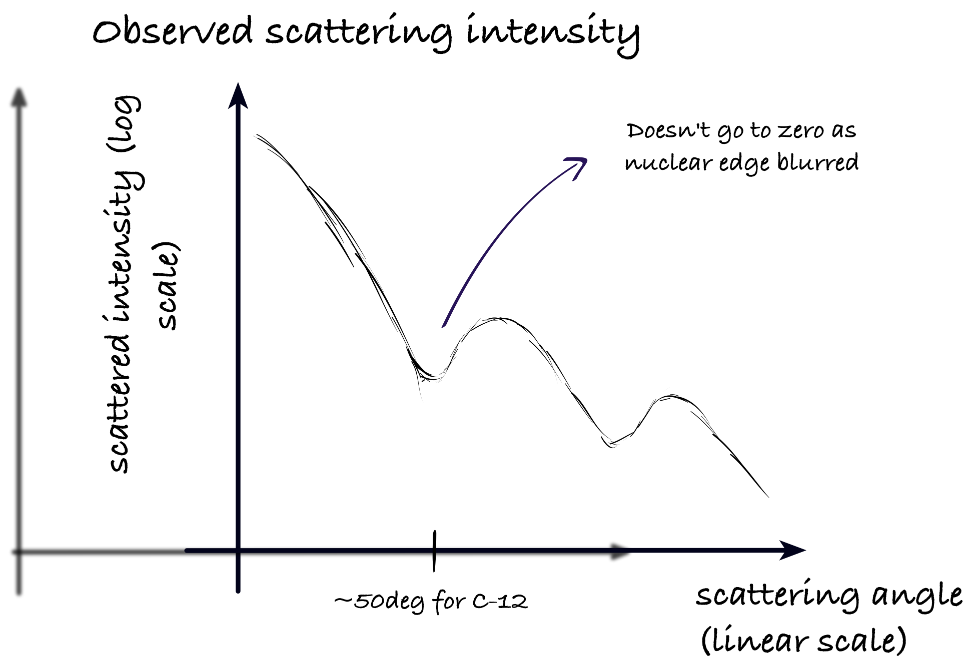


Figure 5: A sketch of the observed scattering intensity for electrons scattering off a nucleus. The first minimum depends on the target nuclei and the energy of the e^-. For example, for ^{12}C this is $\sim 50^\circ$ for e^- at 420 MeV.

2.1.1 Scattering Cross Section

The existence of the nucleus was actually found by Rutherford, who scattered α particles (^{4}He) off gold foil. The fact that some ware scattered by large angles – occasionally right back towards the source – implied the positive charge in the atoms were highly concentrated, unlike the negatively charged electrons. In this simplest approximation this is given by the Rutherford scattering formula, which assumes the nucleus is an immovable point, and the incident particle is non-relativistic, and just taken into account Coulomb scattering. We will look at this case first, and build up from that.

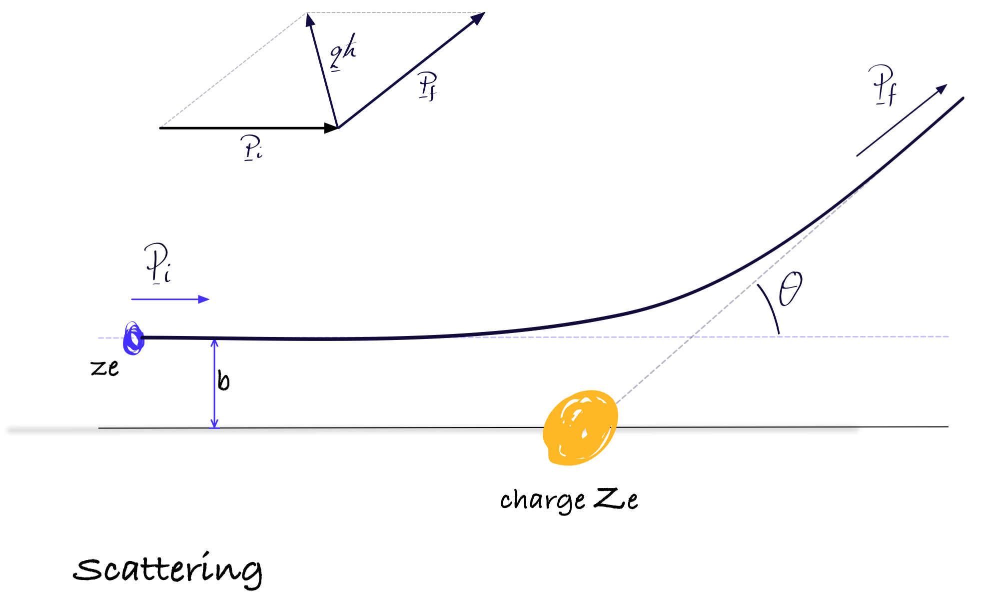


Figure 6: Scattering of a charge ze off a heavy nucleus, charge Ze. We assume azimuthal symmetry about the axis shown.

First though, what quantity do we want to calculate? In optics with a constant light source, we might look at the diffraction pattern on a screen. When we do that, we effectively see a flux of photons – number of photons per unit time – on each part of the screen. Thus we would want to know the number of photons scattered into a solid angle per unit time. For scattering experiments, we use the **differential cross section**, which is defined as the ratio of number of particles scattered into direction (θ, ϕ) per unit time and solid angle. This is related to the reaction rate or reaction probability of an interaction in general and is directly measurable, so for us we're interested in how the measured differential cross section depends on the scattering angle for differing nuclear models.

How does the differential cross section depend on scattering angle? Let us start with Rutherford's formula for a point source:

$$\frac{d\sigma}{d\Omega} = \left(\frac{zZe^2}{16\pi\epsilon_0 E_{kin}}\right)^2 \frac{1}{\sin^4(\frac{\theta}{2})} \,. \tag{8}$$

You can find a derivation in many textbooks, but it is derived from conservation of energy and momentum of the incident particle. This is the scattering fraction for a uniform beam of particles of fixed energy. The kinetic energy for the scattered particle is $E_{kin} = \frac{1}{2}mv_0^2$. From this equation you can see that the higher the energy, the smaller the cross section is. The key feature for us is the angular dependence $\sim 1/\sin^4(\frac{\theta}{2})$. The graph in Fig. 7 shows the shape of $\frac{1}{\sin^4(\frac{\theta}{2})}$.

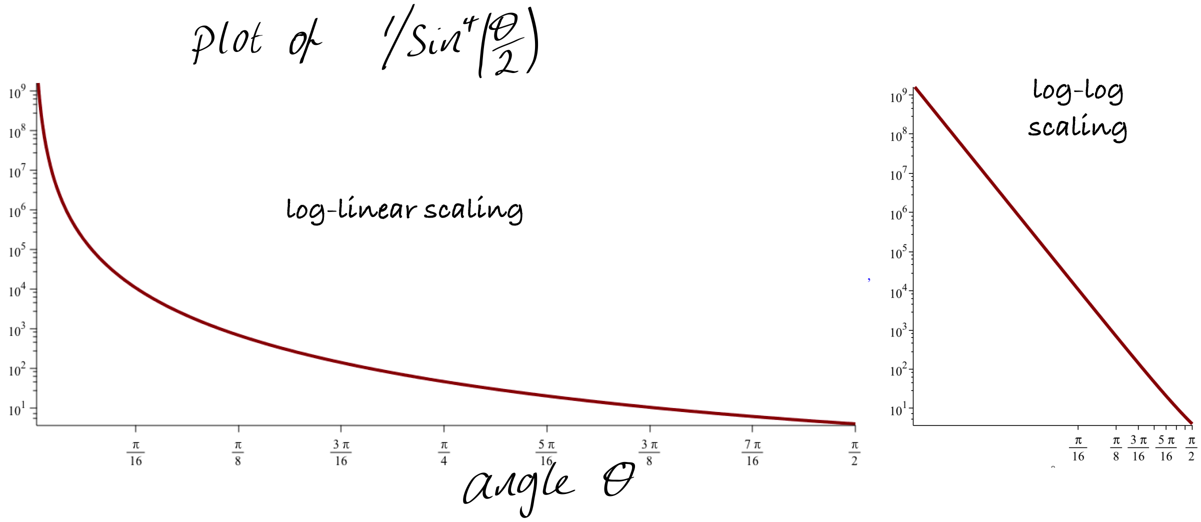


Figure 7: Plots of $1/\sin^4(\frac{\theta}{2})$, shown on 2 different scalings. We often use $\log_{10}$ scaling to reveal how functions behave over many powers of 10 which we can't see on a linear-linear plot.

This is the simplest scattering formula because it assumes a point source, non-relativistic speeds and also a heavy target with zero spin etc. This cross section is key for nuclear probes, but needs modified in a number of ways to make it more realistic.

Firstly we need to account for spin and relativistic speeds, which result in the **Mott scattering formula**

$$\left(\frac{d\sigma}{d\Omega}\right)_{Mott} = \left(\frac{d\sigma}{d\Omega}\right)_{Rutherford}\left[1 - \left(\frac{v}{c}\right)^2 \sin^2\frac{\theta}{2}\right] \,. \tag{9}$$

This is a minor correction as far as we're concerned, giving a slight suppression at large angles for large velocities, which we mention for completeness. There are other corrections to account for the recoil of the nucleus and so on, but we will ignore these.

Most importantly for probing nuclear shapes is to account for the nuclear charge distribution, which we do using a **Form factor** F:

$$\frac{d\sigma}{d\Omega} = \left(\frac{d\sigma}{d\Omega}\right)_{Mott} |F(\underline{q})|^2 \,. \tag{10}$$

Here, $\hbar \underline{q}$ is the momentum change of the scattered particle of wavenumber q. This can be calculated from the conservation of momentum (assuming elastic scattering) and is,

$$q \simeq \frac{2mv}{\hbar} \sin\left(\frac{\theta}{2}\right) . \tag{11}$$

Now, the form factor $F(\underline{q})$ is the **Fourier transform** of the nuclear charge distribution $\rho_e(\underline{r})$,

$$F(\underline{q}) = \int d^3r\, e^{i\underline{q}\cdot\underline{r}} \rho_e(\underline{r}) . \tag{12}$$

For a spherical distribution we have $\rho_e(\underline{r}) = \rho_e(r)$, which means the angular parts of the integral can be done, giving a factor of 4π,

$$F(q) = \frac{4\pi}{q} \int_0^\infty dr\, r \sin(qr)\, \rho_e(r) . \tag{13}$$

Hence, by measuring $\frac{d\sigma}{d\Omega}$ we can get $F(q)$, and then 'invert' this formula to give $\rho_e(r)$. Alternatively, we can propose **models** of $\rho_e(r)$ and fit the calculated F to data, to see which works best.

Example:
A simple model for the nucleus is a hard sphere, like a snooker ball. This would have a constant density inside, falling sharply to zero at radius R:

$$\rho_e(r) = \begin{cases} \rho_e & \text{if } r \le R\,, \\ 0 & \text{if } r > R\,. \end{cases} \tag{14}$$

This gives the form factor, after integration by parts:

$$F(q) = \frac{4\pi\rho_e}{q} \int_0^R dr\, r \sin(qr) = \frac{4\pi\rho_e}{q^3} (\sin(qR) - qR\cos(qR)) \tag{15}$$

Now, since $q \sim \sin\theta/2$, this gives a complicated function of scattering angle. This has zeros at $\tan(qR) - qR = 0$. See Fig. 8.

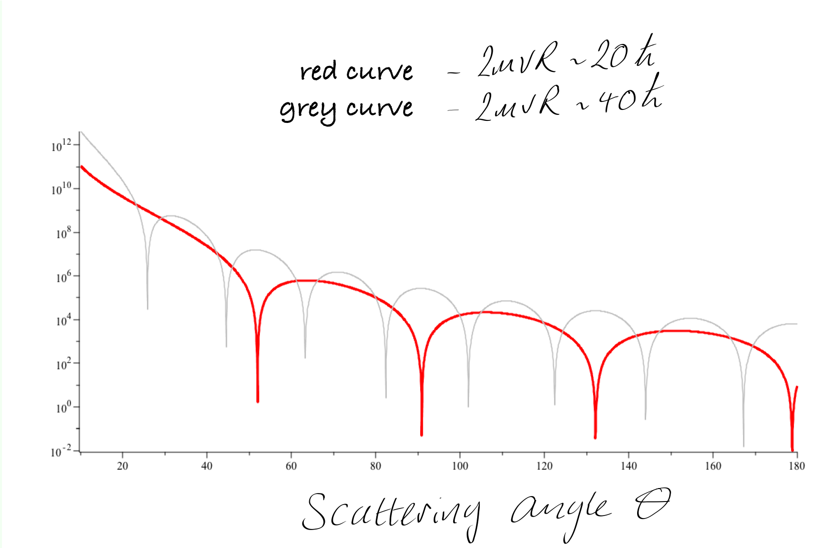


Figure 8: Plots of the differential scattering cross section for a hard sphere in two simple cases. (The overall scaling on the vertical axis is arbitrary.) The spikes correspond to exact zeros of the form factor – clearly unrealistic.

An example is a hard sphere which we can compute exactly (see the Example box). This is now closer to the observed signals, but real signals don't have exact zeroes, just minima which show up as spikes in the log scaling of the plots. This implies that nuclei have **blurred edges**.

A better approximation for $\rho_e(r)$ is the **Wood-Saxon form**,

$$\rho_e(r) = \frac{\rho_0}{1 + e^{(r-R)/a}}, \tag{16}$$

where, R = radius at half density, a is a 'diffuseness parameter' representing the surface thickness of the nucleus, and $\rho_0 \approx$ central charge density. However, the form factor for this case cannot be computed analytically. This will give a differential cross section with its first minimum non-zero, like in Fig. 5.

Since nuclei have an approximately constant density (more later) which is proportional to A, we can say,

$$\frac{A}{\frac{4}{3}\pi R^3} \approx Const \longrightarrow R_0 \propto A^{\frac{1}{3}}. \tag{17}$$

A fit to data gives $R_0 \simeq 1.07A^{\frac{1}{3}}$ fm, together with the other constants in the Wood-Saxon formula, $a \simeq 0.54$ fm and $\rho_0 \sim 0.07\,\mathrm{e\,fm^{-3}}$.

Note:
For a uniform sphere approximation, $R_0 \simeq 1.2A^{\frac{1}{3}}$ fm.

Of course things are more complicated than these three simple phenomenological models (shown in summary in Fig. 10). An example of a real construction is given in Fig. 11.

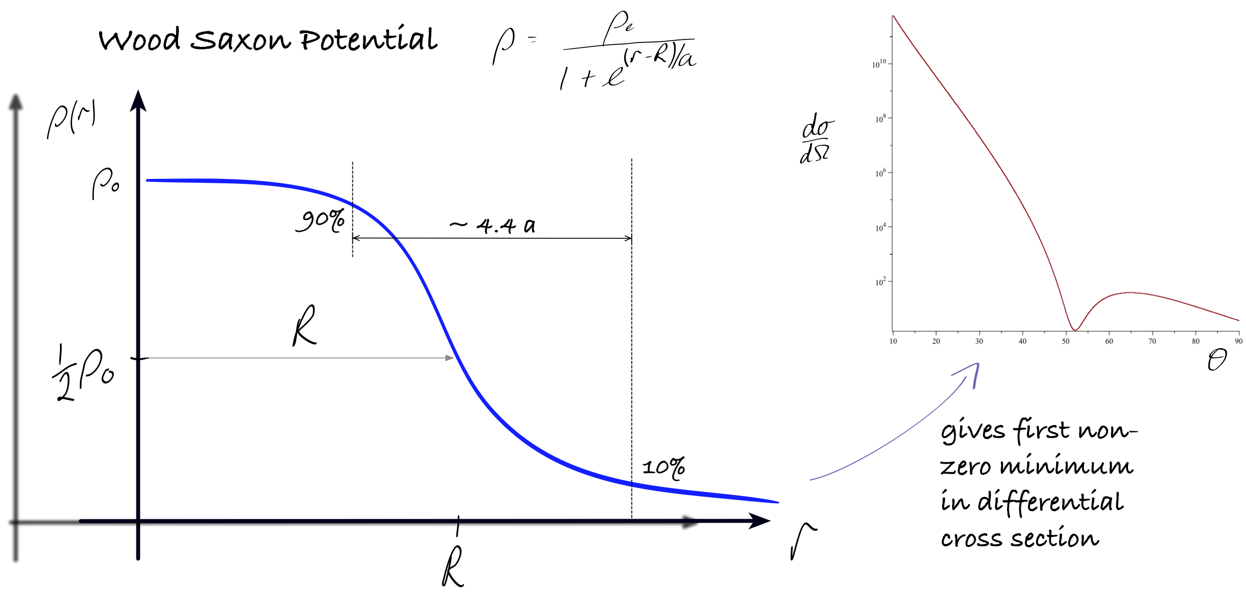


Figure 9: Wood-Saxon form for the charge distribution and an example cross section computed from it.

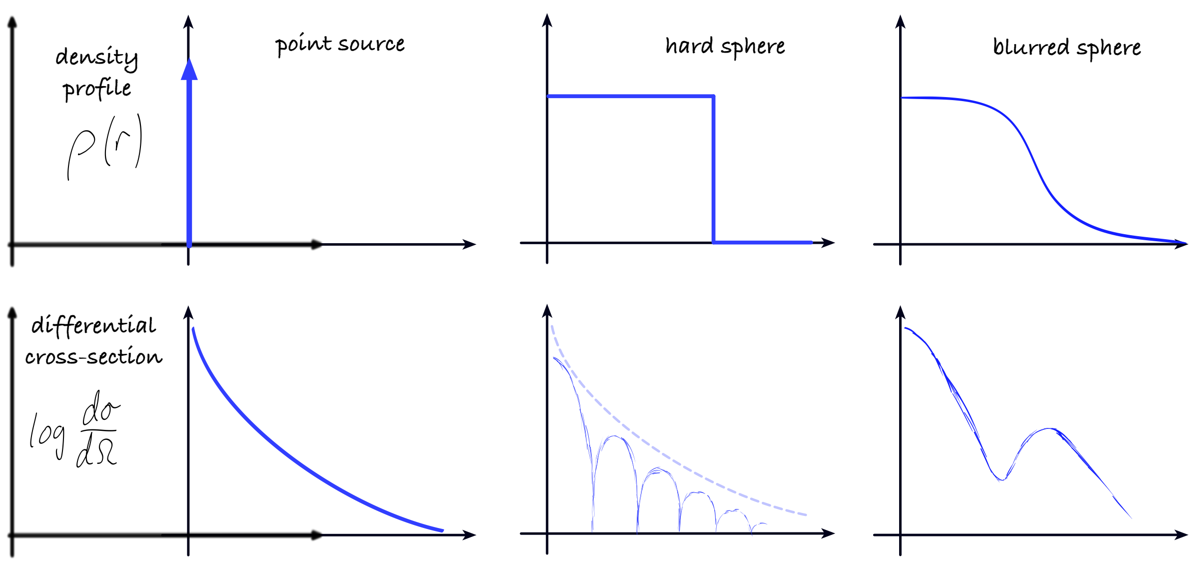


Figure 10: Summary of models of differential cross sections and their models.

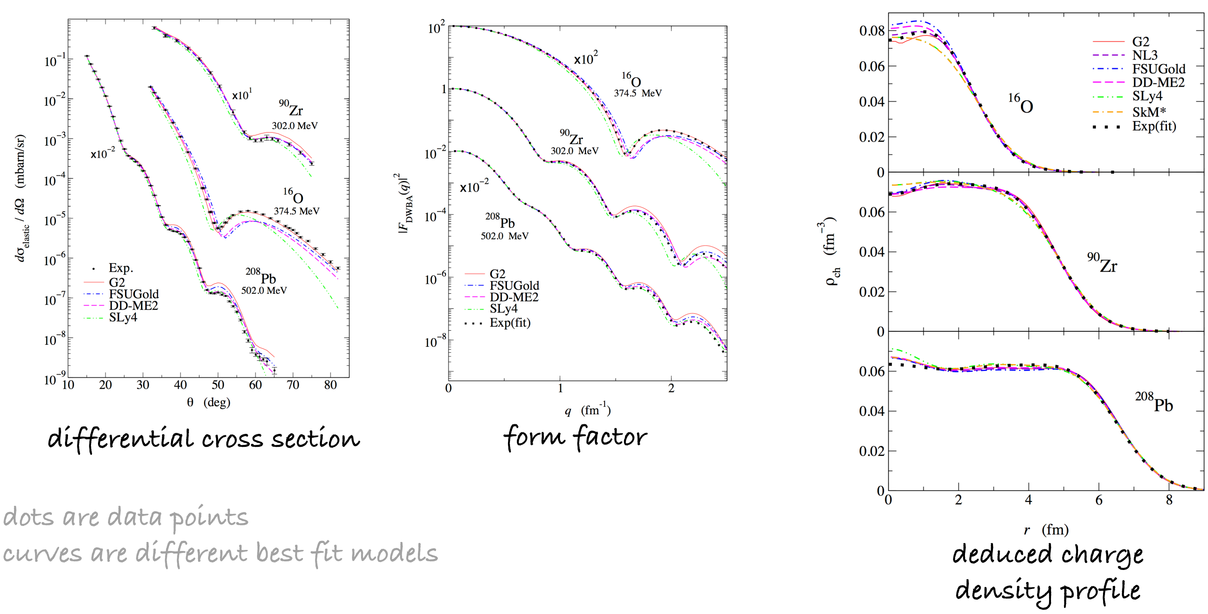


Figure 11: An example of a real charge radius reconstruction from Roca-Maza, X. et. al., arXiv:0808.1252 [Phys.Rev.C78:044332,2008]. The measurements of many scattering experiments result in the figure on the left. The middle figure is the form factor after the ‘simple’ stuff from the Mott formula is removed – this is the Fourier transform of the charge distribution, on the right. (In reality, one makes up a complicated parameterised model for the curves on the right, then performs a best fit to the data on the left to deduce what’s going on.) Note that the Wood-Saxon form gives a reasonable approximation for the differential cross-section of Oxygen, but it looks similar to the other profiles.

2.1.2 Matter Distribution

The matter distribution cannot simply be probed using e^- because they will miss the neutrons in the nuclei. Instead we use protons, neutrons, or light nuclei, which will then probe the distribution of nuclear matter. Although neutrons are difficult to use because they are hard to accelerate to high energies, we can use very high energy α-particles to overcome Coulomb repulsion and probe deep into the structure of the nucleus. For low energies, an α-particle will see a target nucleus as a point source, but once we reach a high enough energy and small impact parameter, the Coulomb repulsion is overcome and a real nuclear interaction will take place (the α-particle may even sometimes be absorbed). In Fig. 12 we see that as the energy is increased the Rutherford formula dramatically breaks down, signifying that the α-particle has penetrated the nuclear radius. We can model the scattering in the way we did before, but in reality the Schrödinger equation is solved using a parameterised potential, which is a sum of Coulomb potentials, an attractive nuclear potential and a spin-orbit potential. This is a bit complicated for this course!

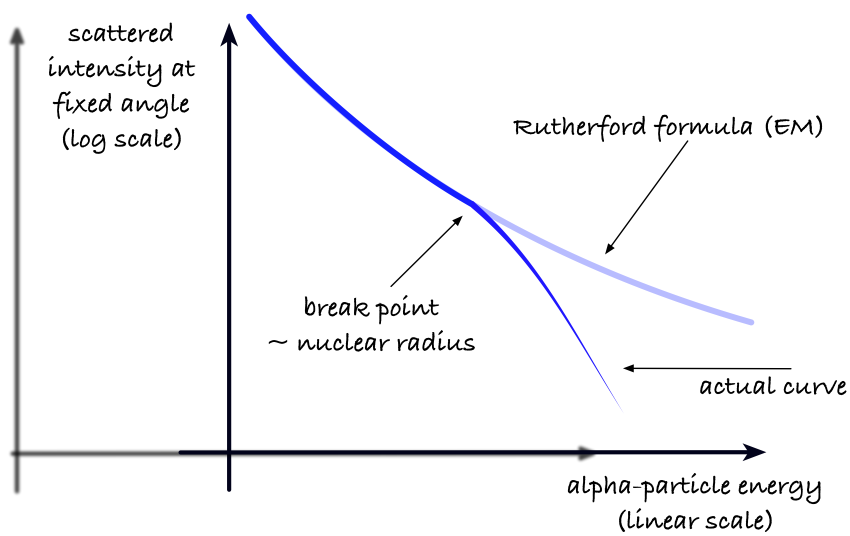


Figure 12: Break down of the Rutherford scattering formula for high energy α-particles.

The key point is that the matter radius is about equal to the charge radius, and the Wood-Saxon form is also roughly appropriate for the nuclear density distribution. Heavy nuclei are neutron rich (see Fig. 13), so protons push outward relative to neutrons, mixing the protons and neutrons internally. In very heavy nuclei a very thin neutron 'skin' can form.

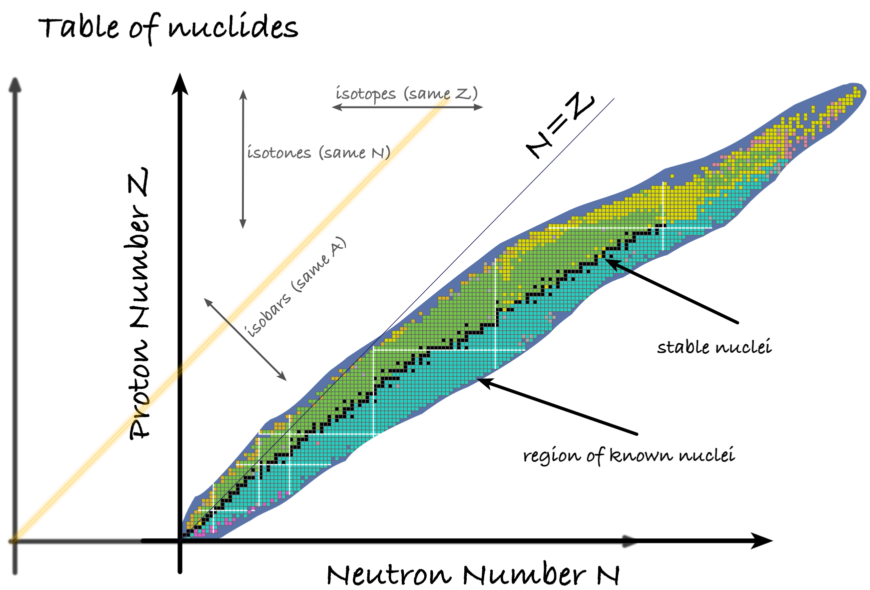


Figure 13: Table of Nuclides. (It you haven't looked one of these up yet do it now!)

2.2 Nuclear Binding Energy

Usually, energy is required to break up a nucleus into neutrons and protons. This energy is called the **nuclear binding energy**. This is similar to electron binding energy holding the electrons to the atom. Similarly, gravitationally bound objects also need energy to separate them (e.g., How much energy is required to remove a person from Earth?). The binding energy will play a really important role in our exploration of nuclear physics and nuclear properties. It is typically a fraction of a percent of the total mass energy of a nucleus, but *really* important.

The atomic mass is of an atom ${}^{A}\mathrm{X}$ is the sum of the masses of the constituent particles, less this nuclear binding energy, $B(Z,A)$:

$$\begin{aligned} m({}^{A}\mathrm{X}) \equiv M(Z,A) &= Z\mathrm{m_p} + N\mathrm{m_n} + Z\mathrm{m_e} - \frac{B}{c^2}\,, \\ &= Z\mathrm{m_H} + N\mathrm{m_n} - \frac{B(Z,A)}{c^2}\,, \end{aligned}$$

where $\mathrm{m_H} = \mathrm{m}({}^{1}\mathrm{H})$ is the mass of a hydrogen atom (a proton and electron). Here we have ignored the binding energy of the electrons in the atom. (Is this ok? Recall the isonisation energy of H is 13.6 eV, while the energy required to separate a deuterium nucleus (a $p+n$ pair) is about 2.2 MeV, so we are probably ok!) We use atomic masses almost always, because nuclear data is historically compiled in this way, and it's obviously very hard to isolate nuclei from an atom (i.e., ionise all the electrons).

Therefore

$$\begin{aligned} B(Z,A) &= (Z\mathrm{m_H} + N\mathrm{m_n} - m({}^{A}\mathrm{X})) \times c^2 \\ &= (Z\mathrm{m_H} + N\mathrm{m_n} - m({}^{A}\mathrm{X})) \times 931.5\,\mathrm{MeV/u} \end{aligned}$$

> ***Note:***
> Sometimes we talk about the **mass defect**
>
> $$\Delta = m({}^{A}\mathrm{X}) - A \times 1\,\mathrm{u} \tag{18}$$
>
> This can either be expressed in u or Δc^2 in MeV.
>
> In addition the **mass deficit** is $\Delta M = -B/c^2$.

The binding energy is approximately proportional to A so we often consider B/A – this is the **binding energy per nucleon**. In Fig. 14 we show a sketch of B/A vs A highlighting the key features, while in Fig. 15 we show the real thing.

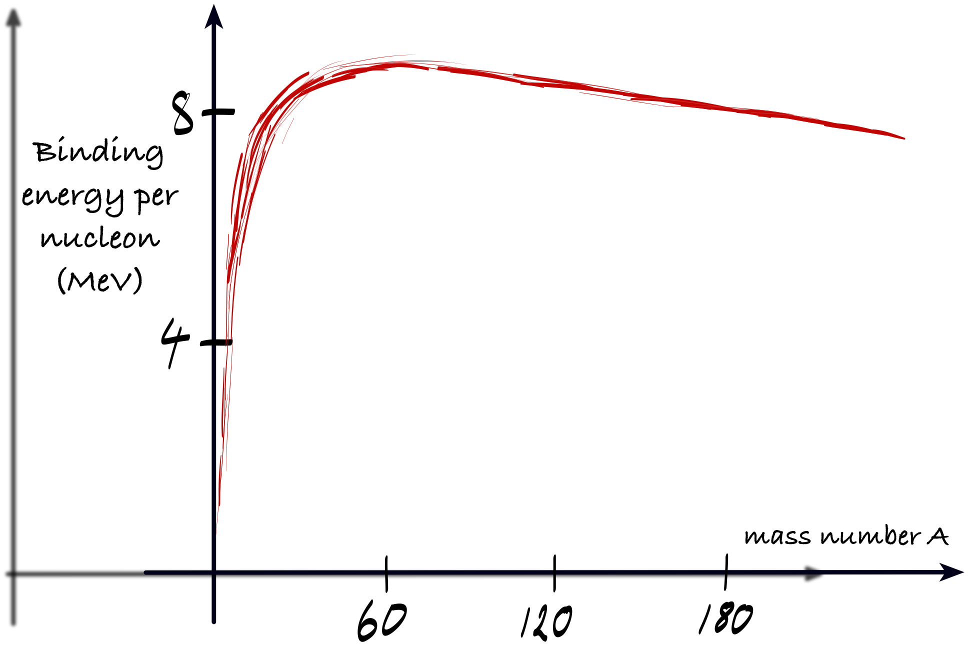


Figure 14: Binding Energy per Nucleon - sketch

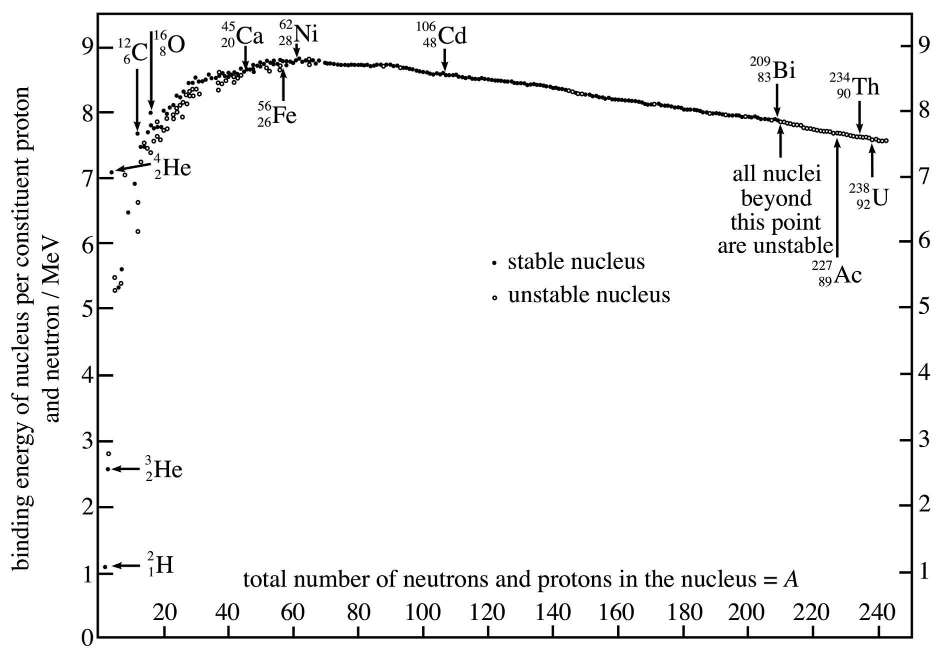


Figure 15: Binding Energy per Nucleon - real data. Note that for a given A we have families of isobars plotted, giving a spread on the vertical axis.

Key Features of the binding energy curve:

- Virtually constant except for few light nuclei $\frac{B}{A} \approx 8\,\text{MeV} \pm 10\%$.
- Higher $\frac{B}{A}$ means that it is **more stable**, i.e. harder to break apart.
- Curve increases for light nuclei: adding nucleons strongly attracts nearby nucleons, making the whole nucleus more tightly bound.
- Curve peaks near $A \approx 60$. The wide stable region from Mg to Xe is because the nucleus is larger than the extent of the nuclear force, so the force saturates, implying that adding more nucleons doesn't increase the binding energy per nucleon.
- Very gradual decay for $A > 100$. This is because the nuclei are so large the Coulomb forces across the nucleus are stronger than the attractive nuclear forces, decreasing the strength of the binding.

  These features above are mainly determined by the competition between the attractive but short range nuclear force, and the long range repulsive Coulomb force.
- Several sharp spikes for ${}^{4}\text{He}$, ${}^{8}\text{Be}$, ${}^{12}\text{C}$, ${}^{16}\text{O}$, ${}^{20}\text{Ne}$, ${}^{24}\text{Mg}$, which have higher $\frac{B}{A}$ than their immediate neighbours. This is a curious quantum effect we will come to later.
- Above $A \approx 120$, $\frac{B}{A}$ decreases. This implies that splitting nuclei can in principle release energy, a process which we call **fission**.

  Example:

  Say we split ${}^{238}\text{U} \to 2 \times {}^{119}\text{Pd}$. From a table of nuclides we can find: $\frac{B}{A}({}^{238}\text{U}) \approx 7.6\,\text{MeV}$ and $\frac{B}{A}({}^{119}\text{Pd}) \approx 8.5\,\text{MeV}$.

  Before:
  $$m_{\text{U}} = Z\text{m}_{\text{H}} + N\text{m}_{\text{n}} - 7.6 \times 238\,\text{MeV}/c^2\,.$$
  After:
  $$2m_{\text{Pd}} = Z\text{m}_{\text{H}} + N\text{m}_{\text{n}} - 2 \times 8.5 \times 119\,\text{MeV}/c^2\,.$$
  Therefore:
  $$\begin{aligned} 2m_{\text{Pd}} &= -2 \times 8.5 \times 119\,\text{MeV}/c^2 + 7.6 \times 238\,\text{MeV}/c^2 + m_{\text{U}} \\ &= m_{\text{U}} - 2033\,\text{MeV}/c^2 + 1809\,\text{MeV}/c^2 \\ &= m_{\text{U}} - 214\,\text{MeV}/c^2 \end{aligned}$$
  This means that when the ${}^{238}\text{U}$ is split, about 214 MeV is released into kinetic energy, plus decay products like n, γ, e^- ...

- Similarly, combining light elements together also releases energy, which we call **fusion**.

Example:
If we look at the **nuclear reaction** $^2\text{H} + {}^2\text{H} \to {}^4\text{He} + \gamma$, where:

$$m(^2\text{H}) = 2.014\,\text{u} = \text{m}_\text{H} + \text{m}_\text{n} - \frac{B_\text{H}}{c^2}\,,$$
$$m(^4\text{He}) = 4.003\,\text{u} = 2\text{m}_\text{H} + 2\text{m}_\text{n} - \frac{B_\text{He}}{c^2}\,.$$

We can then say that the mass before is $2m(^2\text{H}) = 4.028\,\text{u}$ and the mass after is $m(^4\text{He}) = 4.003\,\text{u}$. Therefore a total of $\Delta m = 0.025\,\text{u}$ is released as γ and kinetic energy:

$$\begin{aligned} E = \Delta m c^2 &= 0.025 \times 931.5\,\text{MeV} \\ &= 23.5\,\text{MeV}\,. \end{aligned}$$

(Remember Eq. 4!)

The binding energy per nucleon is related to the separation energy, which is required to remove a proton or a neutron.

Example:

$$^A_Z\text{X}_N \to {}^{A-1}_{\ \ Z}\text{X}_{N-1} + n \tag{19}$$

has a **neutron separation energy:**

$$\begin{aligned} S_n &= \left(m(^{A-1}_{\ \ Z}\text{X}) + \text{m}_\text{n} - m(^A_Z\text{X})\right)c^2 \\ &= B(^A_Z\text{X}) - B(^{A-1}_{\ \ Z}\text{X}) \\ &= B(Z,A) - B(Z,A-1)\,. \end{aligned}$$

Similarly it has a **proton separation energy:**

$$\begin{aligned} S_p &= B(^A_Z\text{X}) - B(^{A-1}_{Z-1}\text{X}) \\ &= B(Z,A) - B(Z-1,A-1)\,. \end{aligned}$$

This is related to $\frac{B}{A}$ in an average sense and is analogous to ionisation energy for atoms. It also shows evidence for a **shell structure** – more on this later.

2.3 The Nuclear Force

The binding energy per nucleon tells us that there are forces between nucleons; there is stability of nuclei; and we also know that that there is energy balance in decays and reactions. But how do we model it?

The nuclear force is the 'residual part' of **strong force**. The strong force holds quarks together to form protons and neutrons, but it is *very* complicated! The nuclear force is similar to forces between neutral atoms, such as Van der Waals or London forces. (These arise from transitory interactions between atoms or molecules where the electrons in one molecule are momentarily 'closer' to the nucleus of a neighbouring atom, thereby attracting them.) In general the interactions between pairs of nn, np, pp is mediated by meson exchange (quark-anti-quark pairs), and in big nuclei it's a very complicated N-body problem. There is still no complete theory, therefore we use simplified models.

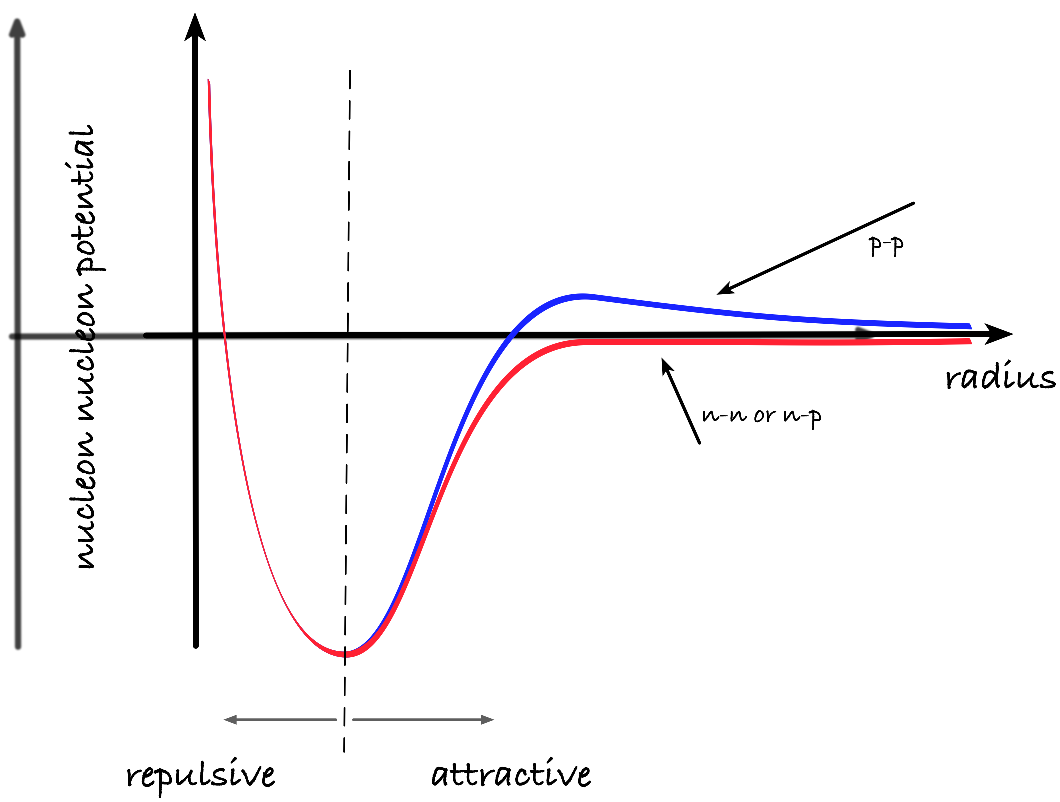


Figure 16: Sketch of the nucleon-nucleon potential.

Key Features of the nuclear force:

- Short range – it's strong only over a few fm.
- Nearly spherical, which means we can model it with a central potential.

Digression: Van der Waals force [from wikipedia https://en.wikipedia.org/wiki/Van_der_Waals_force]
In physical chemistry, the van der Waals forces, named after Dutch scientist Johannes Diderik van der Waals, are distance-dependent interactions between atoms or molecules. Unlike ionic or covalent bonds, these attractions are not a result of any chemical electronic bond, and they are comparatively weak and more susceptible to being perturbed. van der Waals forces quickly vanish at longer distances between interacting molecules.
. . .
Being the weakest of the weak chemical forces, with a strength between 0.4 and 4kJ/mol they may still support an integral structural load when multitudes of such interactions are present. Such a force results from a transient shift in electron density. Specifically, as the electrons are in orbit of the protons and neutrons within an atom the electron density may tend to shift more greatly on a side. Thus, this generates a transient charge to which a nearby atom can be either attracted or repelled. When the interatomic distance of two atoms is greater than 0.6 nm the force is not strong enough to be observed. In the same vein, when the interatomic distance is below 0.4 nm the force becomes repulsive.

- Repulsive over scales $\lesssim 0.5\,\text{fm}$, due to the **Pauli Exclusion Principle**.
- **Charge symmetric** ('$pp = nn$') and almost **charge independent** ('$pp = nn = np$'). (From energy levels of mirror nuclei, e.g. $^{11}_{5}\text{B}$ and $^{11}_{6}\text{C}$)
- **Spin Dependent**, which means that the force between protons and neutron with parallel spins are stronger than anti parallel spins.

Note:
In Fig. 16 we show the nucleon-nucleon potential, which is the potential felt by one nucleon (a neutron or proton) as it's moved in the potential of another. Make sure you understand the difference between this and the Nuclear potential, shown in Fig. 4.

We are going to use these qualitative features to build 2 very different phenomenological models of a nucleus – the liquid drop model and the shell model, which are both relatively simple, and surprisingly accurate. They are used to describe totally different features, and have very different starting assumptions as we will see.

Recap:
The Pauli Exclusion Principle (PEP)

From quantum mechanics we know that 2 identical **fermions** cannot occupy the same quantum state simultaneously. Fermions are particles which have half integer spins such as protons and neutrons (and electrons) – we refer to their 2 possible states as 'spin-up' or 'spin-down' (↑ or ↓). This is distinct from **bosons** which have integer spin (like photons). This means that fermions can't be in the 'same place' with the same spin, so this gives an effective force as fermions of the same spin push apart. The PEP explains the structure of atoms, and strongly affects how a nucleus behaves...

See Quantum Mechanics course for more!

2.4 The Liquid Drop Model and the Semi-Empirical Mass Formula

Can we use properties of nuclear force to build models of a nucleus? Can we predict the mass $M(Z, A)$, or, equivalently $B(Z, A)$? Our first task will be to build a model to predict these quantities.

A nucleus shares properties with a drop of liquid. It roughly has a constant density, and is roughly spherical. It's probably incompressible. A drop of liquid adjusts its shape to minimize its internal energy due to surface tension. The resulting shape is spherical with a constant density. If the liquid is charged then the binding energy has a volume term ($\sim R^3$), a surface term ($\sim R^2$) and a Coulomb term ($\sim Q^2/R$).

By analogy, to a first approximation the binding energy of a nucleus can be described using the same idea. Using $R \sim R_0 A^{1/3}$ this gives:

$$B(Z, A) = a_V A - a_S A^{\frac{2}{3}} - a_C \frac{Z^2}{A^{\frac{1}{3}}} . \quad (20)$$

The **volume term** ($a_v A$) accounts for the constant density by being $\propto A$. This occurs because the nucleons will interact mainly with their nearest neighbours – remember the nuclear force between nucleons is short range and saturates. The **surface term** ($a_s A^{\frac{2}{3}}$) arises from the nucleons near the surface interact with fewer nucleons. This will reduce the binding energy and is therefore subtracted from the volume term, which has over-counted the attraction of the nucleons near the surface. Finally the **Coulomb term** ($a_z \frac{Z^2}{A^{\frac{1}{3}}}$) arises due to the fact that protons with a charge of Ze push apart, and this will reduce the overall binding energy $\propto \frac{Q^2}{R}$.

Strictly the $\frac{Z^2}{A^{\frac{1}{3}}}$ term should really be $\frac{Z(Z-1)}{A^{\frac{1}{3}}}$ due to the fact that protons only interact with other protons, and not themselves. For Z large this difference is negligible.

The constants a_V, a_S and a_C are assumed to be independent of Z and$A$ and can be determined experimentally. (Ideally one would derive formula for from fundamental principles...)

So far this formula is not very accurate (unless $Z \simeq N$ actually) because it misses key **quantum effects**. The Pauli exclusion principle (PEP) states that two identical fermions cannot occupy the same state. Now, at fixed energy level there are only a fixed number of available states. This arrises from the quantum nature of the system (and in your quantum mechanics course you will see how this is derived from the Schroedinger equation in simple cases). This is in contrast to a liquid drop where the energy level is a continuous degree of freedom (for example, excited vibrational modes of a drop can be of any amplitude).

How does this affect a nucleus? Schematically, we can think of each nucleon existing in a potential well from the interactions with all the other nucleons. The details of this well don't matter for now. This potential well gives rise to fixed energy eigenstates within the nucleus. A neutron or proton can 'exist' on one of these energy levels only, and cannot have an energy in between. How many nucleons can live on one level? From the PEP we can count 4: 2n (one spin-up and one spin-down), and 2p (one spin-up and one spin-down). Could a 5th live there two? No, because of the PEP – it would have to be a neutron or proton in a spin up or down state!

This implies an important new quantum feature of nuclei: if there's a differing number of neutrons to protons (an **asymmetry**), it must occupy higher energy levels compared to a nucleus with the same number of neutrons to protons, but with the same overall number – see Fig. 17.

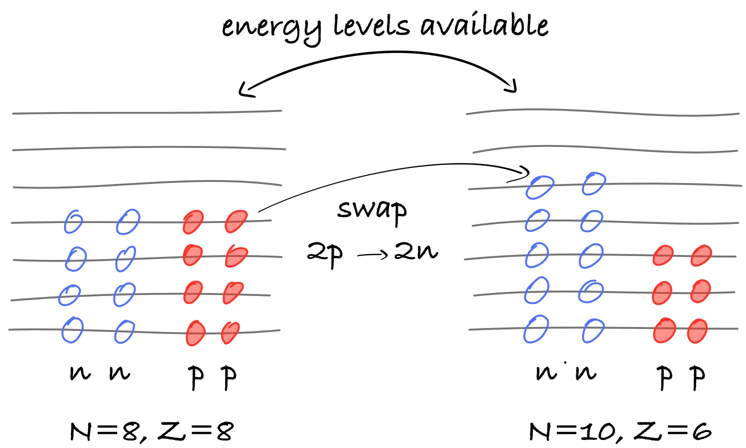


Figure 17: Energy Level Asymmetry. Both nuclei have $A = 16$, but on the right a higher energy state is occupied, by the PEP.

Figure 17 shows that higher energy levels must be occupied. Hence, nuclei with **asymmetric** number of protons and neutrons have **higher** energy than it needs to be. This means that the binding energy is reduced:

$$\text{Asymmetry Term} = -a_A \frac{(N-Z)^2}{A} = -a_A \frac{(A-2Z)^2}{A}\,. \tag{21}$$

We can think of it as $-a_A \times A \times (\text{fraction of neutrons} - \text{fraction of protons})^2$. A justification for the form of this term is given in Fig. 18 which we will now try to understand.

Say we have energy levels evenly spaced by ΔE. Start with a symmetric number of n and p, and sequentially swap $n \to p$. Each swap utilises a higher energy level. Every second swap involves going to a higher level than went before, in the sequence

$$1, 1, 3, 3, 5, 5, 7, 7, \ldots$$

The cumulative effect of this is $1, 2, 5, 8, 13, 18, 25, \ldots \times \Delta E$ for $|N - Z| = 2, 4, 6, 8, 10, 12, 14, \ldots$. Therefore the energy change is $\approx \dfrac{(N-Z)^2 \Delta E}{8}$. (Why? Try it! eg, $N - Z = 10$ implies $100^2/8 \approx 13$ etc. ...) Finally we can estimate

$$\Delta E \propto \frac{1}{\text{Volume of potential well}} \simeq A^{-1}\,.$$

This gives the form of the asymmetry term.

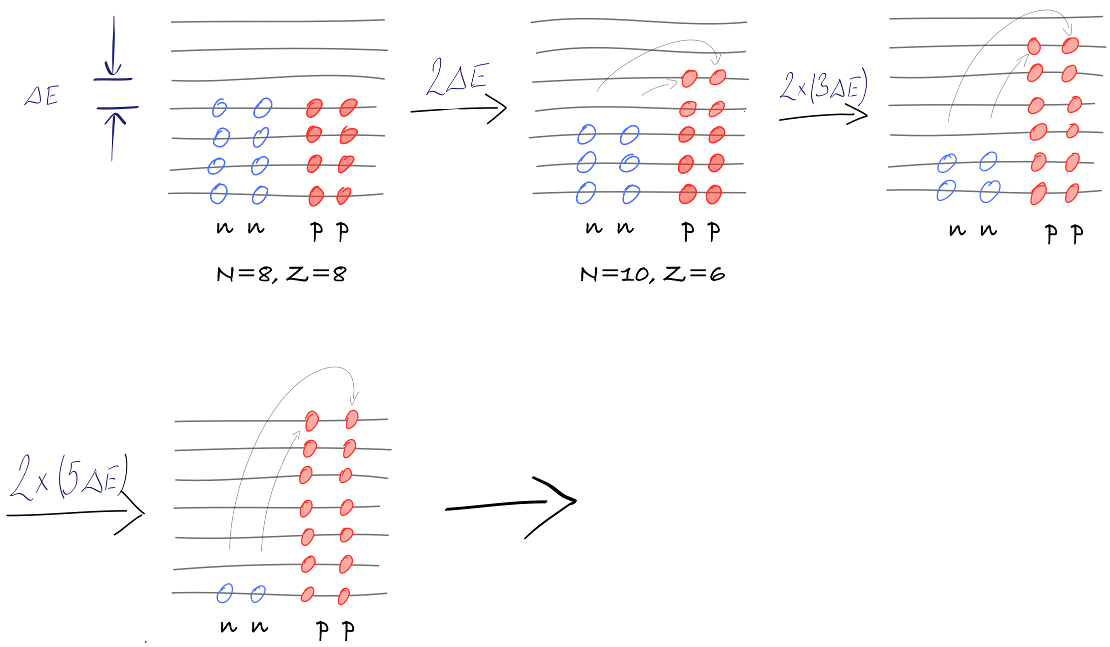


Figure 18: Justification of Asymmetry Term.

Finally we add a **pairing term** : $\delta(Z, A)$. It is found experimentally that proton-proton and neutron-neutron bonds are stronger than neutron-proton bonds (after accounting for Coulomb interactions between $p - p$). This shows that like nucleons 'pair'.

For A **odd**: (Z odd, N even) - 'oe' or (Z even, N odd) - 'eo' : we define $\delta = 0$

For A **even**: (Z odd, N odd) - 'oo' or (Z even, N even) - 'ee' : 'ee' has no neutron-proton pairs, and 'oo' has one. Therefore 'ee' is stronger than 'oo', and the 'ee' case has a slightly stronger binding energy than 'oo'. The physical origin is that pairs with net spin-0 are tightly bound.

So,

$$\delta(Z, A) = \begin{cases} \frac{a_P}{A^{\frac{1}{2}}}, & \text{'ee'.} \\ 0, & \text{'eo' or 'oe'.} \\ -\frac{a_P}{A^{\frac{1}{2}}}, & \text{'oo'.} \end{cases} \tag{22}$$

Note the inverse square-root exponent – this is found from experimental binding energy data. (Older texts sometimes use $1/A^{3/4}$ which gives a different a_P.) This term gives an oscillating feature to the binding energy as a function of A and gives rise to some interesting features.

Note:
if Z is even, then either N is odd and A is odd or N is even and A is even. Whereas if Z is odd, then either N is odd and A is even or N is even and A is odd.

Combining all these contributions, we have the **Semi-Empirical Mass Formula (SEMF)**:

$$B(Z,A) = a_V A - a_S A^{\frac{2}{3}} - a_C \frac{Z(Z-1)}{A^{\frac{1}{3}}} - a_A \frac{(A-2Z)^2}{A} + \delta(Z,A)\,. \tag{23}$$

This is often referred to as the liquid drop model too.

We can find the constants by fitting to the data for $A \gtrsim 20$ (more structure for light nuclei).

Given B, we can now calculate the Nuclear Mass using:

$$M_N(Z,A) = \mathrm{Zm_p} + (A-\mathrm{Z})\mathrm{m_n} - \frac{B(Z,A)}{c^2}\,, \tag{24}$$

or the Atomic Mass using:

$$m(Z,A) = \mathrm{Zm}(^1\mathrm{H}) + (A-\mathrm{Z})\mathrm{m_n} - \frac{B(Z,A)}{c^2}\,. \tag{25}$$

The typical values for the terms in the SEMF are:

$$\begin{aligned} a_V &\simeq 15.8\,\mathrm{MeV} \\ a_S &\simeq 18.3\,\mathrm{MeV} \\ a_C &\simeq 0.714\,\mathrm{MeV} \\ a_A &\simeq 23.2\,\mathrm{MeV} \\ a_P &\simeq 12.0\,\mathrm{MeV} \end{aligned}$$

These values were obtained from Wikipedia but they can vary, owing to the range of A used in the fit, so you will find differing numbers. We will use the values:

$$a_V = 15.56\,\mathrm{MeV},\ a_S = 17.23\,\mathrm{MeV},\ a_C = 0.697\,\mathrm{MeV},\ a_A = 23.28\,\mathrm{MeV},\ a_P = 12.0\,\mathrm{MeV}\,. \tag{26}$$

The contributions of each term are shown in Figure 19. This formula works well for $A \gtrsim 20$, giving B/A to within ~ 0.1 MeV, which means that our assumptions are OK!

Let us use the SEMF to make our first predictions about the nuclear landscape – it should be telling us something about Fig. 13. Can we learn anything about this?

Let us consider $\dfrac{B(Z,A)}{A}$ as a function of Z, but with A held constant – moving to increasing Z moves along isobars in the nuclear landscape. For a **fixed** A, $\dfrac{B(Z,A)}{A}$ is an upside-down parabola.

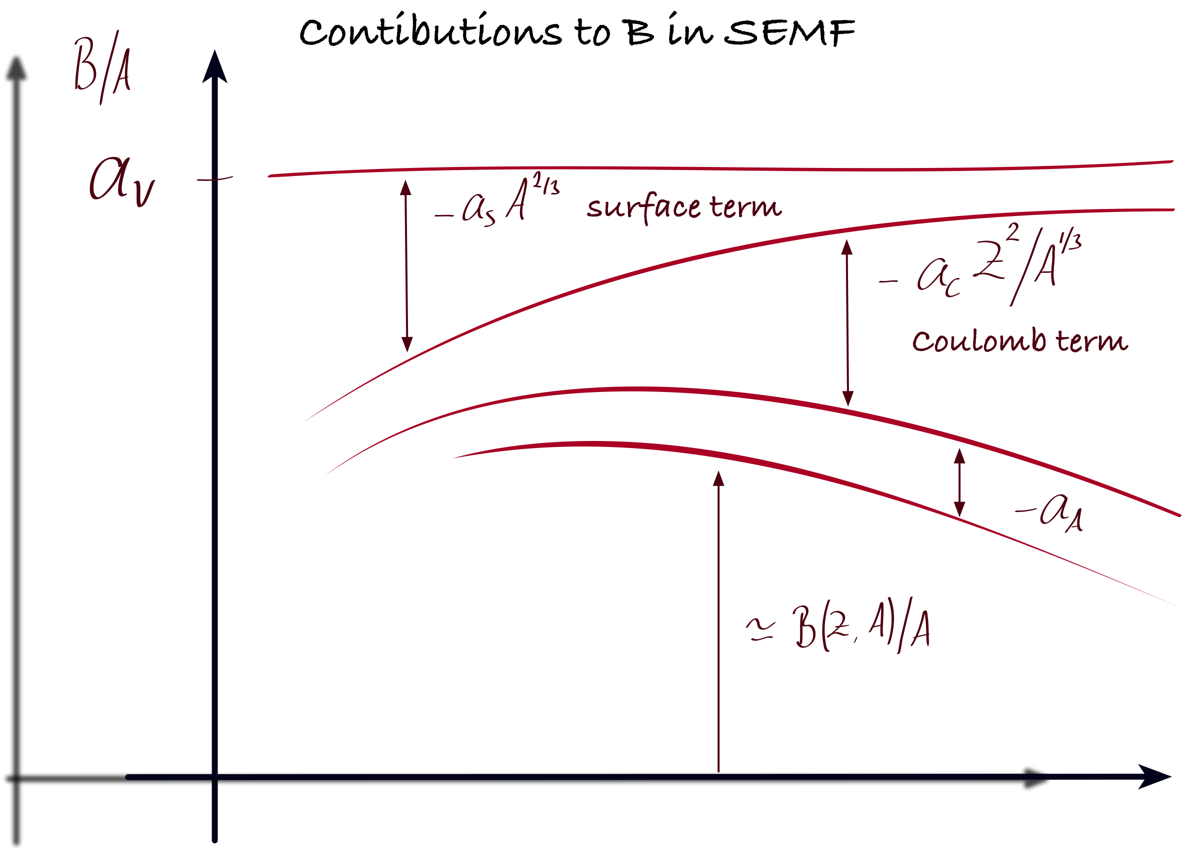


Figure 19: Contributions to the binding energy in the SEMF.

For A **odd** the formula behaves as

$$\frac{B}{A} \sim -(\text{stuff}) \times Z^2 + (\text{other stuff}) \times Z + (\text{more stuff}) \,. \tag{27}$$

The mass isobar corresponding to this is sketched in Figure 20. This implies that each odd-A isobar has a distinct minimum mass per nucleon: this will imply a stronger binding energy and consequently a more stable nucleus. This is why there's a region in the table of nuclides which is most stable – known as the valley of stability because of this parabolic nature in the SEMF. We will see this again when we look at β-decay.

Now, for A **even** it's a bit more complicated. This brings in the alternating δ term as we change Z because we keep swapping from 'ee' to 'oo'. This means that nuclei lie on two separate parabolas, which looks like Figure 21.

Where does the minimum occur? We need to solve $\left.\frac{\partial M}{\partial Z}\right|_{A=\text{Const}} = 0$ which implies

$$Z_{min} \simeq \frac{A}{2} \frac{1}{1 + \frac{1}{4}\frac{a_C}{a_A} A^{\frac{2}{3}}} \tag{28}$$

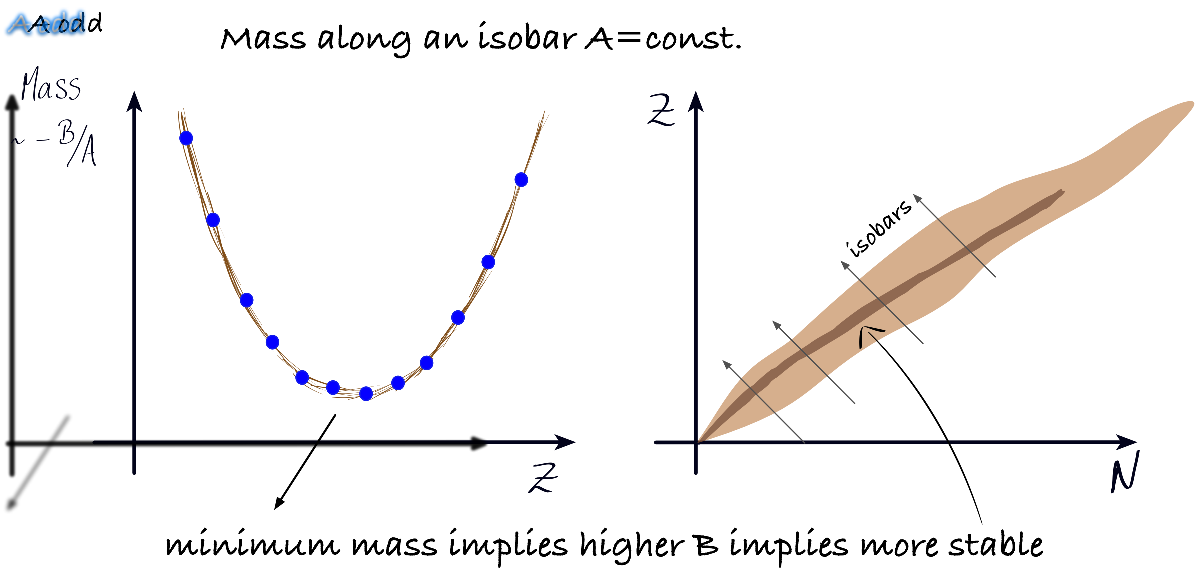


Figure 20: For A = odd we have isobars with a fixed maximum in B/A, with a corresponding minimum mass per nucleon.

This shows that for small A, we tend to find that $Z \simeq \dfrac{A}{2}$ or $N \simeq Z$. For large A we find that $Z_{min} < \frac{A}{2}$. E.g. for $A \sim 200$, $Z \sim 0.4A$. Looking back to Fig. 13, we see that stable nuclei do tend to lie below the $N = Z$ line and are consequently neutron rich. This is really because the Coulomb force for large nuclei becomes much more important because it extends over the whole nucleus, in contrast to the nuclear force which acts over a much shorter range. Therefore for a fixed A, a nucleus with more neutrons is more stable than with more protons (up to a point!).

We show pictorially some SEMF predictions in Fig. 22 – Can you reproduce these plots?

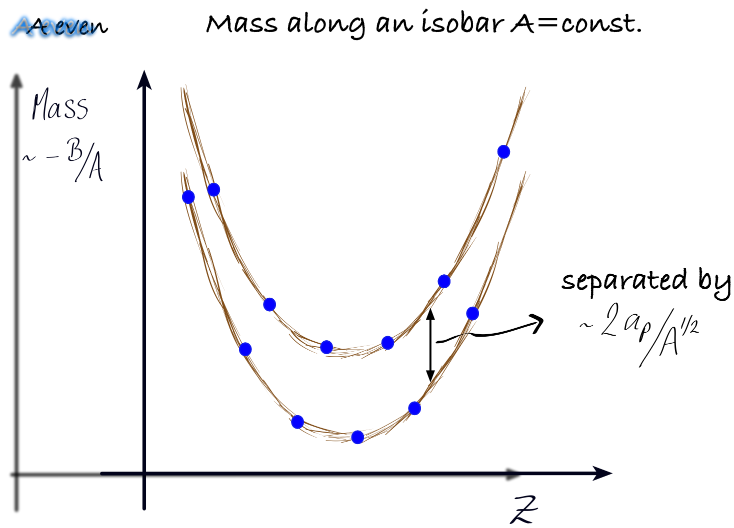


Figure 21: For $A =$ even we now form two distinct parabolas.

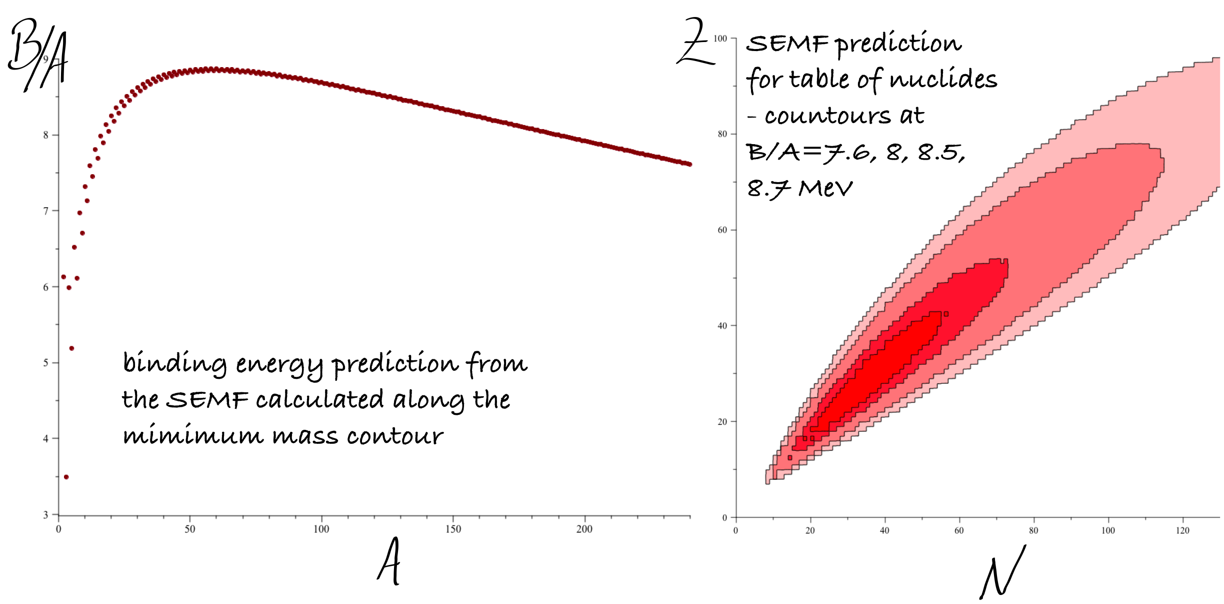


Figure 22: Predictions from the SEMF. Left we have B/A with Z given by (28) – compare to Fig. 15. Right we show the predicted binging energy strength on a Z, N plot – compare to Fig. 13. The jagged edges are due to the pairing term.

2.5 The Shell Model

The SEMF predicts overall $M(Z, A)$ well, but doesn't make any other predictions such as:

- Spins and 'parities'
- Ground and excited states
- Magic Numbers
- Magnetic moments, density, values of SEMF coefficients ...
- 'Scatter' of low A $B(Z, A)$ – compare Figs. 15 to 22.

Let's have a look at some strange things the SEMF can't predict.

First up: How accurate is the SEMF? A plot of the real binding energy versus the SMEF one is given in Fig. 23. Two features are apparent: it overestimates B for light and heavy nuclei, and underestimates it for medium mass nuclei. The most important feature for us are the spikes where the binding energy is really strong. Why?

Alternatively, recall neutron and proton separation energies:

$$S_n = B(Z, A) - B(Z, A - 1)\,, \tag{29}$$
$$S_p = B(Z, A) - B(Z - 1, A - 1)\,. \tag{30}$$

These also have peculiar features – see Figs. 24 and 25. In Fig. 24 we see neutrons being harder to knock out of a nucleus as we increase N, until a certain point, then it gets easier again: the jumps are indicated by the arrow. The interesting thing is that they are at the same neutron number as in Fig. 23

Similar features exist for isotopes with different N. In Fig. 25 we plot the neutron separation energy of $_{56}$Ba. Sawtooth shape from the alternating pairing term in the SEMF. There is also a sharp drop at $N = 82$ which is **not** in the SEMF. We see that $N = 82$ is much more resilient to having a neutron knocked off than for $N = 83$. Why?

We see similar features for proton separation energies. The key finding is that there are special numbers in the nuclear landscape where the binding energy is very strong. These are the :

Nuclear Magic Numbers

If either N or Z takes on the values:

$$2, 8, 20, 28, 50, 82, 126$$

These are called magic. For example, for protons, these correspond to the elements helium, oxygen, calcium, nickel, tin, lead, while the last has not yet been seen experimentally (though it has for neutrons – e.g. radon 212 or lead-208).

Usually the binding energy is very strong. The main features associated with these numbers are:

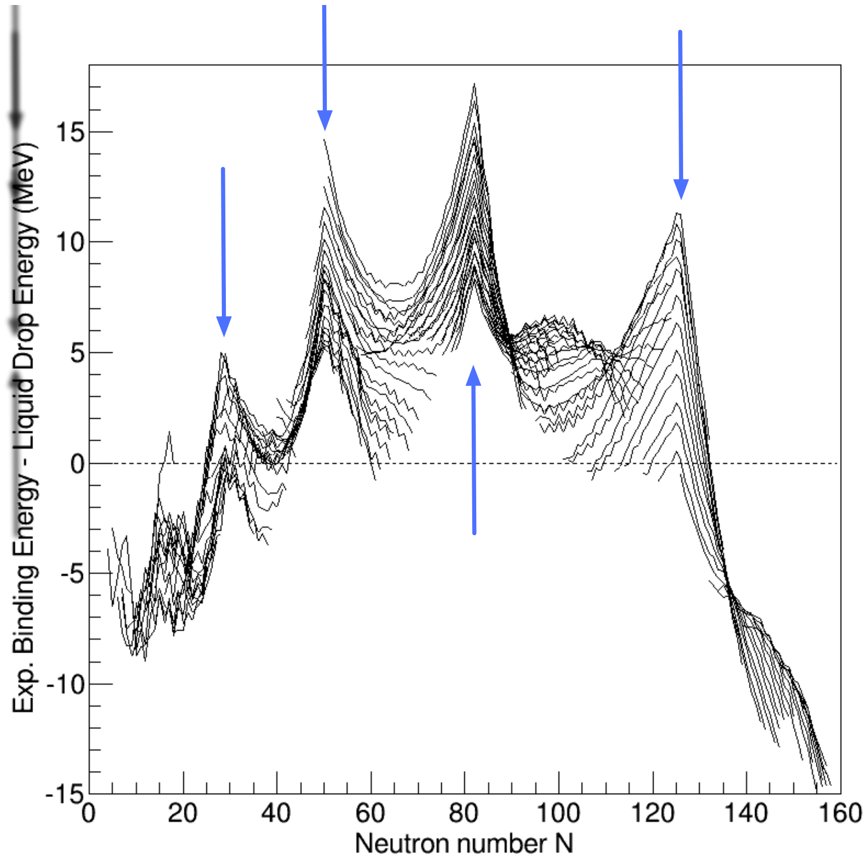


Figure 23: Accuracy of the SEMF as a function of neutron number, for a variety of nuclei. . It's accurate (note this is B not B/A), but also misses especially strongly bound nuclei where indicated.

- Increased B if Z or N is magic.
- If N is magic there are more isotones.
- If Z is magic there are more isotopes.
- If both N and Z are magic, the nucleus is very stable (e.g. ^{4}He or ^{16}O).
- Elements with Z magic have higher natural abundances.
- Higher excitation energies.

'Doubly magic' isotopes include helium-4, oxygen-16, calcium-40, calcium-48, nickel-48, nickel-78, and lead-208.

Why does this happen? And how to we model this?

A key insight is that atomic ionization energies follow the same pattern as $S_{2n(2p)}$ – see Fig. 26. Spikes occur for the noble gases helium, neon, argon, krypton, xenon, radon and oganesson. Hence, we could think of analogous 'atomic magic numbers' as 2, 10, 18, 36, 54, 86 and 118. This is explained by **shell structure** of electrons in an atom. When the outermost valence shells of the atom are full, the configuration is very stable.

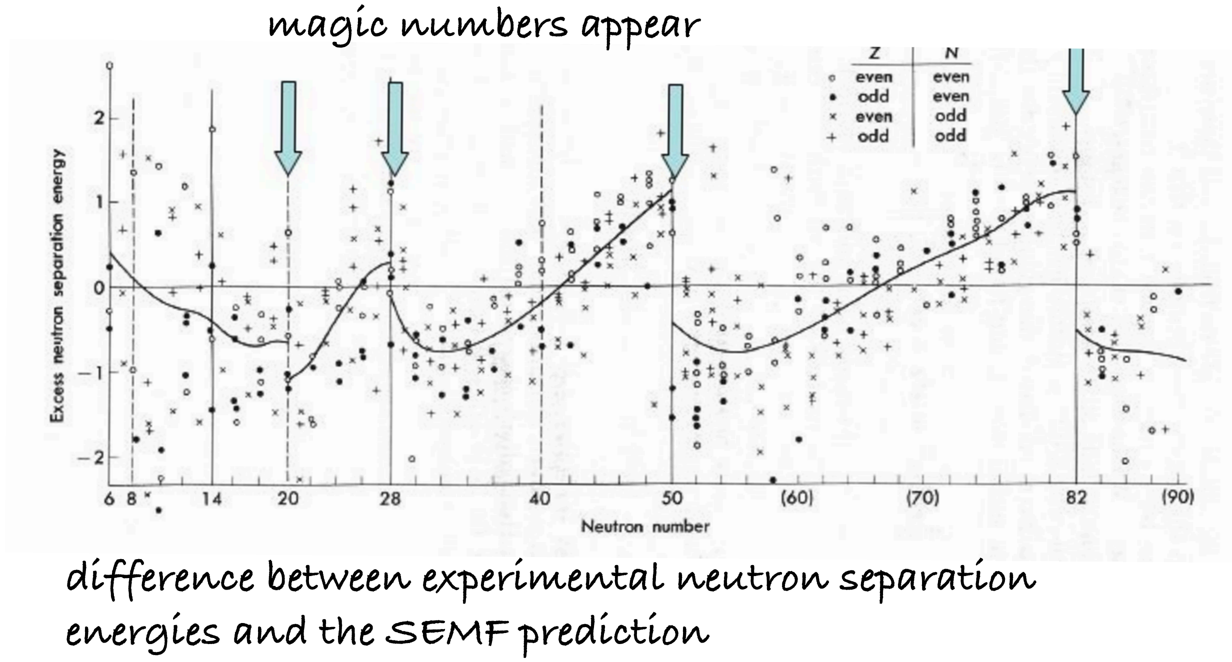


Figure 24: Neutron separation energies vs the prediction from the SEMF: clear discontinuities emerge.

We use this idea to form a **shell model of a nucleus**. The basic idea of this is that each nucleon moves as a single particle in a potential well, given by the average of other nucleons. The potential well is used in Schrödinger's equation, which predicted **quantised energy levels**. We fill the energy levels with neutrons and protons according to the Pauli Exclusion Principle.

The trick then is to find the form of the potential such that the magic numbers appear. These will appear when there are large gaps between two energy levels, and all possible 'spaces' are filled up to that level. We will try this phenomenologically, as the detailed calculations are too messy for this course. Once we have this **shell model** it will actually allow us to calculate nuclear spin and parity.

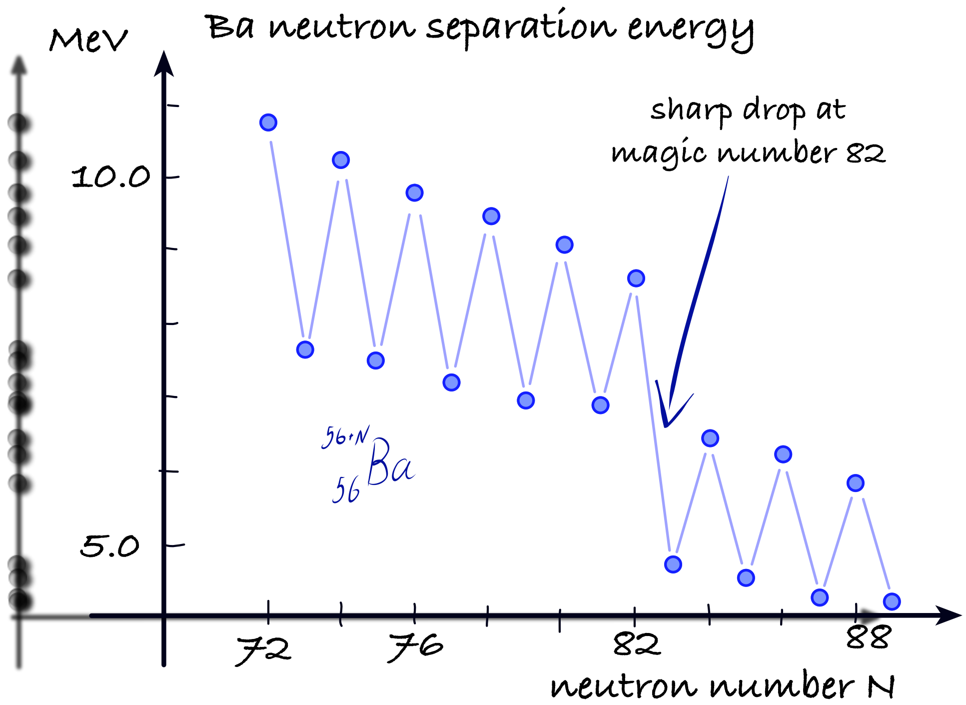


Figure 25: Neutron separation energy of $_{56}$Ba.

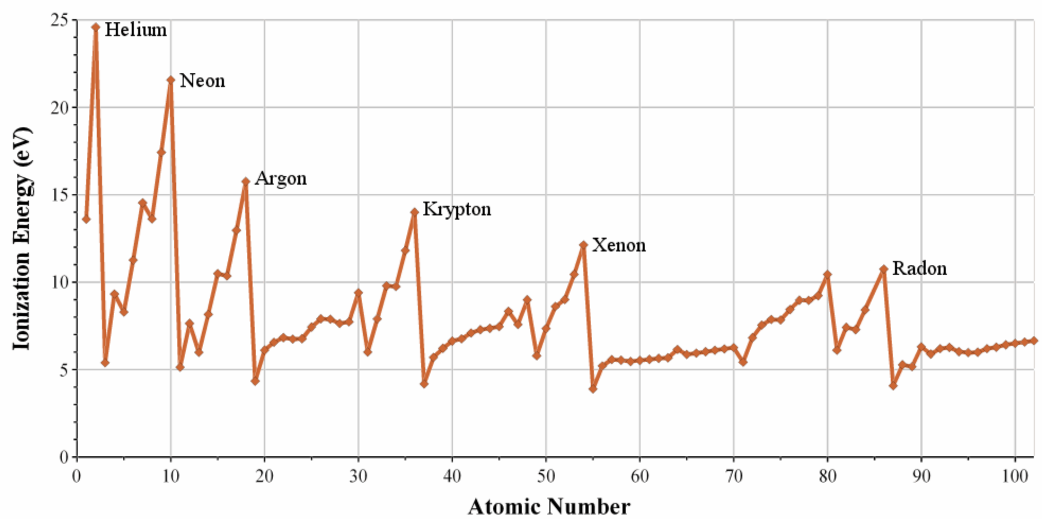


Figure 26: Ionization energies of the elements have a similar spiky feature to neutron separation energies.

Recap:
Schroedinger's Equation and all that.

The 3D stationary Schröedinger equation for a single particle mass M is:

$$\left(-\frac{\hbar^2}{2M}\nabla^2 + V(\underline{r})\right)\Psi(\underline{r}) = E\Psi(\underline{r})\,.$$

We are interested in the case when V is spherical so that $V(\underline{r}) = V(r)$, in which case we can write this more simply using spherical harmonics $Y_{lm}(\theta, \phi)$:

$$\Psi(\underline{r}) = \sum_{lm} \frac{1}{r} R_l(r) Y_{lm}(\theta, \phi)\,,$$

where $R_{lm}(r)$ are radial wavefronts, $Y_{lm}(\theta, \phi)$ are spherical harmonics, $l = 0, 1, 2, 3, \ldots$ is the orbital angular momentum quantum number and m is the 'magnetic' quantum number ($-l \leq m \leq l$). With this the Schröedinger equation becomes an ODE:

$$-\frac{\hbar^2}{2M}\frac{d^2R_l}{dr^2} + \left(V(r) + \frac{l(l+1)\hbar^2}{2Mr^2}\right) R_l(r) = E_{nl} R_l(r)\,.$$

All the angular parts appear through the integer l, and the spherical harmonic parts have factored out. (This is because Schröedinger's equation is linear and separable.)

The number $l(l+1)$ is an eigenvalue of $\underline{L}^2$, the **orbital angular momentum**. The **total angular momentum** includes the particle spin $\underline{S}$, and is $\underline{J} = \underline{L} + \underline{S}$ – it is **always conserved**. E_{nl} is an Energy eigenvalue where n is the quantum number.

Examples of $V(r)$ and $R(r)$ can be seen in Fig. 27. We start with a square well potential for $V(r)$. For $l > 0$ we have an **effective potential** $V(r) + \dfrac{l(l+1)\hbar^2}{2Mr^2}$ shown in the figure.

The **occupancy number** of the nl level is $2 \times (2l+1)$ and this means that $2(2l+1)$ neutrons or protons can fit in each level according to the Pauli exclusion principle. The factor $2l+1$ comes from the $m = -l, (-l+1), ..., -1, 0, 1, ..., (l-1), l$ states possible – each one is a separate state corresponding to the different orientations of the angular momentum. Then the factor 2 comes from the up spin and the down spin of the nucleon. This is known as the **degeneracy** of a state.

Magic Numbers for the Harmonic Potential

The harmonic oscillator has a simple potential which is a useful starting point:

$$V(r) = \frac{1}{2}M\omega^2 r^2\,. \tag{31}$$

Wavefunctions are of the form (polynomial $\times$ exponential in r) $\times\, Y_{lm}(\theta, \phi)$. The energy levels are eigenvalues:

$$E_{nl} = (2n + l - \frac{1}{2})\hbar\omega\,. \tag{32}$$

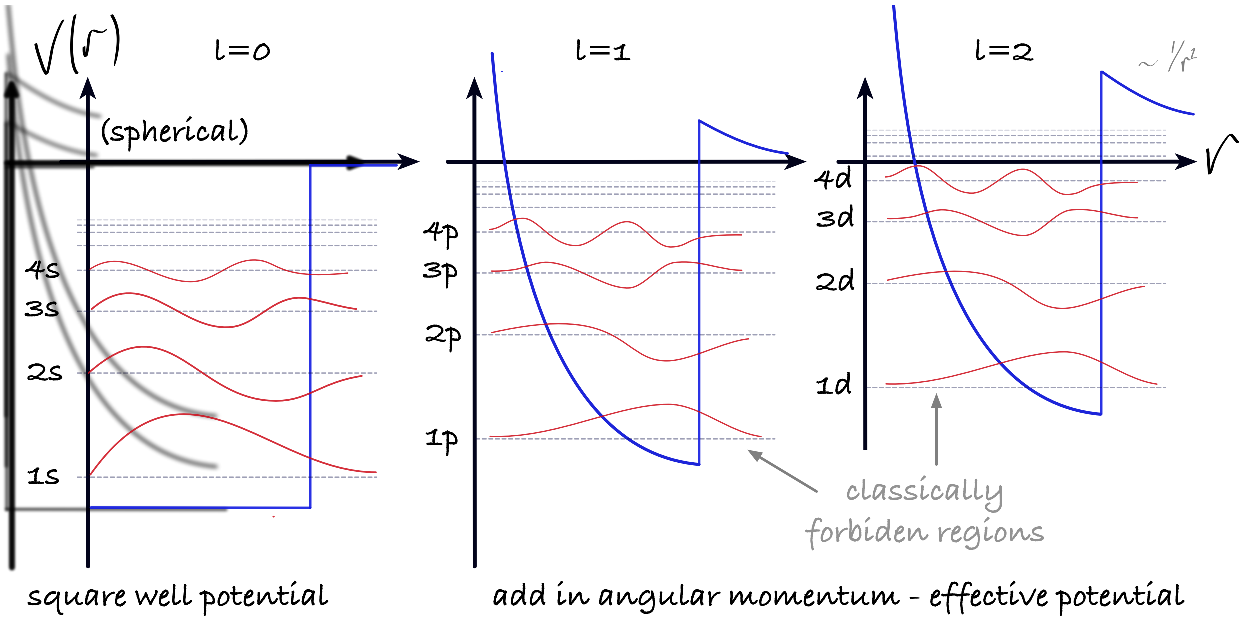


Figure 27: The lowest radial wave functions for a square well potential. For non-zero l, the effect is to give an effective potential with an **angular momentum barrier**.

> ***Note:*** We use a **stup**id**-f** labelling for the angular momentum states
>
> $$\begin{aligned} l = & 0, 1, 2, 3, 4, 5, 6, 7, \ldots \\ & s, p, d, f, g, h, i, k, \ldots \end{aligned}$$
>
> (Originally from the description by early spectroscopists of certain series of alkali metal spectroscopic lines as **s**harp, **p**rincipal, **d**iffuse, and **f**undamental.) Note j is skipped...

How do we see magic numbers from this? Let's sketch the energy levels on a diagram for each l – Fig. 28. Then, for each value of E, count how many neutrons or protons can fit into each level. We simply count $2(2l + 1)$ for each bar shown in the figure, and then add up the total number. The accumulated occupancy then is the total number of neutrons or protons assuming all the levels up to that one have been filled.

So, imagine a nucleus formed from this potential. As we add in neutrons or protons, we add them into the same level until it's full, then we jump up to the next energy level, and start filling them up. We can imagine the levels which are full are somehow stabler than those with empty spaces. So, in a handwaving way we can see that the magic numbers 2, 8 and 20 appear in this model, but thereafter its no good. In reality the particles are not trapped because we can remove a proton or a neutron which is impossible in a potential like this. Hence a realistic potential goes to 0 at large r, and should be something like Fig. 29.

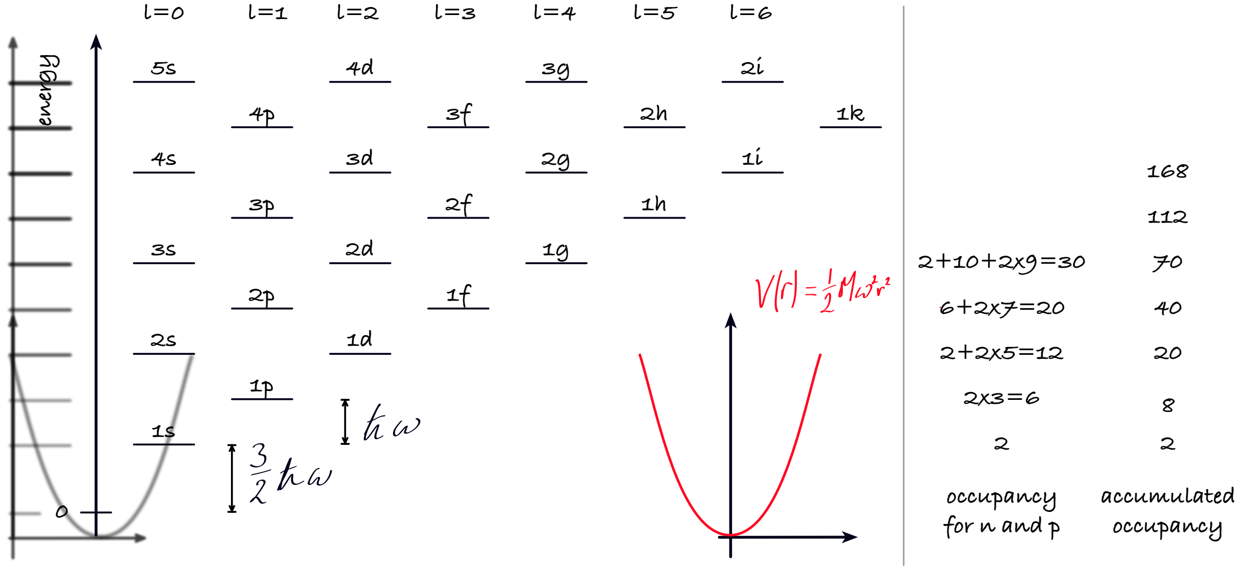


Figure 28: Energy levels for the harmonic potential. The occupancy number is the total of $2(2\ell+1)$ for each energy eigenvalue $E_n\ell$, and the accumulated is the sum of these from the lowest energy state.

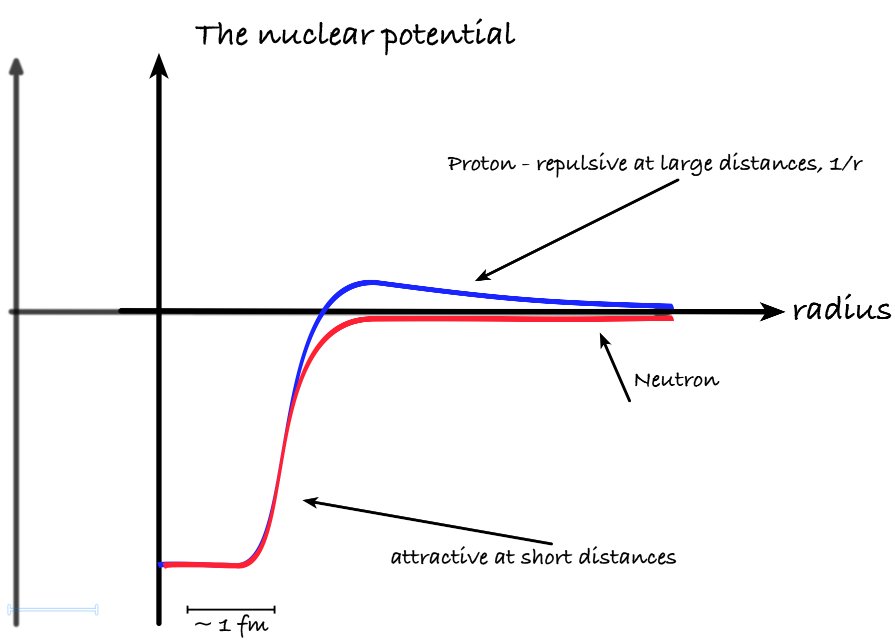


Figure 29: The Nuclear Potential, which approaches zero for large radii. At large distances the potential for protons becomes positive owing to the Coulomb repulsion between the proton (charge $+e$) and the nucleus (charge $+Ze$).

Note:

$$\begin{aligned} &s,\ p,\ d,\ f,\ g,\ h,\ i,\ k,\ldots \\ l =&0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ldots \\ \text{occupancy} = 2(2l+1) =&2, 6, 10, 14, 18, 22, 26, 30, \ldots \end{aligned}$$

Realistic Shape for the Potential

What is a sensible form to choose? We have a rough idea of the matter density and can guess that the potential might be similar, so we use the Woods-Saxon form (for neutrons; for protons its a bit different):

$$V_{central}(r) = -\frac{V_0}{1+e^{(r-R)/a}}\,. \tag{33}$$

What are the energy levels for this like, and do the magic number appear? Compared to the harmonic oscillator this **lowers** energy levels because the wave functions spill over the edge of the potential; and it breaks some degeneracies – we say the energy levels '**split**'. However, this potential still misses magic numbers, which means that the central (spherical) potential assumption that we are using at the moment is not enough.

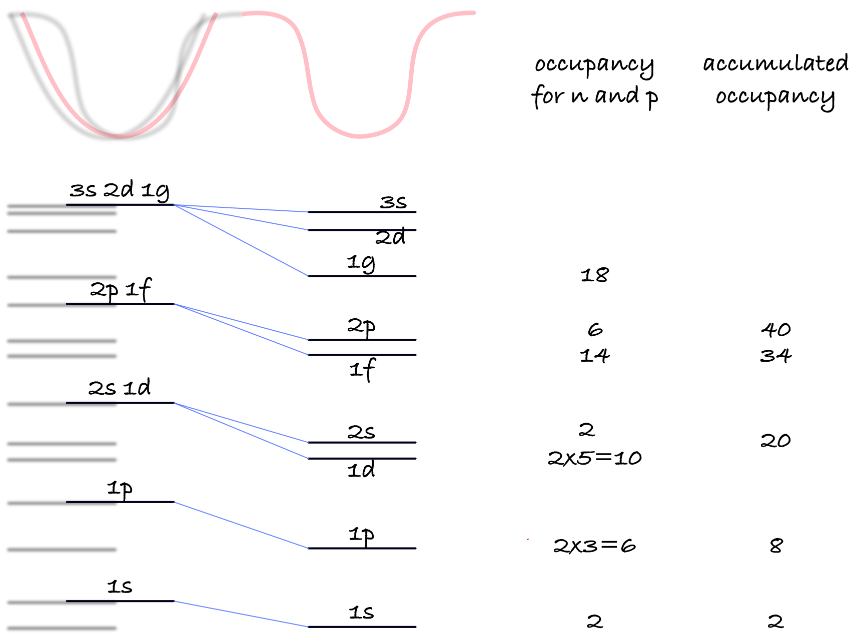


Figure 30: Energy levels for the Woods-Saxon potential

Adding in Angular Momentum into the Potential

Symmetry of the potential must be broken by orbital angular momentum $\underline{L}$ and spin $\underline{S}$, both of which must feed into the potential in some way. This effect is important near the surface. The orbital angular momentum $\underline{L}$ and nucleon spin $\underline{S}$ interact, in such a way which **conserves** the total angular momentum ($\underline{J} = \underline{L} + \underline{S}$). For a given l, $\underline{J}^2$ has an eigenvalue of:

$$j(j+1) \;\text{ where }\; j = l \pm \frac{1}{2} \quad (\text{since } s = \pm\frac{1}{2})\,. \tag{34}$$

That is, for each l there are two possible values for the total angular momentum quantum number, depending on the spin of the nucleon.

Spin and angular momentum couple together – called **spin-orbit coupling** – which alters the potential via a term which must be of the form (since we must form a scalar from 2 vectors!):

$$V(r) = V_{\text{Central}}(r) + V_{\text{Spin-Orbit}}(r)(\underline{L} \cdot \underline{S}) \,. \tag{35}$$

The form of $V_{\text{Spin-Orbit}}(r)$ is not important, but we do need the expectation value of $(\underline{L} \cdot \underline{S})$. Using the trick that $J^2 = L^2 + S^2 + 2(\underline{L} \cdot \underline{S})$ we can get the expectation value as:

$$\langle \underline{L} \cdot \underline{S} \rangle = \frac{1}{2}\hbar^2 (j(j+1) - l(l+1) - s(s+1)) \tag{36}$$

$$= \hbar^2 \begin{cases} \dfrac{l}{2} & j = l + \frac{1}{2} \quad \text{spin up}\,, \\ -\dfrac{l+1}{2} & j = l - \frac{1}{2} \quad \text{spin down}\,, \end{cases} \tag{37}$$

(note that $s = 1/2$ is the spin in both cases). This splits every energy level with $l \geq 1$ into 2 separate levels, with energy difference:

$$\Delta E_{ls} = \frac{2l+1}{2}\hbar^2 \langle V_{\text{Spin-Orbit}} \rangle \,, \tag{38}$$

where $V_{\text{Spin-Orbit}}$ is negative, so that the $j = l - 1/2$ state lies above the $j = l + 1/2$ state. These splittings are large, and also increase as l increases. These energy levels now become as in Fig. 31.

As you can see there energy levels have now clustered, and in the gaps are the magic numbers! You can also see that these energy levels also cross at some points. The occupancy number is determined by j: there are $2j + 1$ neutrons or protons allowed – it's no longer $2\times$ as up and down spins are separated onto their own level.

The neutron and proton levels are very similar, but are slightly different because the potential for the protons is slightly higher owing to the Coulomb repulsion. The ordering of these levels are not absolute, they change with large A, which also means that the magic numbers are not totally fixed.

The idea of how this model works is to fill up energy levels with neutrons or protons, until they are all used up, for a given N, Z. If either N or Z is magic, it will have a **singly closed shell** (see Fig. 32). Similarly if both N and Z are magic then it will have a **doubly closed shell**. Otherwise it's just a normal nucleus. We can however, use the shell model for more than just discussing the magic numbers.

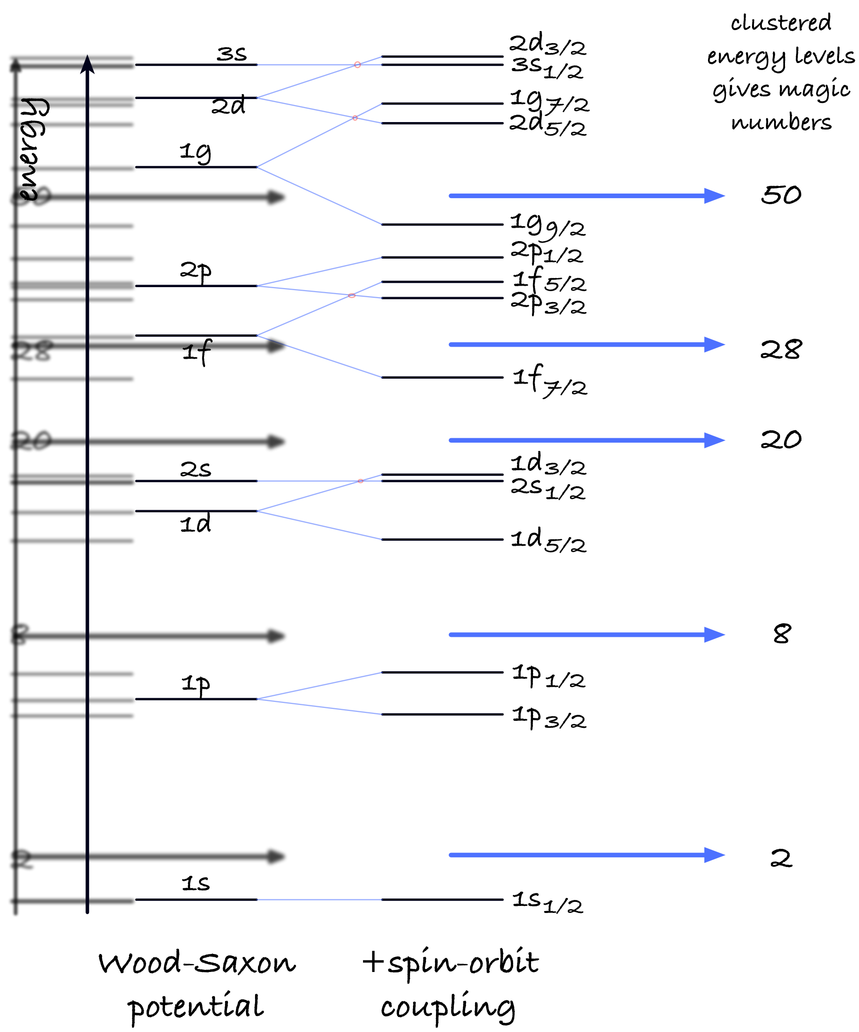


Figure 31: Energy levels for the Woods-Saxon potential with a spin-orbit coupling term included. The energy levels swap (small circles) and cluster, with big gaps appearing. The notation used for each level is nl_j. The accumulated occupancy up to each gap reveals the **magic numbers**.

2.5.1 Nuclei Configurations

We can actually use the shell model to predict a variety of properties of the structure of a nuclei:

- Nuclear spin J, which is a sum over all the internal angular momenta and spins
- Parity π
- Some excited states

We shall have a look at the first two of these, which give the **ground state** which we write as J^π. To do this we need to first figure out the **configuration** of neutrons and protons, which means the shells that are filled in the shell model. We start by drawing out the lowest energy levels and then fill with neutrons and protons (according to the Pauli exclusion principle). An example of what this looks like is Fig. 32. We sketch the energy levels for both neutrons and protons (for large A

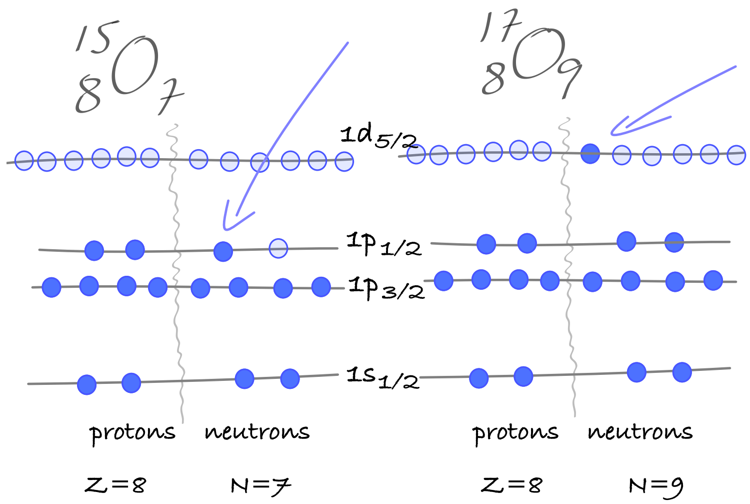


Figure 32: An example using the shell model. Here we have the isotopes $^{15}_{8}O_7$ and $^{17}_{8}O_9$. Filled circles represent a space on each shell which is **filled**, an empty circle is **unfilled**. This has a magic number of protons, but the neutrons in both isotopes have an unfilled shell – the **last unpaired nucleus** is indicated.

the arrangement is slightly different for each), and note how many each level can take – this is just $2j + 1$ (the numerator on the subscript of the energy level plus 1). Then fill up the levels with n & p. This gives the **configuration** of neutrons and protons.

Nuclear Spin:

The ground state nuclear spin can be found from the last odd nuclei, which in the above example is the last neutron. This is because a filled level contributes nothing to the nuclear spin because the z components of the total angular momentum cancel out (the occupancy $2j + 1$ is an even number, so for each m_j there is an equal and opposite m_j). An immediate prediction from this is that all doubly magic nuclei have zero spin, and this is indeed the case! More generally, all nuclei with the last energy level completely filled (even if not magic) will have zero spin.

Digression: Parity:
In general parity is the behaviour of a quantum state under spatial reflection through the origin $\underline{r} \to -\underline{r}$ [in 3d]. In classical physics things like a billiard ball don't change under this transformation, whereas a corkscrew changes its handedness (you would have to screw the 'wrong way' to use it). This symmetry is preserved in the strong force, so parity is an important quantum number.

In fact we can go further for nuclear ground states: **all even-even nuclei have zero spin**. This is found experimentally, from which we can make the **pairing hypothesis**: pairs of neutron and pairs of protons in each energy level **pair up** to give zero total angular momentum, even if that shell is not filled.

It follows immediately that in odd-A nuclei the spin J is just the j-value of the **last unpaired nucleon** – i.e., the j-value of the shell it's in. (Since A is odd there can be only one!). For odd-odd nuclei however, there are two unpaired nucleons: how do we know the angular momenta? We don't: given angular momentum add vectorially, we can only say J of the nucleus is between $|j_p - j_n|$ and $j_p + j_n$, where j_n (j_p) is the total angular momentum quantum number of the last unpaired neutron (proton).

Example:
${}^{15}_{8}\mathrm{O}_7$ has spin $\frac{1}{2}$, because the last unpaired neutron is in a level with $j = 1/2$,

${}^{17}_{8}\mathrm{O}_9$ has spin $\frac{5}{2}$ because the last unpaired neutron is in a level with $j = 5/2$.

Parity:

The parity of each neutron or proton is $(-1)^l$ (which follows from the behaviour of the spherical harmonic functions under a parity transformation), so the parity of the whole nucleus is of the form of a product of these. So π of even-even nuclei is $+1$ or just $+$, and for odd-A is that of the last unpaired nucleon. Finally, for odd-odd it's $(-1)^{l_p+l_n}$.

Example:
${}^{15}_{8}\mathrm{O}_7$ - parity of p-state ($l = 1$), therefore $(-1)^1 = -1$, $\pi = -$,

${}^{17}_{8}\mathrm{O}_9$ - parity of d-state ($l = 2$), therefore $(-1)^2 = 1$, $\pi = +$.

The **nuclear ground state** is written J^π so ${}^{15}_{8}\mathrm{O}_7$ is $1/2^-$ and ${}^{17}_{8}\mathrm{O}_9$ is $5/2^+$. You can check this on the nuclear data websites – amazingly it's spot on!

In general the shell ordering is:

Protons

$$1s_{\frac{1}{2}} \Big\downarrow_2 1p_{\frac{3}{2}} \,|\, 1p_{\frac{1}{2}} \Big\downarrow_8 1d_{\frac{5}{2}} \,|\, 2s_{\frac{1}{2}} \,|\, 1d_{\frac{3}{2}} \Big\downarrow_{20} 1f_{\frac{7}{2}} \Big\downarrow_{28} 2p_{\frac{3}{2}} \,|\, 1f_{\frac{5}{2}} \,|\, 2p_{\frac{1}{2}} \,|\, 1g_{\frac{9}{2}} \Big\downarrow_{50} 1g_{\frac{7}{2}} \,|\, 2d_{\frac{5}{2}} \,|\, 1h_{\frac{11}{2}} \,|\, 2d_{\frac{3}{2}} \,|\, 3s_{\frac{1}{2}} \Big\downarrow_{82} 1h_{\frac{9}{2}} \,|\, 2f_{\frac{7}{2}} \,|\, \cdots$$

Neutrons

$$1s_{\frac{1}{2}} \Big\downarrow_2 1p_{\frac{3}{2}} \,|\, 1p_{\frac{1}{2}} \Big\downarrow_8 1d_{\frac{5}{2}} \,|\, 2s_{\frac{1}{2}} \,|\, 1d_{\frac{3}{2}} \Big\downarrow_{20} 1f_{\frac{7}{2}} \Big\downarrow_{28} 2p_{\frac{3}{2}} \,|\, 1f_{\frac{5}{2}} \,|\, 2p_{\frac{1}{2}} \,|\, 1g_{\frac{9}{2}} \Big\downarrow_{50} 2d_{\frac{5}{2}} \,|\, 1g_{\frac{7}{2}} \,|\, 1h_{\frac{11}{2}} \,|\, 2d_{\frac{3}{2}} \,|\, 3s_{\frac{1}{2}} \Big\downarrow_{82} 2f_{\frac{7}{2}} \,|\, 1h_{\frac{9}{2}} \,|\, \cdots$$

(Can you spot the differences?)

Notation: A level with k neutrons or protons is written as $(nl_j)^k$ and called a state. A level is full if $k = 2j + 1$. Given N and Z fill level, you can see what's left.

> ***Example:***
> $^{49}_{20}$Ca has a proton number $Z = 20$ (which is magic and therefore the last energy level is full) and a neutron number $N = 29$:
>
> $$\begin{aligned} \text{p state} =& (1s_{\frac{1}{2}})^2(1p_{\frac{3}{2}})^4(1p_{\frac{1}{2}})^2(1d_{\frac{5}{2}})^6(2s_{\frac{1}{2}})^2(1d_{\frac{3}{2}})^4\,, \\ \text{n state} =& (1s_{\frac{1}{2}})^2(1p_{\frac{3}{2}})^4(1p_{\frac{1}{2}})^2(1d_{\frac{5}{2}})^6(2s_{\frac{1}{2}})^2(1d_{\frac{3}{2}})^4(1f_{\frac{7}{2}})^8(2p_{\frac{3}{2}})^1\,. \end{aligned}$$
>
> We see that in the last line, the last term has $k < 2j + 1$, so represents an unfilled shell. Therefore, because $(2p_{\frac{3}{2}})$ is the last unpaired neutron, J^π is $\frac{3}{2}^-$. In practice, we can write the state as: neutron configuration: $(2p_{\frac{3}{2}})^1$; proton configuration: magic, which is 0^+.

We can also use a 'hole' notation for the last level, where k is negative representing the number of unfilled states in the last level. In the last example, we would have neutron configuration: $(2p_{\frac{3}{2}})^{-3}$, or from earlier, $^{15}_{8}$O has a neutron configuration of $(1p_{\frac{1}{2}})^1$ or $(1p_{\frac{1}{2}})^{-1}$.

Excited States:

The shell model can predict *some* excited states. Essentially, we can think of a nucleon occupying a higher level than it should, leaving a hole in a lower level. (After fission or α-decay a nucleus can be left in an excited state like this.) It can drop back down again editing a high energy photon (a **gamma ray**). Looking at Fig. 33 we can see many different ways to be excited, for example for the neutron configuration:

$$\begin{array}{ccccc} (2d_{\frac{5}{2}})^1 \rightarrow & (2s)^1 & \rightarrow & (1d_{\frac{3}{2}})^1 & \\ & \downarrow & & \downarrow & \\ & \frac{1}{2}^+, & & \frac{3}{2}^+, & \frac{7}{2}^-, \quad \dots \end{array}$$

The actual state is more complicated than this picture, however. The excited states are actually found as **superposition** of possibilities!

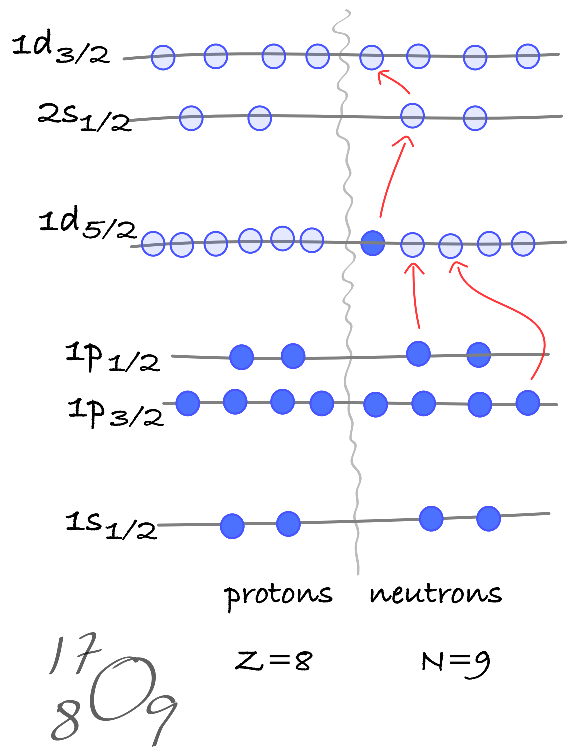


Figure 33: The shell model can predict some excited states. Here's some random ways an Oxygen isotope can be excited.

3 Radioactive Decay and Nuclear Instability

Nuclei can sometimes undergo a spontaneous transition from one state to another. In fact, we have seen that nuclei occur in a very narrow region of the nuclear landscape, and stable nuclei occupy a thin band in the middle of the region of known nuclei. All other nuclei are **unstable** and decay in various ways.

Example:

$$(Z, A) \rightarrow (Z+1, A) \text{ and a } \beta^- \text{ particle,}$$
$$(Z, A) \rightarrow (Z-2, A-4) \text{ and an } \alpha \text{ particle,}$$

or it could emit a γ ray. Representations using energy levels is shown in Fig. 34

Typically the resulting nucleons will have a higher binding energy, so there will always be extra energy released, often as recoil energy.

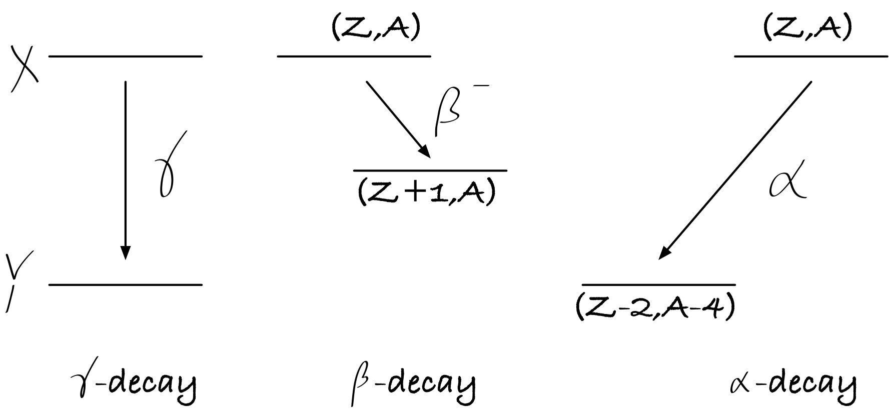


Figure 34: Pictures of some decay modes. The change in height represents the energy released and the left/right offset the change in proton number.

The simplest type of decay happens for very large nuclei where the Coulomb repulsion overcomes the nuclear force. A small chunk of the nucleus may be ejected as an α-particle, which is a very stable Helium-4 nucleus, though other decays are possible (and it may just split into two pieces in a spontaneous fission reaction). Other types of decay happen in nuclei which are very neutron (proton) rich can gain binding energy by converting a neutron (proton) into a proton (neutron), in a β-decay process. Sometimes a proton or neutron can be directly ejected. These are the common decays which are responsible for the shape of the table of nuclides, continually pushing nuclei towards the **valley of stability** and to higher binding energies.

When these processes are possible they occur **randomly**. We will look into the processes later, but first we will look at statistics of decay.

3.1 Radioactive Decay

The probability per unit time a nucleus of given type will decay in a particular way (a **decay mode**) is the **decay constant** λ. Say we have large number of nuclei $N(t)$. The number decaying in a short time Δt is ΔN, with $\Delta N \propto N$:

$$\Delta N = -\lambda N \Delta t\,, \tag{39}$$

which becomes

$$\frac{dN}{dt} = -\lambda N \quad \text{as} \quad \Delta t \to dt\,. \tag{40}$$

> ***Note:***
> $A = \lambda \times N$ is the expected decays per unit time, and is called the **activity** of a sample.
> λ is a constant since its equally likely to decay at any instant.

Now, in order to find $N(t)$ we need to integrate this **differential equation**. This is a first-order linear homogeneous ordinary differential equation which can be formally integrated using separation of variables. This means that we move all the bits in the equation involving N onto the left hand side (the dN tells us that we will be integrating with respect to N), and all the bits involving t (only the dt here!) to the right hand side, and then integrate:

$$\int_{N_0}^{N} \frac{dN}{N} = -\lambda \int_{t_0}^{t} dt\,, \quad \Rightarrow \ln N - \ln N_0 = -\lambda(t - t_0)\,, \tag{41}$$

which implies

$$N(t) = N_0 e^{-\lambda(t-t_0)}\,, \tag{42}$$

where $N_0 \equiv N(t_0)$, which is the initial number in the sample. We also usually set $t_0 = 0$. This is the **radioactive decay law**, and gives the expected number at some time t later than t_0. This is close to the actual number for large N. We show this in Fig. 35. Here we associate some useful quantities:

- $\tau = \dfrac{1}{\lambda}$ is the **mean lifetime**, which is just the average lifetime of a nucleus. Note that $N = N_0 e^{-t/\tau}$.
- $t_{1/2}$ is the **half life**: $N(t_{1/2}) = \dfrac{1}{2}N_0 = N_0 e^{-\lambda t_{1/2}}$, $t_{1/2} = \dfrac{\ln(2)}{\lambda} = \tau \ln(2) \quad (\ln(2) \simeq 0.693)$. This is the time taken for a sample to reduce by a factor of 2. Half-lives can vary from $> 10^{10}$ yrs to just 10^{-24} s.
- The **activity** ($A = \lambda N$) is measured in Becquerels (Bq) which is defined as the as the activity of a quantity of radioactive material in which one nucleus decays per second. We also use Curie (Ci)= 3.7×10^{10} Bq (this is decays/s of $^{226}_{88}$Ra per gram). It's a bit of an odd unit as formally 1 Bq=1 s^{-1}, but really this means 1 Bq is 1 radioactivity event per second (a bit like 1 Hz is 1 cycle per second for frequency).

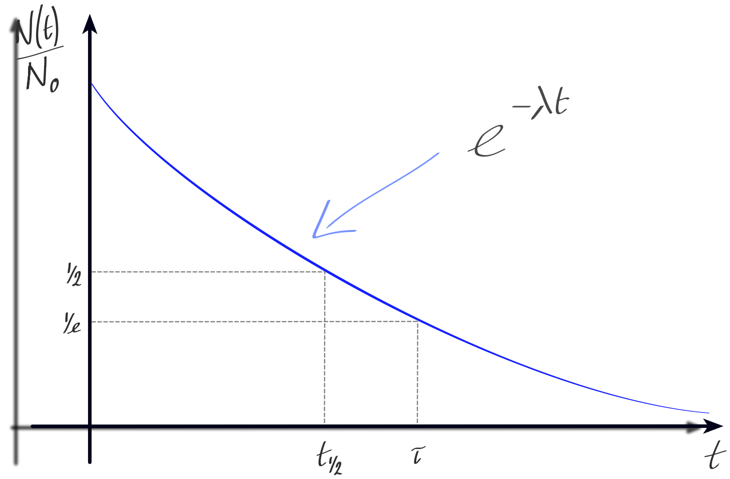


Figure 35: The radioactive decay law.

Example:
The Curie:

What is the half-life of ${}^{226}_{88}\text{Ra}$? We can find this knowing that 1 Ci$= 3.7 \times 10^{10}$ Bq is the decays/s of ${}^{226}_{88}\text{Ra}$ per gram. The number of nuclei in 1g is found as follows. First calculus the mass in grams:

$$M({}^{226}_{88}\text{Ra}) = 88m_p + (226-88)m_n \simeq 3.77 \times 10^{-25}\,\text{kg} \quad \text{(Neglect B!)},$$

so the initial number is,

$$N_0 \simeq \frac{10^{-3}\,\text{kg}}{3.77 \times 10^{-25}\,\text{kg}} \simeq 2.67 \times 10^{21}\,.$$

Now, given the activity of this 1 gram is

$$3.7 \times 10^{10}\,\text{decay/s} \to \lambda \simeq \frac{3.7 \times 10^{10}}{2.67 \times 10^{21}}\text{s}^{-1} = 1.39 \times 10^{-11}\,\text{s}^{-1}\,.$$

Therefore,

$$t_{1/2} \simeq \frac{0.693}{1.39 \times 10^{-11}}\text{s} \simeq 1620\,\text{yrs}$$

Example:
Carbon Dating.

Carbon appears naturally as the isotopes ${}^{12}\text{C}$ (stable), ${}^{13}\text{C}$ (stable) and ${}^{14}\text{C}$ (decays $t_{1/2}$ =

Digression: on random numbers:
If we expect ΔN events, the 'error' or expected fluctuation around this number is $\sqrt{\Delta N}$. This means that there is approximately a 68% probability that the actual number of events is between $\Delta N - \sqrt{\Delta N}$ and $\Delta N + \sqrt{\Delta N}$ (this is called 1 'standard deviation'). There's also a 95% probability that the actual number of events is between $\Delta N - 2\sqrt{\Delta N}$ and $\Delta N + 2\sqrt{\Delta N}$. This follows because the events are random and follow a Gaussian (normal) probability distribution. Note that for ΔN large, $\sqrt{\Delta N} \ll \Delta N$, so when we calculate $N(t)$ this will be very close to the values measured in experiments.

5730 yrs $\sim 0.25\,\mathrm{Bq/g}$). ^{14}C produced in the atmosphere from cosmic rays in the **reaction**:

$$n + {}^{14}_{7}\mathrm{N} \to {}^{14}_{6}\mathrm{C} + p\,. \tag{43}$$

This isotope combines with oxygen into CO_2 which finds its way into plants and then animals, which maintain a fixed fraction of ${}^{14}_{6}$C, assuming the flux of cosmic rays is constant. Once the plant or animal dies it no longer exchanges carbon with the environment, as CO_2 from the atmosphere stops being taken up and it slowly decays, changing the ratios of carbon isotopes in the dead plant or animal. So, ^{14}C acts as clock as ^{14}C decays relative to ^{12}C.

With a bit of thought you can figure out that the ratio today of ^{14}C to ^{12}C decays as $R = R_0 e^{-t/8267}$, where t is in years and R_0 is assumed to be the same as in the atmosphere.

Note:
The activity of a sample is rate at which decays occur: $A(t) = \lambda N(t)$. If a sample consists of $N_1 \to N_2$, and $N_1 + N_2 = N_0$ =constant, then $N_1 = N_0 e^{\lambda_1 t}$ and $N_2 = N_0(1 - e^{\lambda_1 t})$

Multi model Decays

A nuclei may decay in different **modes**.

Total decay rate from 2 modes is:

$$\frac{dN}{dt} = \left.\frac{dN}{dt}\right|_1 + \left.\frac{dN}{dt}\right|_2 = -\lambda_1 N(t) - \lambda_2 N(t) = -\lambda N(t)\,. \tag{44}$$

So, the total number is $N(t) = N_0 e^{-\lambda t}$ and it only depends on $\lambda_1 + \lambda_2$. The **branching fractions** $f_1 = \frac{\lambda_1}{\lambda}$, and $f_2 = \frac{\lambda_2}{\lambda}$ are the fractions decaying by mode 1 and 2 respectively.

Decay Chains

Often the products of a decay are radioactive too, so we get a decay chain of $1 \to 2 \to 3 \to 4$... This can lead to more complicated behaviour over time than the simple radioactive decay law.

The first species $1 \to 2$ just obeys:

$$N_1(t) = N_0 e^{-\lambda_1 t}\,. \tag{45}$$

For species 2, we have:

$$\frac{dN_2}{dt} = -\lambda_2 N_2 + \lambda_1 N_1\,, \tag{46}$$

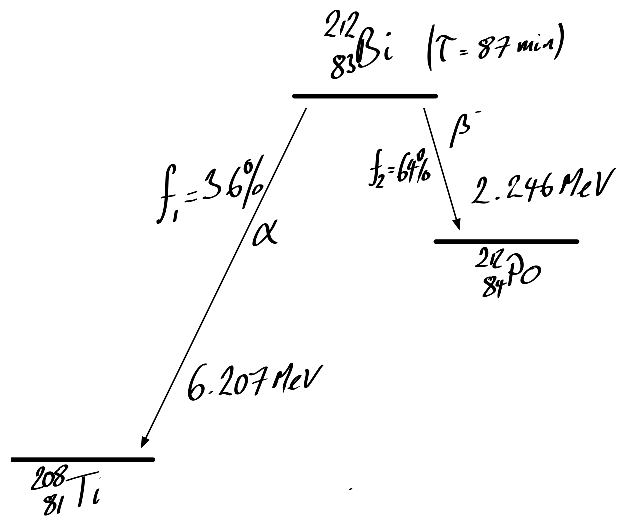


Figure 36: An example of a bimodal decay with branching fractions expressed as a percentage. Can you work out the individual decay constants for each branch?

where the $\lambda_2 N_2$ is the usual decay of species 2 and $\lambda_1 N_1$ is the number of species 1 which have decayed into species 2 at time t.

The solution to this with $N_2(0) = 0$ is given by,

$$N_2(t) = N_0 \frac{\lambda_1}{\lambda_2 - \lambda_1}(e^{-\lambda_1 t} - e^{-\lambda_2 t}) \,. \tag{47}$$

etc. The general solution is sum of $e^{-\lambda_i t}$. Can you work out the number for species 3?

Example:
Strontium-Rubidium-Krypton-Bromine decay chain:

$$
\begin{array}{ll}
{}^{79}_{38}\mathrm{Sr} \rightarrow {}^{79}_{37}\mathrm{Rb} + e^+ + \nu_e & (2.25\,\mathrm{min}) \\
\quad\quad\quad\downarrow & \\
\quad\quad\quad{}^{79}_{36}\mathrm{Kr} + e^+ + \nu_e & (22.9\,\mathrm{min}) \\
\quad\quad\quad\downarrow & \\
\quad\quad\quad{}^{79}_{35}\mathrm{Br} + e^+ + \nu_e & (34.04\,\mathrm{hr})
\end{array}
$$

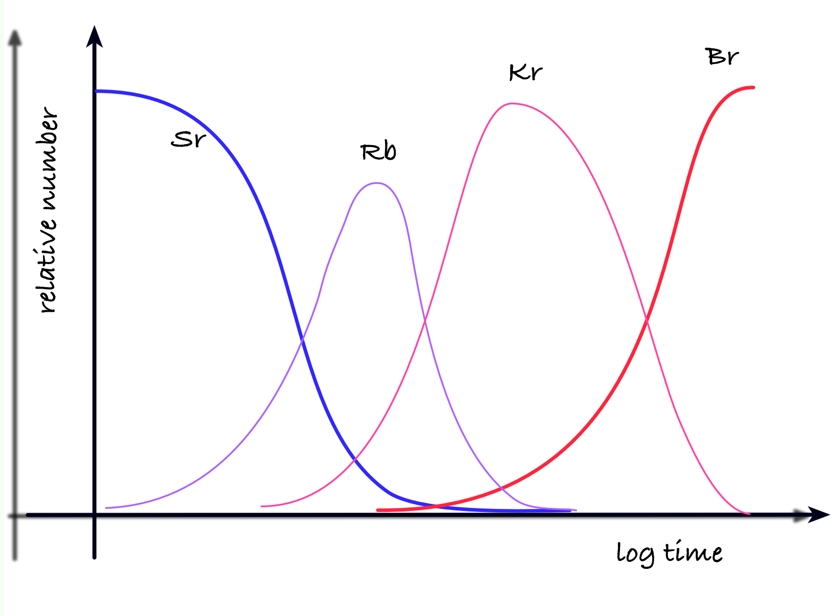


Figure 37: The evolution over time of the nuclei numbers in the decay chain Sr→Br.

Types of Decays

Let us now have a brief look at some of the common types of decay.

α-Decay

Nucleus emits an α-particle ($={}^4_2$He nucleus). ^{4}He is very tightly bound (doubly magic) so the kinetic energy released is maximised, compared to release of similar nuclei.

$$(Z, A) \to (Z-2, A-4) + {}^4_2\text{He} \tag{48}$$

Example:
Radium to Radon

$$^{226}_{88}\text{Ra} \to {}^{222}_{86}\text{R} + \alpha \quad \text{(half-life 1600 yr)} \tag{49}$$

Here the kinetic energy released is 4.8 MeV.

We will model some aspects of α-decay, and derive the **Geiger-Nuttal rule** which correlates lifetime with the energy released in a decay.

β-Decay

A neutron or a proton converts to a proton or a neutron, and sometimes involves a new particle the **positron** e^+:

$$n \to p + e^- + \; ? \qquad \beta^- \text{ decay} \tag{50}$$

$$p \to n + e^+ + \; ? \qquad \beta^+ \text{ decay} \tag{51}$$

$$p + e^- \to n + \; ? \qquad \text{electron capture } (\epsilon) \tag{52}$$

Another new particle must be involved from experimental conservation of energy - this is the **neutrino**.

We will investigate how β-decay is critical in understanding the **valley of stability** in the table of nuclides. **γ-Decay**

Decay from an excited state emits high energy photon. It usually follows α or β decay. Mostly short lived ($\sim 10^{-9}$ s), though sometimes there are some long ($\sim 10^{-6}$ s). **Isomeric transitions** result in metastable states - isomers.

Fission

Spontaneous split of heavy nucleus into 2 halves...

Nucleon emission

Very far from valley of stability.

Digression: There are lots of decay mechanisms! Here's a list [adapted from Wikipedia]

Mode of decay	**Participating particles**	**Daughter nucleus**
Decays with emission of nucleons:		
Alpha Decay	An alpha particle ($A = 4, Z = 2$) emitted from nucleus	$(A-4, Z-2)$
Proton Emmision	A proton ejected from nucleus	$(A-1, Z-1)$
Neutron Emmision	A neutron ejected from nucleus	$(A-1, Z)$
Double Proton Emission	Two protons ejected from nucleus simultaneously	$(A-2, Z-2)$
Spontaneous Fisson	Nucleus disintegrates into two or more smaller nuclei and other particles	-
Cluster Decay	Nucleus emits a specific type of smaller nucleus (A_1, Z_1) which is larger than an alpha particle	$(A-A_1, Z-Z_1)$ $+(A_1, Z_1)$
Different Modes of β Decay:		
β^- Decay	A nucleus emits an electron and an electron antineutrino	$(A, Z+1)$
Positron Emission (β^+ Decay)	A nucleus emits a positron and an electron neutrino	$(A, Z-1)$
Electron Capture	A nucleus captures an orbiting electron and emits a neutrino; the daughter nucleus is left in an excited unstable state	$(A, Z-1)$
Bound State Beta Decay	A free neutron or nucleus beta decays to electron and antineutrino, but the electron is not emitted, as it is captured into an empty K-shell; the daughter nucleus is left in an excited and unstable state. This process is a minority of free neutron decays (0.0004%) due to the low energy of hydrogen ionization, and is suppressed except in ionized atoms that have K-shell vacancies.	$(A, Z+1)$
Double Beta Decay	A nucleus emits two electrons and two antineutrinos	$(A, Z+2)$
Double Electron Capture	A nucleus absorbs two orbital electrons and emits two neutrinos and the daughter nucleus is left in an excited and unstable state	$(A, Z-2)$
Electron Capture with Positron Emission	A nucleus absorbs one orbital electron, emits one positron and two neutrinos	$(A, Z-2)$
Double Positron Emission	A nucleus emits two positrons and two neutrinos	$(A, Z-2)$
Transitions between states of the same nucleus:		
Isomeric Transition	Excited nucleus releases a high-energy photon (gamma ray)	(A, Z)
Internal Conversion	Excited nucleus transfers energy to an orbital electron, which is subsequently ejected from the atom	(A, Z)

3.2 α Decay

An α decay process for a nucleus $X(Z, A)$ decaying to a daughter nucleus $Y(Z-2, A-4)$ will look something like:

$$(Z, A) \to (Z-2, A-4) + \alpha\,, \tag{53}$$
$$^{A}_{Z}\mathrm{X} \to {}^{A-4}_{Z-2}\mathrm{Y} + \alpha\,. \tag{54}$$

Let us start investigating this by noting that both the energy and the momentum are conserved. This will allow us to calculate the fraction of energy which goes into the kinetic energy of the α-particle and daughter nucleus.

Energy

The equation for the energy conservation of this process is (assuming the nucleus is at rest to begin with):

$$m_X c^2 = m_Y c^2 + m_\alpha c^2 + T_Y + T_\alpha\,, \tag{55}$$

where T_Y and T_α are the kinetic energies of the daughter nucleus and the alpha particle respectively. We now define a new value called the **Q-factor** This Q-factor is the energy associated with the change in mass (see Fig. 38):

$$Q = (m_Z - m_Y - m_\alpha)c^2 \tag{56}$$
$$= B_\alpha + B_Y - B_X \tag{57}$$
$$= B(2,4) + B(Z-2, A-4) - B(Z, A)\,. \tag{58}$$

The last line tells us the Q-factor is the binding energy after minus the binding energy before. So, more energy is released when the daughter nucleus is very stable.

Note:
A decay *can* occur only if $Q > 0$. This implies that

$$B(2,4) > B(Z,A) - B(Z-2, A-4) \sim 4\frac{dB}{dA} = 4\left(A\frac{d}{dA}\frac{B}{A} + \frac{B}{A}\right) < 28.3\,\mathrm{MeV} \tag{59}$$

where the last approximation is for $Z \sim N$ (i.e., the difference of a function value between nearby points is approximately its derivative divided by the distance between the points), and the second step converts to B/A which is what we are used to plotting. For large A the slope of B/A is about $-7.7\,\mathrm{keV}$ which implies

$$B/A \lesssim (7.1 + 7.7\times10^{-3}A)\mathrm{MeV} \tag{60}$$

which is satisfied for $A \gtrsim 150$ – see Fig. 15.

Momentum

As stated previously the momentum is conserved. This is a non-relativistic decay, and if we assume that the mother nucleus was stationary, then we can say:

$$T = \frac{p^2}{2m} \longrightarrow m_Y T_Y = m_\alpha T_\alpha\,. \tag{61}$$

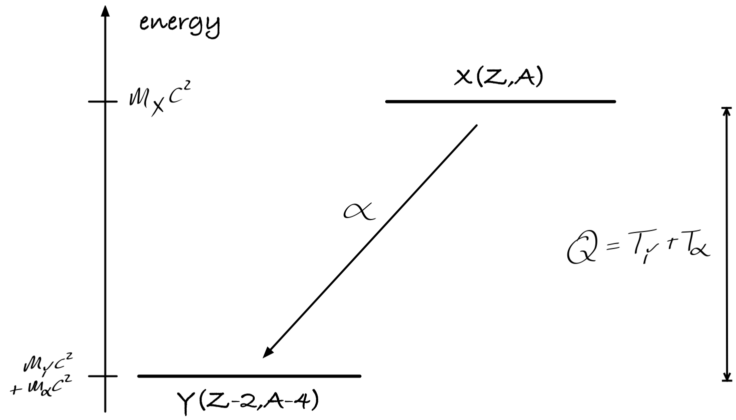


Figure 38: Energy diagram for α-decay.

Now,

$$\frac{m_\alpha}{m_Y} \simeq \frac{4}{A-4} \longrightarrow Q \simeq \left(\frac{4}{A-4}+1\right) T_\alpha = \left(\frac{A}{A-4}\right) T_\alpha \,. \tag{62}$$

(We have neglected the binding energies in the first approximation, and assumed $m_n \simeq m_p$.) Therefore,

$$T_\alpha \simeq \left(1-\frac{4}{A}\right) Q \,. \tag{63}$$

So, the heavier the nucleus the higher the kinetic energy of the α-particle.

Example:
What is the kinetic energy of α in ${}^{228}_{90}\mathrm{Th} \to {}^{224}_{88}\mathrm{Ra} + \alpha$?

Firstly we find (or lookup) the binding energies of the different elements which are:

$$\begin{aligned} {}^{228}_{90}\mathrm{Th} &= 1.743\,077\,\mathrm{GeV} \\ {}^{224}_{88}\mathrm{Ra} &= 1.720\,301\,\mathrm{GeV} \\ \alpha &= 28.296\,\mathrm{MeV}\,. \end{aligned}$$

With this we can then calculate Q:

$$Q = 1720.301 + 28.296 - 1743.077 \simeq 5.52\,\mathrm{MeV} \tag{64}$$

(Remember to carry lots of digits for these calculations!)

Finally we can then calculate T_α:

$$T_\alpha = \frac{224}{228} \times 5.52 = 5.42\,\mathrm{MeV}\,. \tag{65}$$

Typical lifetimes for α decays are about 10^{-7} s to 10^{10} yr! This is approximately correlated with the Q-value for heavy nuclei. In fact this correlation is so striking it was noticed over 100 years ago, and is known as the **Geiger-Nuttall rule**.

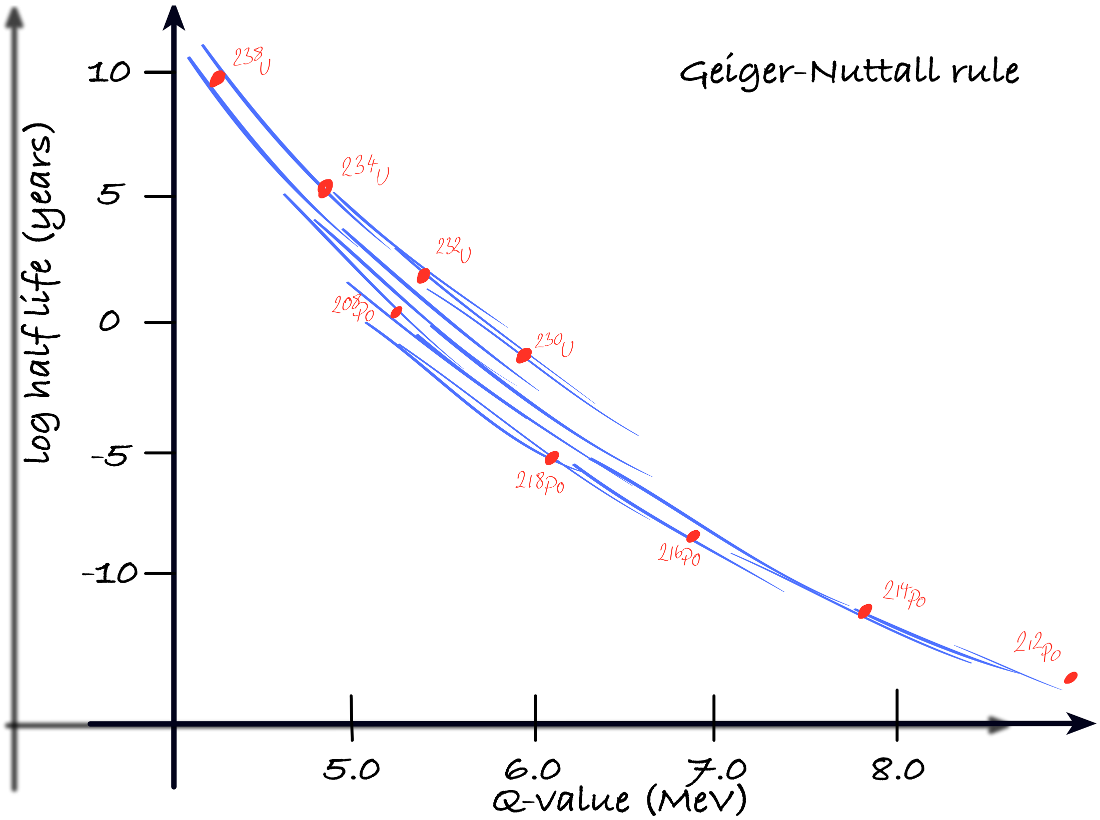


Figure 39: The strong correlation of lifetime of a decay versus the kinetic energy of the emitted α-particle for $A \gtrsim 200$. Note the massive range of lifetimes. Some example nuclei are shown.

3.2.1 Decay Mechanism and a calculation of $t_{1/2}(Q)$

Here we are going to try to build a model for α-decay, and see if we can understand where the Geiger-Nuttall rule comes from. The key insight for this is to note that decays happen only for very heavy nuclei. If we think in terms of the shell model, we can think of the 2 highest protons and neutrons forming a 'quasi-bound state'. In some sense then we can imagine the α-particle as ready formed trying to escape the nucleus. So we can think of the α particle moving in potential well of the daughter nucleus.

The key point is that the α particle can tunnel through the potential barrier which is a Quantum Mechanical process! We can calculate the **transition probabilities**, and it turns out that if the Q-factor is higher, there is a higher probability of tunnelling – we can see this from Fig. 41. This in turn lowers the half life ($t_{\frac{1}{2}}$), just because the process is more likely.

To calculate the probability of barrier penetration we start with:

$$P = e^{-2G}\,, \tag{66}$$

where the Gamow factor is:

$$G = \sqrt{\frac{2m_\alpha}{\hbar^2}} \int_R^b (V(r) - Q)^{\frac{1}{2}} dr\,. \tag{67}$$

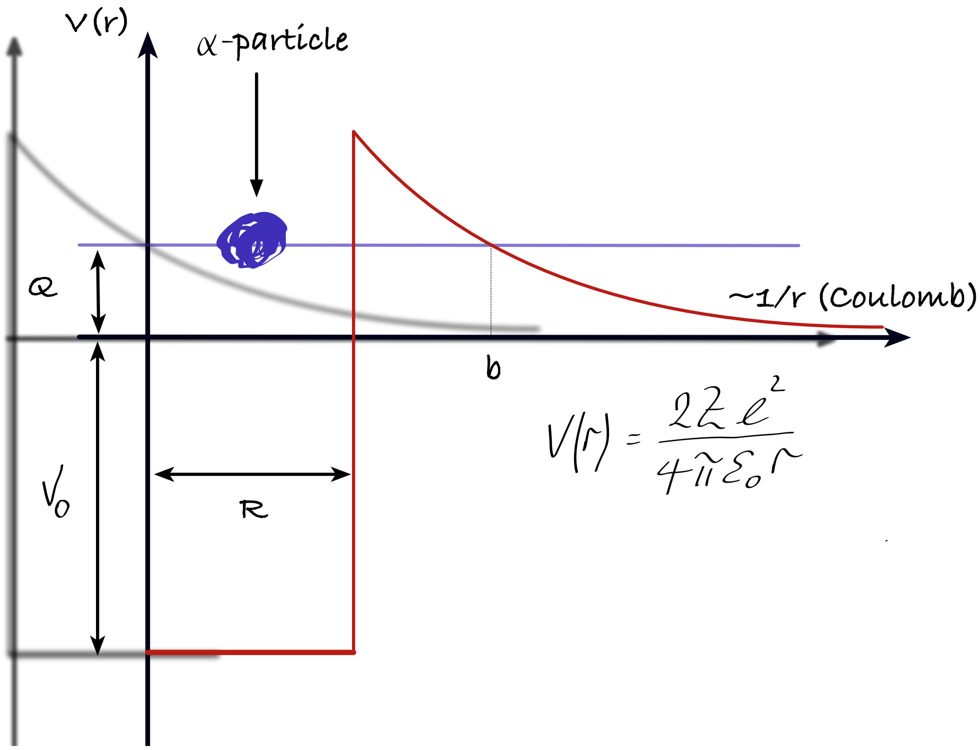


Figure 40: Approximate potential for a preformed α-particle in a nuclear potential. Its Q-value will be below the top of the potential barrier. At large distances the potential is just a Coulomb potential.

Since

$$V(r) = \frac{2Ze^2}{4\pi\epsilon_0 r}, \tag{68}$$

we can say,

$$V(b) = Q = \frac{2Ze^2}{4\pi\epsilon_0 b}, \tag{69}$$

and rearranging we find b in terms of Q which we use below,

$$b = \frac{2Ze^2}{4\pi\epsilon_0 Q}. \tag{70}$$

So, we can rewrite G as

$$G = \sqrt{\frac{2m_\alpha}{\hbar^2}\frac{2Ze^2}{4\pi\epsilon_0}} \int_R^b \left(\frac{1}{r} - \frac{1}{b}\right)^{\frac{1}{2}} dr\,. \tag{71}$$

If we use the substitution of $r = b\cos^2\theta$, the integral part of the above equation becomes,

$$\sqrt{b}\cos^{-1}\sqrt{\frac{R}{b}} - \sqrt{R - \frac{R^2}{b}} \approx \sqrt{b}\left(\frac{\pi}{2} - 2\sqrt{\frac{R}{b}}\right), \tag{72}$$

where the approximation applies when $b \gg R$ (which is usually true), using $\cos^{-1} x \approx \frac{\pi}{2} - x$, for

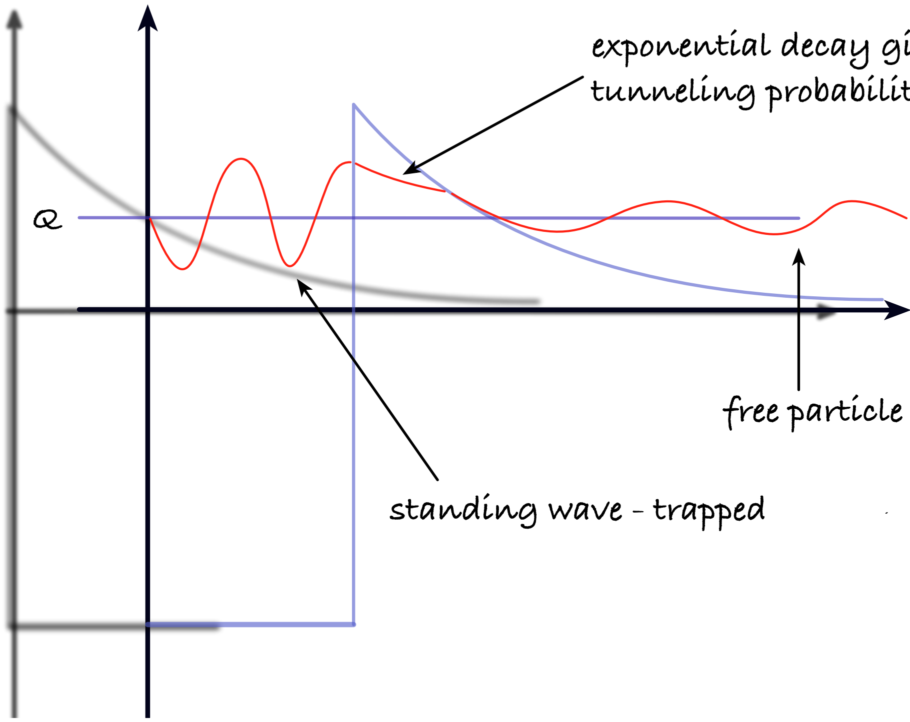


Figure 41: Wavefunction for the potential shown. For an energy Q below the top of the potential barrier there is still a tunnelling probability for the α-particle .

$x \ll 1$ and the binomial expansion on the second term. Then,

$$G \simeq 2\sqrt{\frac{m_\alpha Z e^2 b}{\pi\epsilon_0 \hbar^2}}\left(\frac{\pi}{2} - 2\sqrt{\frac{R}{b}}\right) . \tag{73}$$

How does this relate to half-life? Now, $\lambda = \dfrac{1}{\tau}$ = frequency of α 'hitting the barrier' $\times P$. This frequency is the travel time across the nucleus which is $\sim \frac{v}{2R}$. Since $\frac{1}{2}m_\alpha v^2 \sim Q + V_0$ inside the nucleus (not outside),

$$v \sim \sqrt{\frac{2T_\alpha}{m_\alpha}} \sim \sqrt{\frac{2(Q+V_0)}{m_\alpha}} . \tag{74}$$

This gives us an expression for half-life:

$$t_{\frac{1}{2}} \simeq 2\ln 2 \times R\sqrt{\frac{m_\alpha}{2(V_0+Q)}} \times \exp\left(\sqrt{\frac{2m_\alpha}{\hbar^2 Q}\frac{Ze^2}{\pi\epsilon_0}}\left(\frac{\pi}{2} - 2\sqrt{\frac{R}{b}}\right)\right) . \tag{75}$$

Note:

$$\frac{R}{b} = Q \times \frac{4\pi\epsilon_0}{2Ze^2} ,$$

so the form of $t_{\frac{1}{2}}$ is:

$$\sim \frac{1}{(\text{const.}+Q)^{\frac{1}{2}}} \exp\left(\frac{1}{Q^{\frac{1}{2}}} - \text{const.}\right).$$

Now all we need to do is tidy up! To do this we use:

$$\text{fine structure constant } \alpha = \frac{e^2}{4\pi\epsilon_0 \hbar c} = \frac{1}{137} \tag{76}$$

$$m_\alpha c^2 \simeq 3.7\,\text{GeV}, \tag{77}$$

$$R \simeq 1.2\,\text{fm} \times A^{\frac{1}{3}}, \tag{78}$$

$$\therefore \frac{R}{c} \simeq 0.4 \times 10^{-23}\,\text{s} \times A^{\frac{1}{3}}, \tag{79}$$

$$\frac{4\pi\epsilon_0}{2e^2} = \frac{1}{2}\frac{1}{\alpha\hbar c} = \frac{1}{2\alpha} \times \frac{1}{197.33\,\text{Mev fm}} \simeq 0.347\,\text{MeV}^{-1}\,\text{fm}^{-1}, \tag{80}$$

$$\therefore \frac{R}{b} = \frac{Q}{\text{MeV}}\frac{A^{\frac{1}{3}}}{Z} \times 0.416. \tag{81}$$

For the term $\sqrt{1/(V_0+Q)}$ we can use $V_0 \sim 35\,\text{MeV}$, and since the Q dependence of this term will be very weak, we can just take $Q \sim 5\,\text{MeV}$ for this part. Then,

$$t_{\frac{1}{2}} \sim 5 \times 10^{-23} A^{\frac{1}{3}} \exp\left(\frac{2.5Z}{\sqrt{Q}}\left(1.57 - 1.3\sqrt{\frac{A^{\frac{1}{3}}}{Z}Q}\right)\right)\text{s}, \tag{82}$$

where the Q-factor is in MeV. This reproduces the **Geiger-Nuttal** Rule. It is sometimes written as,

$$\log_{10}\frac{t_{\frac{1}{2}}}{1\,\text{s}} \sim -22.3 + 0.14\ln A - 1.4A^{\frac{1}{6}}\sqrt{Z} + 1.72\frac{Z}{\sqrt{Q}}\text{MeV}^{\frac{1}{2}}, \tag{83}$$

or,

$$\ln(\lambda \times 1\,\text{s}) \sim 128 - 3.97\,\text{MeV}^{\frac{1}{2}}\frac{Z}{\sqrt{Q}}. \tag{84}$$

3.3 β-Decay

β-Decay is the set of decays which involve electrons e^- and their anti-particle, the positron e^+. From conservation of charge this means turning neutrons into protons and vice versa. This is not a nuclear reaction of the sort we have been looking at, and instead involves a new force of nature called the **weak force**. First though, we can predict a new particle!

Consider the decay,

$$ {}^{14}_{6}\mathrm{C} \rightarrow {}^{14}_{7}\mathrm{N} + e^- \qquad (5730\,\mathrm{yr}) \tag{85}$$

The Q-factor for this appears to be 0.15 MeV. In the case of α-decay, that number would be split precisely between the kinetic energies of the decay products with an exact prediction, which is just a result of conservation of energy and momentum. However, this decay brings some puzzles which tells us that, as it's written, it's not the whole picture. Firstly the e^- is measured with a continuous spectrum of energies, not discrete as in α-decay – see Fig. 42. Another problem is that when we consider spins it goes from 0 to $1 \pm \frac{1}{2}$. This means that angular momentum appears not to be conserved!

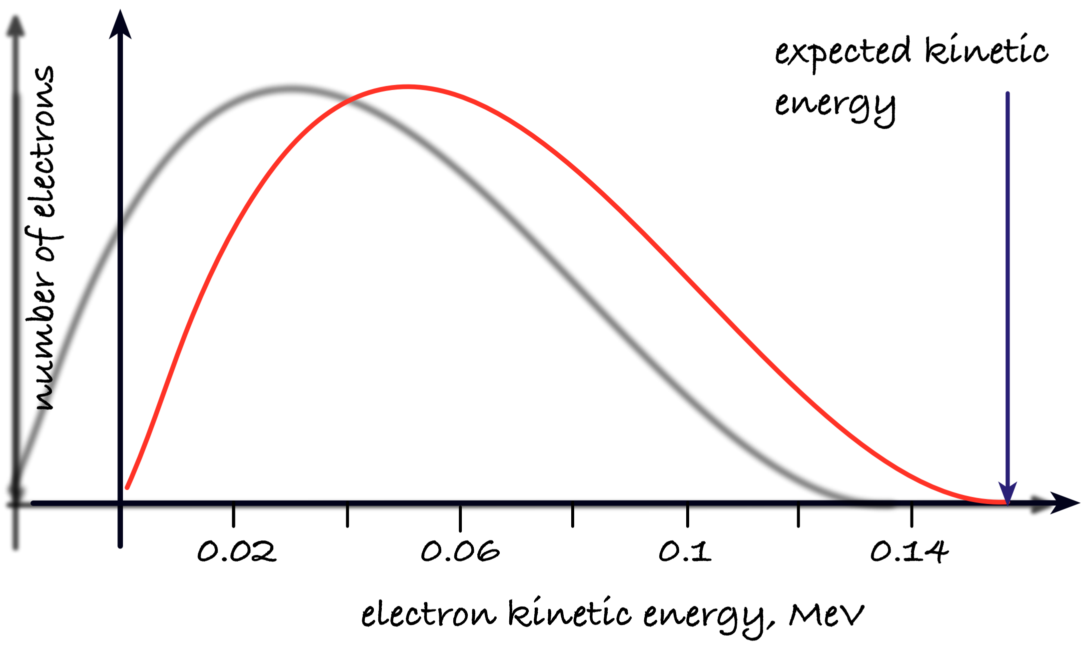


Figure 42: Distribution of the kinetic energies of the electron emitted in the decay ${}^{14}_{6}\mathrm{C} \rightarrow {}^{14}_{7}\mathrm{N} + e^-$. If there were no other particle involved this would be a sharp spike (shown).

The solution to this was proposed by Pauli, who postulated a new, very light, neutral, spin-1/2, particle which is also emitted in the process. This is the **neutrino** ν_e, with its anti-particle, the **anti-neutrino** $\bar{\nu}_e$. In the reaction above its an anti-neutrino which is emitted. The idea was that the undetected anti-neutrino carries some energy away shared with the e^- which causes the continuous spectrum, because a different fraction of energy goes into the electron and anti-neutrino for each decay. For this to work, neutrinos should have a spin of $\frac{1}{2}$ and no charge. It should also not interact with the nuclear (strong) force, like the e^-. This makes them very hard to detect, and were not detected until the 1950's.

There are 3 possible decays, which all have A=const and lifetimes of $\sim 1\text{s} \to 10^{10}$ yr:

β^-:

$$(Z, A) \to (Z+1, A) + e^- + \bar{\nu}_e\,. \tag{86}$$

A neutron converts into a proton, an electron and an electron type anti neutrino. The decay

$$n \to p + e^- + \bar{\nu}_e + 0.782\,\text{MeV}$$

happens naturally for free neutrons with $\tau = 898\,\text{s}$. $Q_\beta = 0.782\,\text{MeV}$ is the total kinetic energy of $p + e^- + \bar{\nu}_e$, and is shared among them.

β^+:

$$(Z, A) \to (Z-1, A) + e^+ + \nu_e\,. \tag{87}$$

A proton converts into a electron, a positron and an electron type neutrino. This can only happen for a proton inside a nucleus, as the decay

$$p \to n + e^+ + \nu_e \tag{88}$$

has a negative Q-value.

ε:

$$e^- + (Z, A) \to (Z-1, A) + \nu_e\,. \tag{89}$$

Electron capture, usually from lowest shell in atom. The electron and a proton converts into a neutron. Again, this can only happen for a proton inside a nucleus (which usually has some electron shells filled), as the decay

$$e^- + p \to n + \nu_e \tag{90}$$

has a negative Q-value.

Energetics

β-decays are *possible* only if the total mass on the left hand side is greater than the right hand side. We typically write the Q-value using the initial and final **nuclear** masses. We have to be really careful about using atomic versus nuclear masses in these calculations. To use atomic masses we need to start from the basic formula for the Q values – which is in terms of nuclear masses – and carefully convert to atomic masses. This is because atomic mass is

$$m(Z, A) = M_N(Z, A) + Zm_e + \text{electron binding energy term}$$

The electron binding energy term is O(keV) and can be ignored.

Example:
In β^--decay,

$$Q_{\beta^-} = [M_N(Z, A) - M_N(Z+1, A) - m_e - m_{\bar{\nu}_e}]c^2 \tag{91}$$

$$\simeq (m(Z, A) - m(Z+1, A))c^2\,, \tag{92}$$

if we treat the mass of the neutrino as 0 and ignore the binding energy of the electrons.

Similarly in β^+-decay (Note: $m_{e^+} \equiv m_{e^-}$)

$$Q_{\beta^+} \simeq (m(Z,A) - m(Z-1,A) - 2m_e)c^2 \,. \tag{93}$$

using atomic masses is often easier but you have to take care! What is the equivalent formula for ε-capture?

Note that Q-values for $p \to n + e^+ + \nu_e$ and $e^- + p \to n + \nu_e$ are negative and therefore there is no electron capture in ^{1}H (otherwise we would never find neutral Hydrogen). This also means that free protons are stable, in contrast to neutrons.

Note:
You have to carry lots of digits in these calculations. The reason is that nuclear masses are typically GeV while the mass of the electron is

$$m_e c^2 \approx 0.511\,\text{MeV}\,.$$

The energy release in a decay is typically a few MeV or less, so the mass differences of the nuclei involved in the decays will come at the 4th or 5th significant figure.

3.3.1 The Valley of Stability

β-decay helps us finally understand the origin of the valley of stability, which we saw earlier in when we were exploring the SEMF. Have a look at Fig. 43.

β-decay occurs along **isobars** because A remains constant. We looked at this case before. Recall the SEMF with the Z dependence explicit:

$$M(Z,A) = \alpha - \beta Z + \gamma Z^2 - \frac{\delta}{c^2}\,, \tag{94}$$

where,

$$\alpha = a_S A^{\frac{2}{3}} + A(a_A - a_V) + m_n\,,$$
$$\beta = \frac{a_C}{A^{\frac{1}{3}}} + 4a_A - m_p + m_n\,,$$
$$\gamma = \frac{a_C}{A^{\frac{1}{3}}} + 4\frac{a_A}{A}\,.$$

This is a parabola with a minimum at $Z_{\text{min}} = \frac{\beta}{2\gamma}$ (ignoring δ). For even-A nuclei where the pairing term is important, this is a family of two parabolas. (Make sure you understand why this is the case.)

Lets investigate how β-decay looks on an isobar, using the SEMF as a guide. We shall need to look at odd and even-A separately. It will be useful to recall Figs. 20 and 21 at this point – these were rough sketches of masses along isobars using the SEMF. We shall now look at these **mass chains** in 2 real examples.

First consider the case of A odd, and investigate the mass chain $A = 111$ – Fig. 44. Nuclei to the left of ^{111}Cd decay via β^--decay, with ever longer half-lives as they move closer to the valley of stability. Nuclei to the right of ^{111}Cd decay in two possible ways via β^+-decay or electron capture. Again, the half-lives get shorter the further from ^{111}Cd.

In Fig. 45 we show the same picture but now for an even mass chain. The zig-zag placing of the points is from the pairing term (see Fig. 21), with odd-odd nuclei lying on a higher parabola than even-even. Decays happen only towards lower masses indicated by the arrows (as in Fig 44). The key difference with the A-odd case is that *there can be two stable nuclei in an even mass chain*. In this example, ^{102}Ru and ^{102}Pd are stable. Furthermore, theres a nucleus which can potentially decay in all 3 types of β-decay – this is ^{102}Rh.

Here there is the potential for **double β-decay**. It is energetically possible for a decay via $^{102}\text{Mo} \to {}^{102}\text{Ru} + 2e^- + 2\bar{\nu}_e$, but this has never been observed. This kind of decay has only been observed in 10 isotopes, the first being ${}^{82}_{34}\text{Se} \to {}^{82}_{36}\text{Kr} + 2e^- + 2\bar{\nu}_e \quad (\sim 10^{20}\,\text{yr})$. Double electron capture ($2e^- + {}^{102}\text{Pd} \to {}^{102}\text{Ru} + 2\nu_e$) could also occur but also has never been observed!

In summary, for β-decay far from Z_{min}, nuclei decay towards the most stable nucleus.

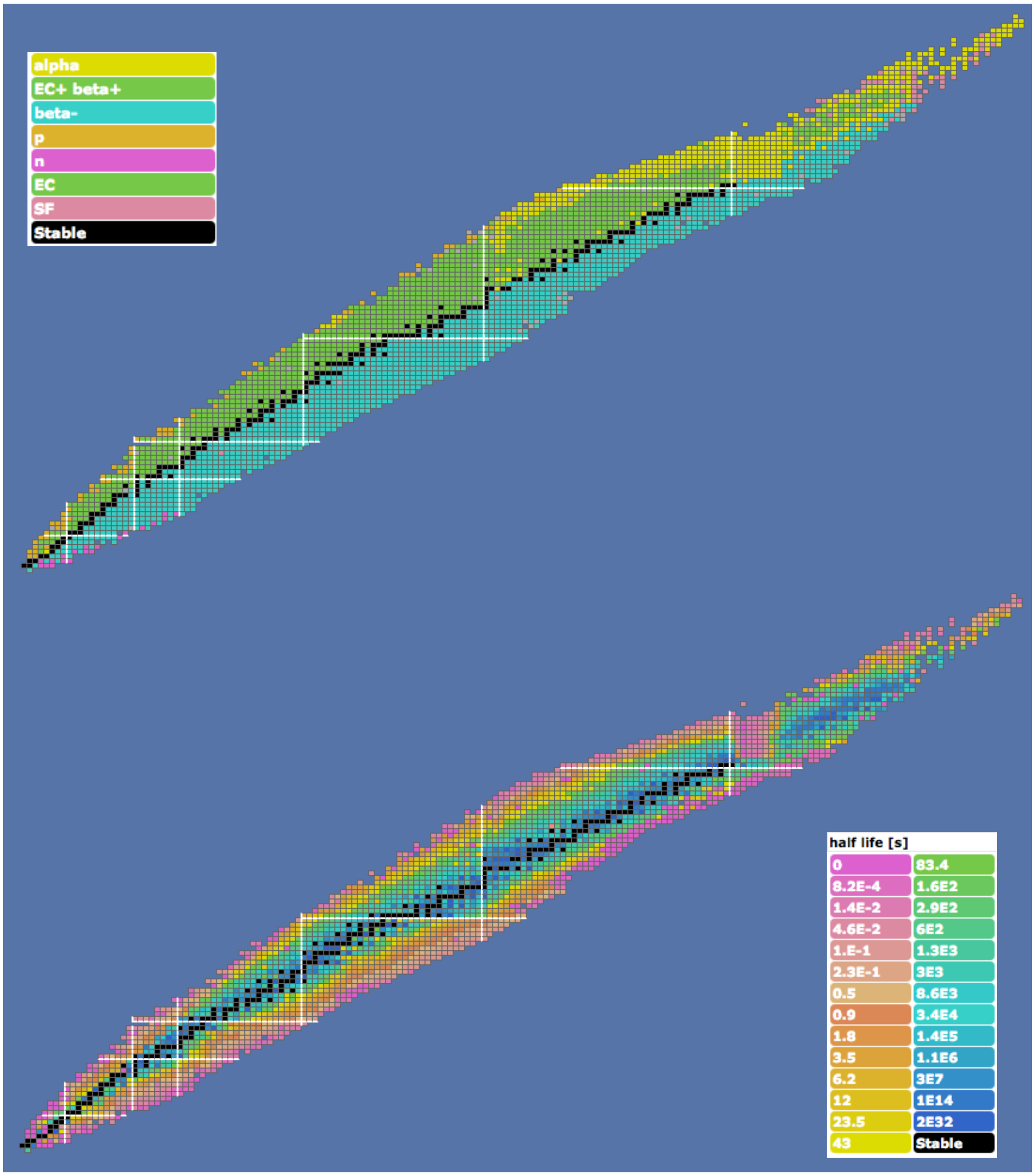


Figure 43: The table of nuclides shown in two colour schemes. On the top we have it coloured by the main decay mode. Nuclei which are coloured black do not decay – this is the valley of stability. What is striking is that above this line most nuclei are green which is for β^+-decay or ε-capture, while below is mostly blue which is β^--decay. (You can see α-decay in yellow for very heavy nuclei.) On the bottom it's coloured by half life – note how far from the valley of stability the lifetimes are very short, while near it they are much longer.

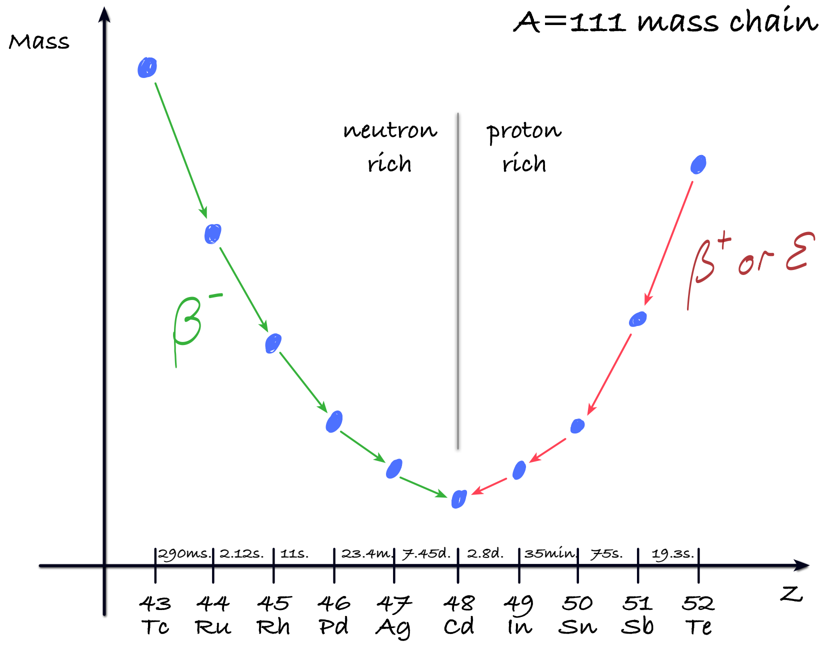


Figure 44: β-decay for an odd mass chain. Note the relative positions of each nuclide is accurate.

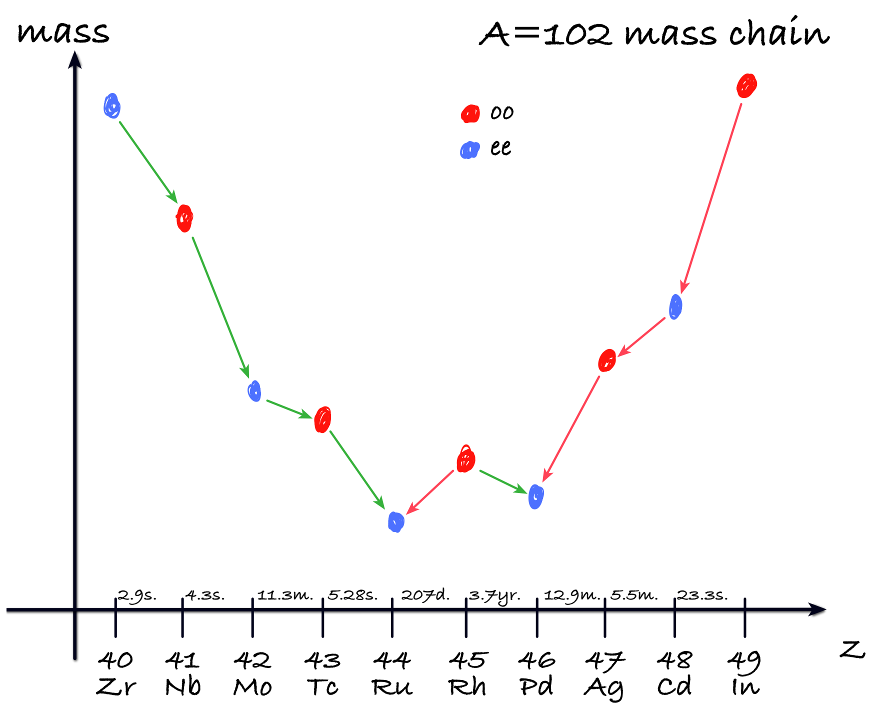


Figure 45: β-decay for an even mass chain. Note the relative positions of each nuclide is accurate. To the right of ^{111}Cd the half-lives shown are for ε-decay.

3.3.2 Neutrinos, Leptons and Weak Force

Understanding of β-decay actually requires introducing a new force of nature - the **weak force**.

$\bar{\nu}_e$, e^- are part of the family of **leptons**: these have spin $\frac{1}{2}$ (they are **fermions**); they do **not** feel strong force (unlike quarks). Charged leptons e^-, muon μ^-, tau τ^- each has associated neutrino ν_e, ν_μ, ν_τ. They are neutral, and have 6 **flavours** with 3 **generations**. They each have an anti-particle pair and interact via the **weak force** as well as the electromagnetic force.

Key Point: Lepton number is **conserved**:

$$L_e = 1 \text{ for } e^-,\ \nu_e \tag{95}$$

$$L_e = -1 \text{ for } e^+,\ \bar{\nu}_e \tag{96}$$

L_e is conserved in β-decay! Note: Parity is **not** conserved.

For β-decay, the key process is interaction with **quarks**. Neutrons and protons are formed with 3 up or down quarks

> ***Note:***
> Neutrinos are **very** light, hardly interact and are very hard to detect. This is due to their tiny **cross section**

3.4 γ-Decay

Nuclei in excited states decay to lower energy states via γ-ray emission.

- $E_\gamma \sim 0.1 - 10\,\text{MeV}$
- Wavelength $\sim 100 - 10^3\,\text{fm}$
- Lifetime $\sim 10^{-12}\,\text{s}$ unless ????stable $\sim 10^{-9}\,\text{s}$
- Rich Structure which allows for γ-ray spectroscopy to probe nucleus structure.
- Decay rate $\lambda \simeq 10^5 E_\gamma^3 A^{\frac{2}{3}}$ ($\propto \int d^3r \psi_f^* e^{ik\cdot r} \psi_i$)

Transitions are subject to selection rules. These include the γ carries spin 1 so that angular momentum is conserved. This means that only certain transitions are allowed. If J_i is initial spin of nucleus, $J_f = J_i \pm 1$ (dipole) or $J_f = J_i \pm 2$ etc. For example $J_i = 0$ and $J_f = 0$ are forbidden.

4 Fission and Fusion

"You know what uranium is, right? It's this thing called nuclear weapons. And other things. " – Donald J. Trump

We now examine the processes by which nuclei can break apart or bond together. The first, **fission**, happens mainly in heavy nuclei and can release a lot of energy if the daughter nuclei have higher binding energies. It's mainly interesting for engineering purposes, as a potential fuel source, but is also used in ridiculous nuclear weapons.

The process whereby nuclei can join together and release energy is **fusion**, and understanding the processes by which this happens allows us to understand the **origin of the elements** which evolved from the primordial soup of fundamental particles in the **big bang**.

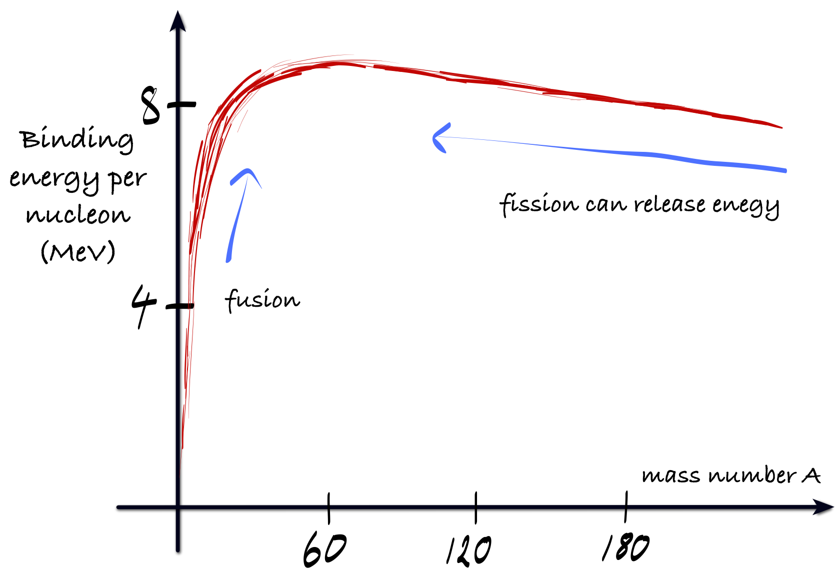


Figure 46: The binding energy per nucleon curve. To the right of the peak fission can release energy, splitting nuclei into two; to the left, fusion of two light nuclei can release energy.

4.1 Fission

We can see from the binding energy curve that heavy nuclei can break into two and release energy in the process – in principle. For example if ^{238}U splits into 2 with $A \simeq 118$, $\sim 214\,$MeV must be released (mainly into kinetic energy, though other particles can be emitted). **Spontaneous fission** decay modes are usually **improbable** however – why?

This is actually due to the Coulomb barrier:

Example:
Just after fission we have 2 ^{119}Pd 'touching', which would have a Coulomb repulsion

$$\begin{aligned} V =& \frac{e^2}{4\pi\epsilon_0}\frac{Z_1 Z_2}{R} \\ =&(1.44\,\text{MeV fm})\cdot\frac{(46)^2}{2\times 1.2\,\text{fm}\,(119)^{\frac{1}{3}}} \\ =&250\,\text{MeV} \end{aligned}$$

where we have assume the radius of the nuclei is $R \approx 1.2A^{1/3}$ fm. This barrier would have to be tunnelled through to release 214 MeV of energy, which makes this event pretty unlikely! In fact, the half life is $\sim 10^{16}$ yr!

Actually, spontaneous fission is unlikely to produce equal mass fragments like this, instead we find a distribution of masses, as in Fig. 47

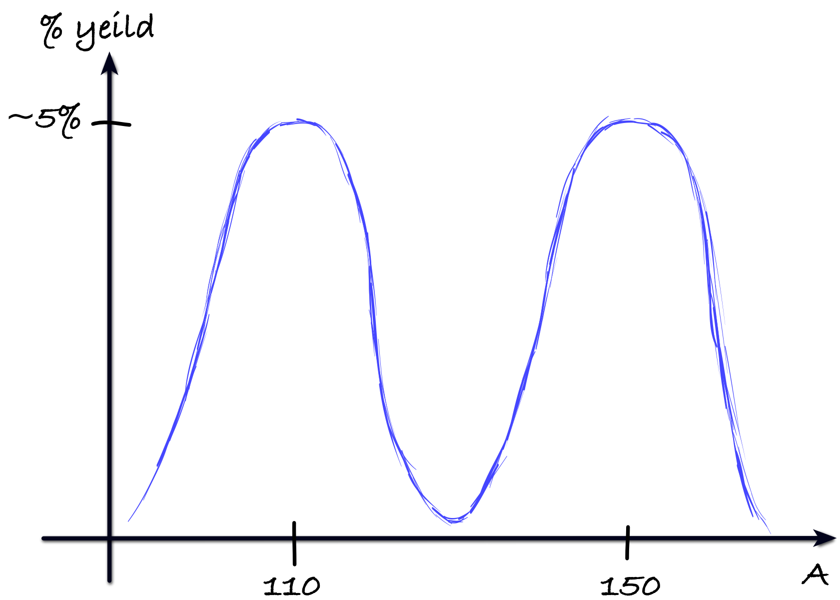


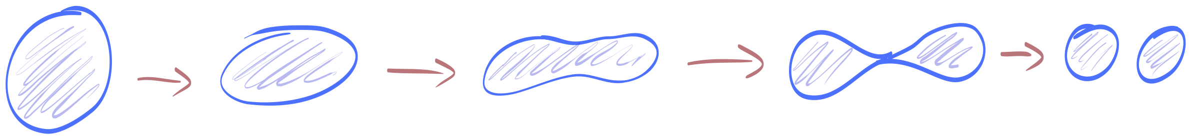
Figure 47: Sketch of a typical mass distribution for the fragments of a spontaneous fission event from a nuclei with $A \sim 250$.

So for ^{238}U a more typical reaction will be:

$$^{238}_{92}\text{U} \rightarrow {}^{145}_{57}\text{La} + {}^{90}_{35}\text{Br} + 3n \tag{97}$$

But this is still very rare!

When might spontaneous fission be more likely? To answer this we use the Semi-Empirical Mass Formula! The SEMF assumes spherical symmetry which minimises energy. If fission happens we expect by analogy with liquid drop for this to happen:



Essentially the drop is deformed initially then splits into two smaller drops. There must be something happening for a nuclei which is about to fission such that the defamed state is energetically preferred to the spherical state. Say we had a spherical drop of water: if it's deformed it bounces back to a spherical shape quickly because of the **surface tension** from all the water molecules pulling towards each other at the surface of the drop which act like a coating membrane – this acts to minimise the surface area. Of course this changes if you add charge to the drop – add enough and it will split apart!

The same idea works for the nucleus. In fact we can use this simple idea to work out roughly for which nuclei spontaneous fission might happen in. For this we need to calculate when the squashed shape might be energetically preferred over the spherical shape. We can do this using **perturbation theory**. This is a calculation method where one essentially performs a Taylor expansion about a simple model, and see what happens at 'lowest order' in the expansion.

So, let's deform the sphere into an ellipsoid, maintaining rotational symmetry about 1 axis for simplicity. Let's define the long axis radius a and the two short ones b. Crucially we don't change the volume when we do this, so $V = \frac{4}{3}\pi ab^2$ is preserved, and therefore $ab^2 = R^3$. If we let $a = R(1+\epsilon)$, $b = \frac{R}{(1+\epsilon)^{\frac{1}{2}}}$, where $\epsilon \ll 1$ how would the SEMF change?

The surface and Coulomb terms change, with $\epsilon \ll 1$, according to

$$E_S = a_S A^{\frac{2}{3}} \left(1 + \frac{2}{5}\epsilon^2 + ...\right) \tag{98}$$

$$E_C = a_C Z^2 A^{-\frac{1}{3}} \left(1 - \frac{1}{5}\epsilon^2 + ...\right) \tag{99}$$

So the change in energy is,

$$\Delta E = (E_S + E_C)_{\text{ellipse}} - (E_S + E_C)_{\text{SEMF}} \tag{100}$$

$$\simeq \frac{\epsilon^2}{5}(2a_S A^{\frac{2}{3}} - a_C Z^2 A^{-\frac{1}{3}}) \tag{101}$$

$$< 0 \quad \text{for spontaneous fission}\,. \tag{102}$$

Therefore,

$$\frac{Z^2}{A} \gtrsim \frac{2a_S}{a_C} \simeq 49 \tag{103}$$

$$\text{With } \frac{Z}{A} \sim 0.4 \to Z \gtrsim \frac{50}{0.4} \sim 120\,. \tag{104}$$

So we see that the surface term acts to pull the shape back to a sphere, and the Coulomb term acts to break it apart. (It might surprise you to look back at the SEMF and notice that the surface term is negative, when the effect of the surface nucleons is to pull the nucleus together, and help it maintain a spherical shape. Remember that the surface term is defined because the volume term over-counted the effect of the nucleons on the surface which have less neighbouring nucleons to interact with.)

Think of the potential energy as fission occurs – i.e., as a nucleus goes through the process of splitting into two pieces, Fig. 48. Unless the nucleus satisfies Eq. 104, the potential energy will

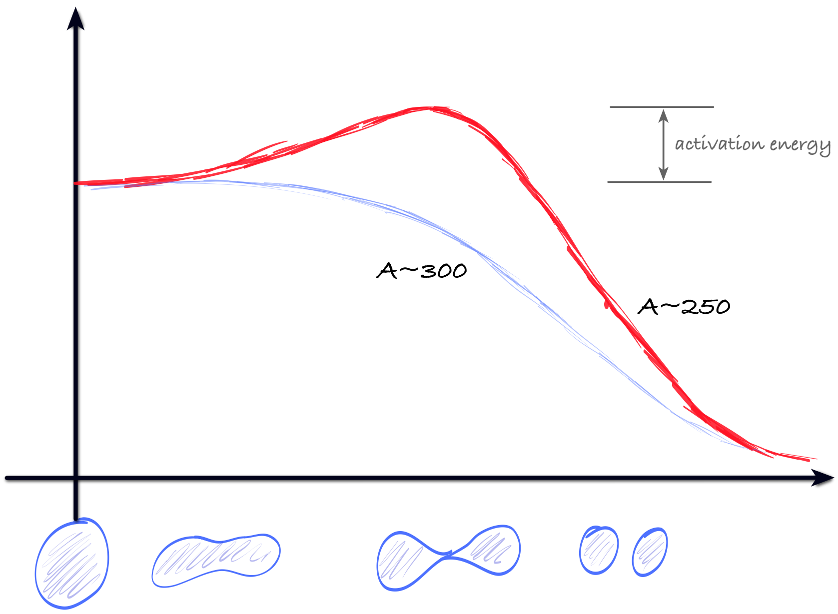


Figure 48: As a nucleus pulls apart an activation energy must be overcome unless it's really heavy.

increase as it's deformed – this is the **fission barrier**. Pull it apart further over the **activation energy**, and it will then fission. For very heavy nuclei satisfying Eq. 104, the activation energy goes to zero, and **spontaneous fission** becomes very likely.

The activation energy is small for nuclei with $\frac{Z^2}{A} \sim 50$, and only a small push is required to achieve this. Neutrons can supply the activation energy, as they can overcome the Coulomb repulsion easily. This leads to ...

4.1.1 Induced Fission

Neutrons don't see the coulomb barrier so they can penetrate the nucleus and bump it up over the activation energy for fission! For example low energy ('**thermal**') neutrons induce the reaction

$$^{235}\text{U} + n \to {}^{93}\text{Rb} + {}^{141}\text{Cs} + 2n\,.$$

This reaction is not unique, and usually undergoes α, β and γ decay chains. Some examples of the distribution of daughter nuclei is seen in Fig. 50

However a seemingly similar nucleus, ^{238}U requires high energy ('fast') neutrons with a kinetic

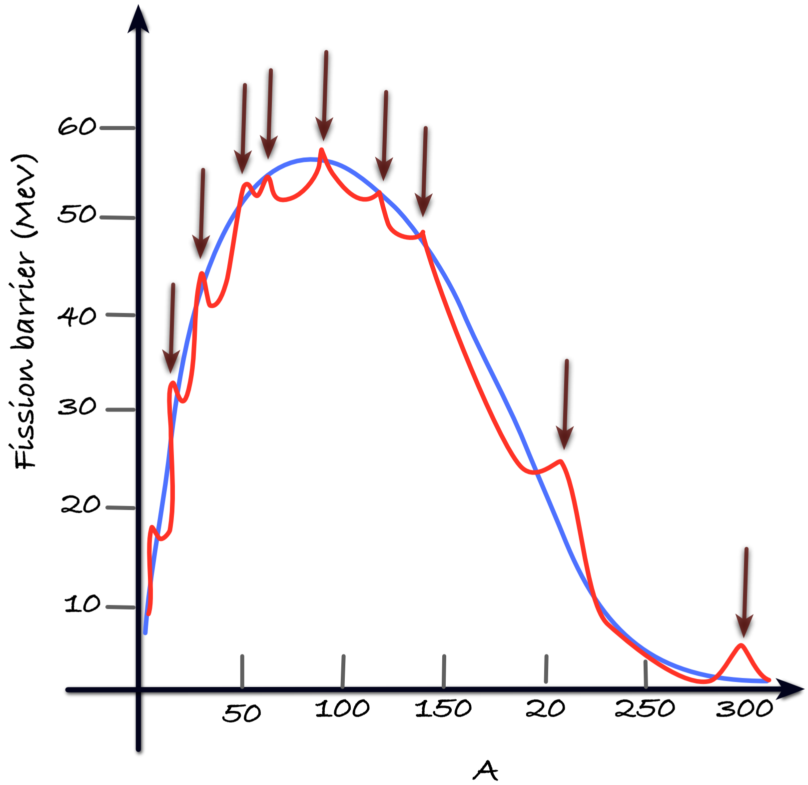


Figure 49: The activation energy as a function of mass number. The blue curve is from a (complicated) prediction from the SEMF, the red a more realistic description. Nuclei with magic number of neutrons and/or protons are indicated.

energy $> 1\,\mathrm{MeV}$. This produces more equal-mass fragments than ^{235}U. Why don't thermal neutrons induce fission in this case?

Energy in fission

If we compare the excited state after n capture to activation energy we get clearer idea of what is going on. The excitation energy is calculated by considering the mass of a nucleus plus a neutron, minus the mass of the new nucleus once the neutron is absorbed.

> ***Example:***
> For ^{235}U, the excitation energy is
>
> $$E_{ex} = [M(^{236}\mathrm{U}^*) - M(^{236}\mathrm{U})]c^2\,, \qquad (105)$$
>
> where, $M(^{236}\mathrm{U}^*) = M(^{235}\mathrm{U}) + m_n = 236.052\,539\,\mathrm{u}$ and $M(^{236}\mathrm{U}) = 236.045\,563\,\mathrm{u}$, which will give us
>
> $$E_{ex} = 6.5\,\mathrm{MeV} > \text{ Activation Energy } \simeq 6.2\,\mathrm{MeV}\,. \qquad (106)$$

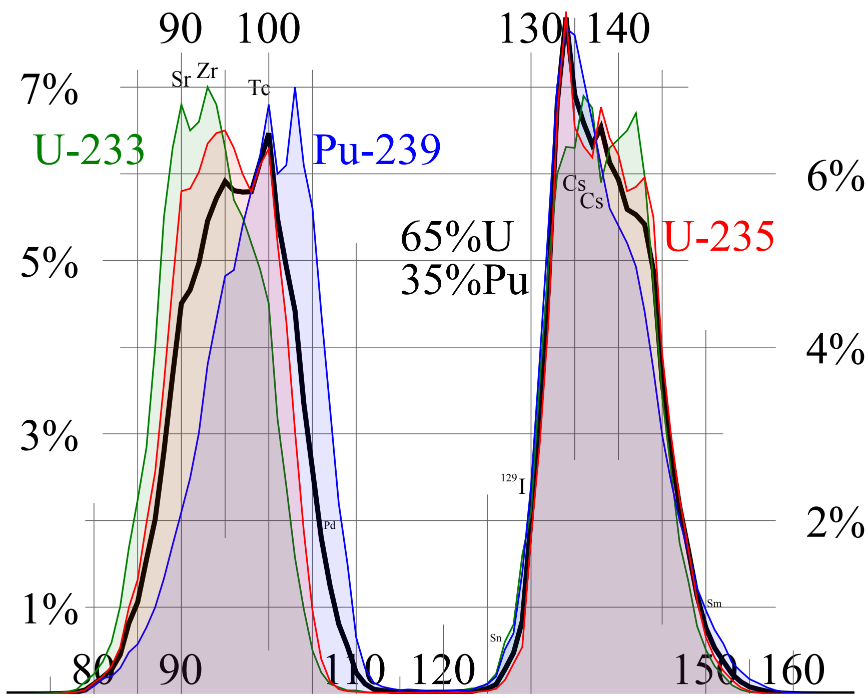


Figure 50: Distribution of fission fragments for a range of nuclei: Fission product yields by mass for thermal neutron fission of U-235, Pu-239, a combination of the two typical of current nuclear power reactors, and U-233 used in the thorium cycle. See https://en.wikipedia.org/wiki/Ternary_fission for more information.

However for $^{238}\text{U} + n$ it has,

$$E_{ex} = 4.8\,\text{MeV} < \text{ Activation Energy } \simeq 6.6\,\text{MeV}\,. \tag{107}$$

This means that the fast neutron absorbed will have to carry enough extra energy to overcome this activation energy.

This pattern in the example above is the same for large A. Odd-A nuclei with an odd number of neutrons like

$$^{233}_{92}\text{U},\ ^{235}_{92}\text{U},\ ^{239}_{94}\text{Pu},\ ^{241}_{94}\text{Pu}$$

are '**fissile**', which means that fission is induced from slow neutrons, because the binding energy once the neutron is absorbed becomes larger than the critical energy for fission to occur. On the other hand, even-A nuclei which are e-e like

$$^{232}_{90}\text{Th},\ ^{238}_{92}\text{U},\ ^{240}_{94}\text{Pu},\ ^{242}_{94}\text{Pu}$$

require energetic neutrons for fission. That is, if they absorb a slow neutron the increase in the binding energy is too low to overcome the activation energy. (These are fissionable but not fissile.)

This can be understood from the pairing term in the Semi-Empirical Mass Formula – nucleons like to pair. In the case of e-e nucleons, they are already comparatively stable so the extra neutron doesn't push them over the edge. However, the e-o nucleus ^{235}U, for example, changes to e-e, because the extra neutron pairs up with the last unpaired one. Now, the excitation energy is the binding energy of the nucleus once the neutron is absorbed (which would then be e-e) minus the binding energy of the initial nucleus in the excited state (which is e-o). This difference has an extra paring factor (from the e-e bit) which is positive – this results in a higher excitation energy by a factor $a_p/A^{1/2} \sim 2\,\text{MeV}$. This extra excitation energy can then activate fission. This is a bit counterintuitive because on the face of it the newly crated e-e should be more stable – but more energy released into the nucleus as it settles into this state which makes it fission.

Note that o-o nuclei are unstable to β-decay – see earlier.

Number of Emitted Neutrons

In a $^{235}\text{U} + n$ induced fission reaction, the fragments are usually contain a number of neutrons. These are shed at the instant of fission ($t \sim 10^{-16}\,\text{s}$) as **prompt neutrons** and the number of these are, for example,

$$\bar{\nu} \simeq 2.42 \quad \text{for} \quad {}^{235}\text{U} \quad \text{or,}$$
$$\bar{\nu} \simeq 2.86 \quad \text{for} \quad {}^{237}\text{Pu}\,.$$

The actual number is a Gaussian distribution about $\bar{\nu}$.

Also there are **delayed neutrons** following β-decay of fission fragments ($t \sim$ s), which have an intensity of $\sim 1\%$ of the total number.

4.1.2 Chain Reactions

In a lump of fissile material, the neutrons released hit other nuclei, which then releases more neutrons which then hit more nuclei and so on... This is called a **chain reaction**. To quantify this in more detail, we define

$$k = \frac{\text{number of neutrons produced in } (i+1)\text{'th stage}}{\text{number of neutrons produced in } i\text{'th stage}}\,. \tag{108}$$

If,

$$k < 1 - \text{sub critical} - \text{dies out}\,,$$
$$k = 1 - \text{critical} - \text{sustained reaction}\,,$$
$$k > 1 - \text{supercritical} - \text{energy grows rapidly!}$$

(In a fission bomb, $k > 2$)

One of the key contributions to k is the mean distance a neutron travels before hitting another nucleus:

$$\bar{l} = \frac{1}{\rho_{nuc}\bar{\sigma}_{tot}} \sim 3\,\text{cm for } {}^{235}\text{U}\,. \tag{109}$$

In the above equation, ρ_{nuc} is the density of the material. In the case of ^{235}U we have $\rho_{nuc} \sim 4.8 \times 10^{28}\,\text{nuclei/m}^3$. We also have $\bar{\sigma}_{tot}$ which is the total cross section, or probability of a collision. Again for ^{235}U we have $\bar{\sigma}_{tot} \sim 7\,\text{barnes}(700\,\text{fm}^2)$.

The time scale for fission is also important to k, which is $\sim 10^{-8}$ s. This is because we need several interactions, each had a probability of $q < 1$ of inducing fission. This leads us to deduce that the **critical size** of ^{235}U is ~ 7 cm!

Another thing we can do is calculate the number of neutrons as a function of time. First we define τ as the mean time before a neutron is absorbed. This depends on the material. We will also define k as the reproduction factor.

So if $\exists$ N neutrons at t, at $t + \tau \exists kN$; at $t + 2\tau \exists k \cdot (kN)$ neutrons etc.

In a short interval dt,

$$dN = (kN - N)\frac{dt}{\tau}\,, \tag{110}$$

which we can integrate to get,

$$N(t) = N_0 e^{(k-1)\frac{t}{\tau}}\,. \tag{111}$$

We can also use the equation for the energy rate, which is

$$dE = Q \times \frac{N}{\tau} dt\,, \tag{112}$$

Where Q is the Q per fission and τ is the mean number of absorbed neutrons. This allows us to get,

$$E \propto e^{(k-1)\frac{t}{\tau}}\,, \tag{113}$$

This means that if,

$$k = 1 \text{ – The energy output is constant}\,,$$
$$k < 1 \text{ – Chain reaction stops and the energy goes to } 0\,,$$
$$k > 1 \text{ – Exponential growth which leads to an explosion!}$$

4.1.3 Fission Reactors

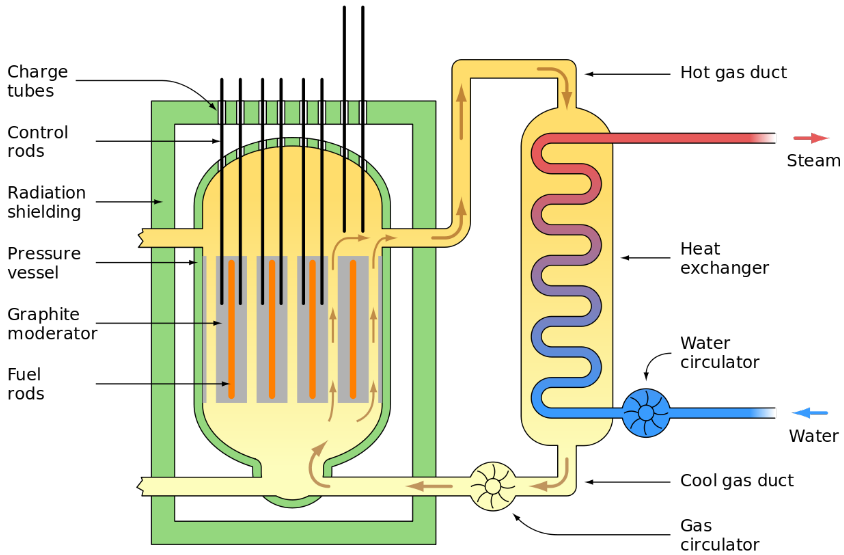
How do we set up a system to keep $k = 1$? The key idea is that the moderator absorbs prompt neutrons, and we allow $k = 1$ only from delayed neutrons. Prompt neutrons have $\tau \sim 10^{-3}$ s for a mix of ^{235}U and ^{238}U. The delayed neutrons have a $\tau \sim 13$ s, so we use the control rods to absorb neutrons if k gets too large.

4.1.4 Fission Bombs

These are very hard to make because you have to assemble a critical mass! Fissile material blows apart from thermal pressure as soon as the chain reactions which stops the process.

Little Boy – Hiroshima Gun assembly

In this bomb there was an explosion which pushed two subcritical masses together which makes one super critical mass. (90,000 - 160,000 people killed)



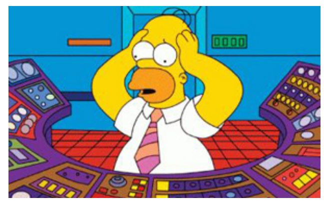


Figure 51: A schematic of a fission reactor and controller. The important bits for us are the moderator, fuel and control rods.

Fat Man – Nagasaki Implosion assembly

This involves a plutonium core which is subcritical surrounded by shaped explosives. These explosives create imploding shock waves which compresses the core to a critical density. (40,000 - 80,000 people killed. Half killed on first day and half killed over months from burns and radiation sickness.)

4.2 Fusion

Light nuclei are often weakly bound and energy can be released when they are fused into heavier nuclei. Example processes are:

$$
\begin{aligned}
{}^2_1\mathrm{H} + {}^2_1\mathrm{H} &\longrightarrow {}^4_2\mathrm{He} + \gamma \quad (Q = 23.6\,\mathrm{MeV})\,, &(114)\\
{}^2_1\mathrm{H} + {}^2_1\mathrm{H} &\longrightarrow {}^3_2\mathrm{He} + n \quad (Q = 3.3\,\mathrm{MeV})\,, &(115)\\
{}^2_1\mathrm{H} + {}^2_1\mathrm{H} &\longrightarrow {}^3_1\mathrm{H} + p \quad (Q = 4.0\,\mathrm{MeV})\,. &(116)
\end{aligned}
$$

These are deuterium-deuterium (D-D or d-d) reactions. The more stable the end product the more energy is released. (Why?) Fusion reactions in many stages are responsible for energy output from the sun and all other stars. It is also responsible for synthesising heavy elements up to about nickel and iron. (The synthesis of nuclei heavier than these requires extreme events such as Supernova or neutron star mergers.)

Fusion reactions have a huge energy release which means that it has great prospects for energy generation! The issue is that is very hard to get going, contain and sustain.

4.2.1 Energy Release

Energy output is the Q value of the reaction $X + a \longrightarrow Y + b$. The initial kinetic energy is small so:

$$Q \simeq \frac{1}{2} m_Y v_Y^2 + \frac{1}{2} m_b v_b^2\,. \tag{117}$$

We can also use the conservation of momentum to say that:

$$m_b v_b \simeq m_Y v_Y\,. \tag{118}$$

Combining these we can get the relation:

$$\frac{1}{2} m_b v_b^2 = \frac{Q}{1 + \frac{m_b}{m_Y}} \quad \text{and} \quad Y \leftrightarrow b\,. \tag{119}$$

4.2.2 Coulomb Barrier

Nuclei typically scatter off each other, which is the key problem in getting fusion to happen. This just because of the Coulomb repulsion between nuclei, where the barrier is roughly:

$$
\begin{aligned}
V_c &= \frac{e^2}{4\pi\epsilon_0} \frac{Z_a Z_X}{R_a + R_X} &(120)\\
&\simeq 1.198 \frac{Z_a Z_X}{A_a^{\frac{1}{3}} + A_X^{\frac{1}{3}}} \mathrm{MeV} \qquad [R \approx 1.2 A^{1/3}\,\mathrm{fm}] &(121)\\
&\simeq 0.15 A^{\frac{5}{3}} \mathrm{MeV} \quad \text{if} \quad A_a \simeq A_X \simeq 2Z_a \simeq 2Z_X\,. &(122)
\end{aligned}
$$

In D-D reactions this is $\simeq 0.5\,\text{MeV}$, which isn't that high, but for accelerated beams most particles will just scatter off each other. A way to overcome this is to heat the mixture of nuclei until thermal energy overcomes V_c. Naively this is something like

$$E = k_b T\,, \tag{123}$$

where k_b is the Boltzmann Constant ($\simeq 8.62 \times 10^{-11}\,\text{MeV/K}$). This estimate means that we'd require a gas with $T \sim 10^{10}\,\text{K}$ which is huge: for example, the Sun's interior is only $\sim 1.5 \times 10^{6}\,\text{K}$. How does this work?

Fusion occurs at lower temperatures for 2 reasons:

1. *Quantum Tunnelling:*

Particles with energies which are below V_c can tunnel through the potential barrier. This is a bit like the reverse of α-decay – have a look at Figs. 40 and 41. The probability of this happening is $= e^{-2G}$ where:

$$G = \frac{e^2}{4\pi\epsilon_0\hbar c}\frac{\pi Z_a Z_X}{v/c}\,, \tag{124}$$

where

$$\alpha = \frac{e^2}{4\pi\epsilon_0\hbar c} \tag{125}$$

is the **fine structure constant** $\alpha \simeq \dfrac{1}{137}$.

We can also write this in terms of the **Gamow energy** E_G:

$$G = \frac{1}{2}\sqrt{\frac{E_G}{E}} \quad \text{where,} \quad E_G = 2mc^2(\pi\alpha Z_a Z_X)^2, \quad m = \frac{m_a m_X}{m_a + m_X}\,. \tag{126}$$

Here, m is the **reduced mass** of the particles, which is the effective mass of the two particles.

Therefore as the energy increases, so does the probability of tunneling. But for example, $p + p$ at $10^7\,\text{K}$ have $E_G \sim 0.5\,\text{MeV}$ and $E \sim \text{keV}$, which leads to a probability of about 10^{-10}! (To understand this function, it's worth making a plot of $e^{-1/\sqrt{x}}$.)

2. *Thermal Distribution is a Maxwell-Boltzmann distribution*:

In an ideal gas the distribution of the velocities of the individual particles follows a statistical distribution known as a **Maxwell-Boltzmann distribution** (or Maxwellian). A gas is a collection of particles flying about in random directions – some of these have small velocities, some have large velocities, most have somewhere in the middle, with a kinetic energy near $k_B T$. The distribution clearly must depend on the temperature (higher temperature means the particles must be moving faster on average), and the velocity distribution must depend on the mass of the particles (higher mass means smaller velocity for the same kinetic energy. The distribution of these velocities is a **probability distribution** which we write as:

$$P(v)dv = \sqrt{\frac{2}{\pi}}\left(\frac{m}{kT}\right)^{\frac{3}{2}} e^{-mv^2/2k_B T} v^2\,dv\,, \tag{127}$$

Here, $P(v)$ is the probability that 2 nuclei picked at random have a relative velocity in the range $v \to v + dv$. (This is derived from the **Boltzmann equation**, which describes how a system of many particles evolves in **phase space**.) Note that the exponential part is the ratio of the particle's kinetic energy to the thermal energy of the gas. The function is shown in Fig. 52. The key point is

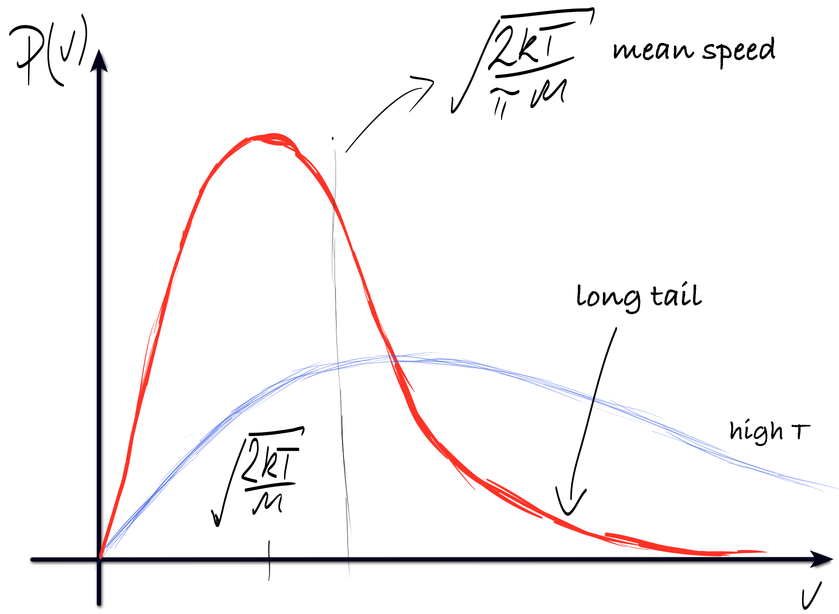


Figure 52: A sketch of the Maxwell Boltzmann distribution, with peak and mean indicated. (Can you derive them?)

that *some nuclei have a very high relative velocity compared to the mean velocity*. This is seen in the long tail of the distribution.

If n_X and n_a are number densities of two species which can undergo fusion, their reaction rate is given by:

$$R_{Xa} = n_X n_a \langle \sigma v \rangle \,, \tag{128}$$

where σ is the fusion cross section which you will recall is basically the probability of collision followed by tunnelling. Here the angle brackets refer to averaging according to the Maxwell-Boltzman probability distribution. This average product is given by:

$$\langle \sigma v \rangle = \int_0^\infty dv \; v \sigma P(v) \,. \tag{129}$$

Since σ is the the probability of collision and tunnelling, this means $\sigma \propto \frac{1}{v^2} e^{-2G}$. Therefore, the reaction rate depends on a convolution of both the Maxwell-Boltzman distribution, and the probability of tunnelling.

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So,

$$R \sim \langle \sigma v \rangle \propto \int_0^\infty e^{-2G} e^{-\frac{mv^2}{2kT}} v \, dv \tag{130}$$

$$\propto \sqrt{\frac{8}{\pi m}} \frac{1}{(kT)^{\frac{3}{2}}} \int_0^\infty e^{\left(-\sqrt{\frac{E_G}{E}} - \frac{E}{kT}\right)} dE \tag{131}$$

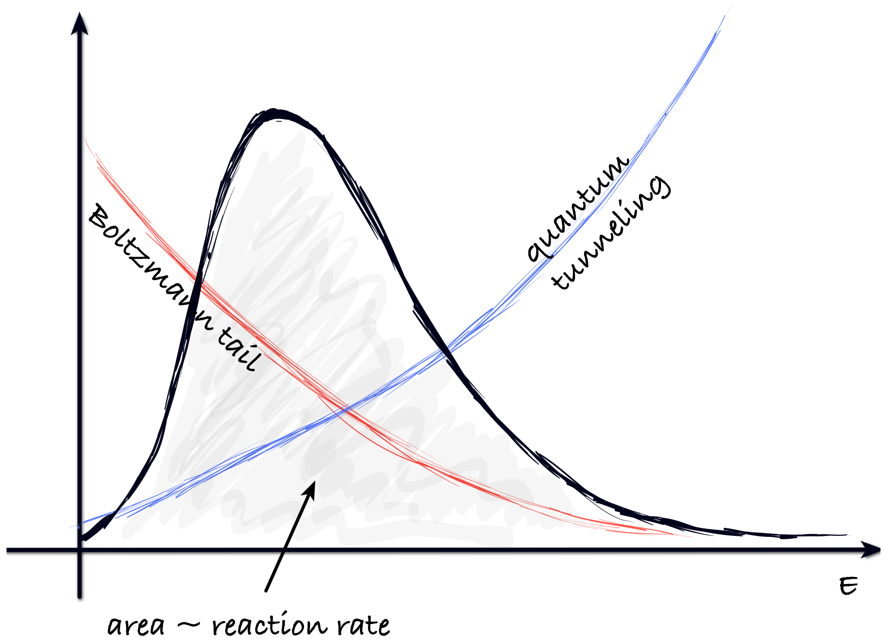


Figure 53: The integrand in the reaction rate is a product of the falling tail of the Boltzmann distribution and the increasing probability of quantum tunnelling. The area under the Gamow peak is proportional to the reaction rate.

The integrand has a peak at $E_0 = \frac{1}{4}(E_G(kT)^2)^{\frac{1}{3}}$. This gives us the optimal temperature for fusion by the long tail in Maxwell distribution and the probability for barrier tunnelling.

Example:
$p + p$ at $T = 2 \times 10^7$ K has $E_G = 500$ keV. We also know that $kT = 1.7$ keV which therfore means that $E_0 \simeq 7.2$ keV. So, fusion actually occurs at much lower temperatures!

The integral can be done:

$$\langle \sigma v \rangle \sim \frac{8}{9} \left(\frac{2}{2mE_G} \right)^{1/2} \tau^2 e^{-\tau} , \tag{132}$$

where $\tau = \frac{3}{2^{2/3}} \left(\frac{E_G}{kT} \right)^{\frac{1}{3}}$.

4.2.3 Fusion Reactors

Consider the D-D reaction:

$$^2_1\text{H} + ^2_1\text{H} \longrightarrow ^4_2\text{He} + \gamma + 23.8\,\text{MeV}\,, \tag{133}$$
$$^2_1\text{H} + ^2_1\text{H} \longrightarrow ^3_2\text{He} + n + 3.27\,\text{MeV}\,, \tag{134}$$
$$^2_1\text{H} + ^2_1\text{H} \longrightarrow ^3_1\text{H} + p + 4.03\,\text{MeV}\,, \tag{135}$$

where $1\,\text{MeV} \to 10^{-13}\,\text{W s}$. Deuterium is found in vast quantities in sea water, so the potential for a ‘free’ energy source is enormous. An even better reaction is D-Tritium:

$$^2_1\text{H} + ^3_1\text{H} \longrightarrow ^4_2\text{H} + n + 17.62\,\text{MeV}\,, \tag{136}$$

The cross-section $\langle \sigma v \rangle$ determines easiest reaction. D-T has significantly higher cross section at ’low’ energy **and** produces more heat. However tritium has to be manufactured and is not stable ($t_{\frac{1}{2}} \sim 4$ years). These reactions produce neutrons which are hard to contain or extract energy from.

Another example of a reaction is:

$$^1\text{H} + ^{11}\text{B} \longrightarrow 3^4\text{He}\,. \tag{137}$$

This would be ideal because $E_0 \sim 120\,\text{keV}$ but it is much harder.

The key problem is that very high temperatures are required for ignition and this means that containment is in turn also very hard!

Energy Production

For this we need a plasma, which is made up of electrons and the fusion materials. The heat is radiated and will cool without fusion, but fusion only happens at high temperatures. The break-even point is called the Lawson Criteria:

$$L = \frac{\text{energy out}}{\text{energy in}} = \frac{n_d^2 \langle \sigma v \rangle}{\frac{3}{2}(4n_d)kT} \times \text{time} \times Q \tag{138}$$

We need $L > 1$ for a useful reactor which is still a holy grail!

> ***Example:***
> for $\langle \sigma v \rangle \sim 10^{-22}\,\text{m s}^{-}1$ at $kT = 20\,\text{keV}$, we have that $n_d \times \text{time} > 10^{19}\,\text{m}^{-}3\,\text{s}$. This means that we have to have both a high density and a very long time.

These reactions are usually contained using a magnetic field. A big reactor called ITER uses a toroidal B-field (‘Tokamak’) to contain the plasma which means it cant touch walls. They aim to produce 500 MW of power for 400 seconds by 2027, but the cost of this project is $\sim 10^{10}$Euros.

4.2.4 Fusion Bombs (Thermo-nuclear)

For ease of delivery and storage the reaction $^6\text{Li} + n \longrightarrow ^3\text{H} + ^4\text{He} + 4.78\,\text{MeV}$ is used. Fusion bombs used to compress and ignite. The blast radius of a bomb like this is about 10 km which is enough for the complete destruction of London.

5 Nuclear Physics in the Universe

One of the most important applications of nuclear physics lies in our understanding of the **origin of the elements**, and in particular their relative **abundances** – see Fig. 54. In the very early universe the universe was $10^{\sim 27}$ K at the end of **inflation** and cooled to about 10^{10} K about 1 second later. Physics during this time is pretty complicated, but the universe went through a series of phase transitions until quarks combine to form protons and neutrons by about 10^{-4}s. After about 1 sec **Big Bang nucleosynthesis** began, and with it formed the lightest of the nuclei. Later heavier nuclei were synthesised in stars of various masses, with the heaviest nuclei formed when stars die and explode, or when neutron stars merge. This section will examine some of these processes – see Fig. 55.

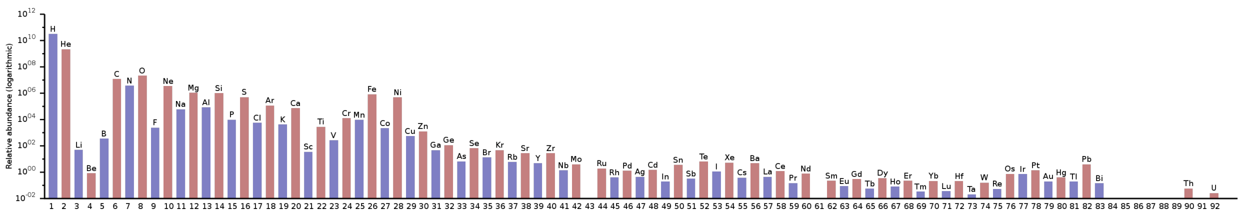


Figure 54: The abundance of the elements in the Solar System as a function of atomic number. Notice the fact that mostly its Hydrogen and Helium, with a relative abundance ratio of 1:4. Why? Other features include: peaks for CNO, Iron and Nickel, as well as an alternating pattern as we move through atomic number. Why?

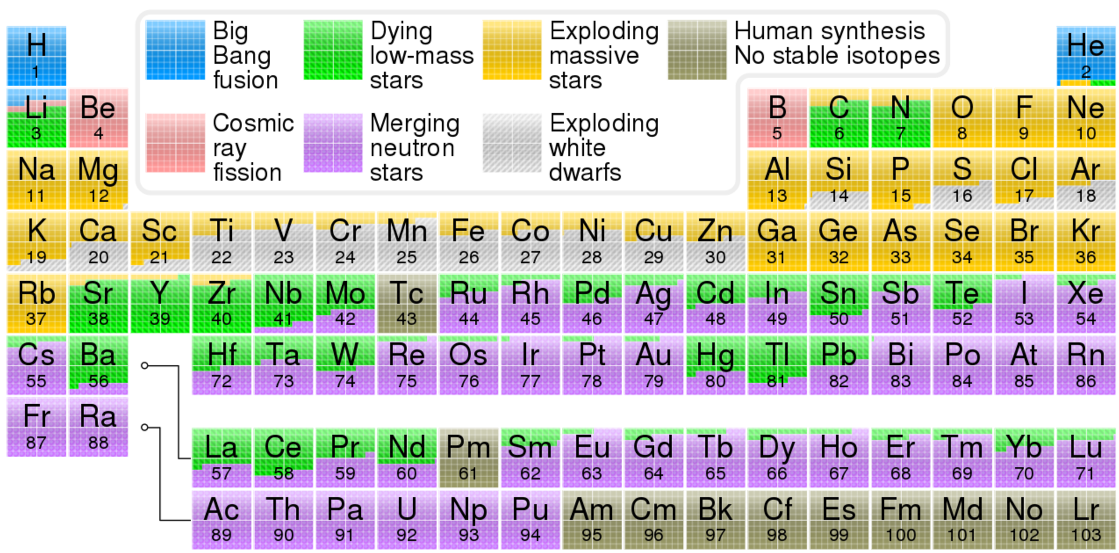


Figure 55: Based on `http://www.astronomy.ohio-state.edu/~jaj/nucleo/`, the periodic table coloured according to the way elements are synthesised.

Digression: The stuff in the universe.
One of the most interesting things we know about the universe is that we don't know much about what's in it! If we add up the energy densities of all the matter and energy in the universe we find that stars, free hydrogen and helium, plus the tiny amount of heavier matter make up only about 5%. The rest is dark: it doesn't interact with normal baryonic matter electromagnetically, so we can't see it. We detect it through other means: for **dark matter** we see galaxies rotate as if they are in the potential well of a much heavier 'halo' of dark matter – we also see the same amounts of dark matter through the **gravitational lensing** of light by massive clusters of galaxies. For **dark energy**, we detect this mainly through the way it affects the expansion of the universe, where it acts like anti-gravity to accelerate the expansion rate.

5.1 Big Bang Nucleosynthesis

To understand Big Bang Nucleosynthesis (BBN) we need to understand how the temperature of the universe scales with time. The universe is expanding, and therefore cooling down – at what rate?

5.1.1 Basics of Cosmology

On large scales we assume that the universe is **homogeneous** (similar everywhere) and **isotropic** (similar in all directions). The Universe can be characterised by the expansion rate as a function of time, $H(t)$ – the '**Hubble rate**'; the **scale factor** $a(t)$ defined such that $H(t) = \dfrac{1}{a}\dfrac{da}{dt}$ and the **redshift** $z\,:\, 1+z = 1/a$, which describes how photons 'stretch'. The scale factor tells us the size of the universe relative to today $t = t_0$, when $a(t_0) = 1$. We also define the expansion rate today as $H(t_0) = H_0$, which is called the **Hubble constant**. This is often written in terms of a dimensionless number h as

$$H_0 = 100\,h\,\mathrm{km\,s^{-1}\,Mpc^{-1}}\,, \tag{139}$$

where from observations $h \simeq 0.7$ and Mpc is a mega-parsec ($= 3.089 \times 10^{19}$ km). Note that H_0 is a *rate* and has dimensions of 1/time. This quantity can be used to define the Hubble length ('size of universe'):

$$\frac{c}{H_0} \simeq \frac{3000}{h}\mathrm{Mpc} \simeq \frac{9.785}{h}\mathrm{Gly}\,, \tag{140}$$

as well as the Hubble time:

$$\frac{1}{H_0} \simeq \frac{9.785}{h}\mathrm{Gyr} \sim \text{age of Universe}\,. \tag{141}$$

To understand how the scale factor evolves we need to know the matter content of the universe and how it scales with the scale factor:

$$\begin{aligned}
\rho_m &= \text{density of dark matter + nuclear matter ('baryon')} \propto 1/\text{volume} \sim a^{-3}\,,\\
\rho_r &= \text{energy density of radiation } (\gamma+\nu) \propto 1/\text{volume} \times \text{redshift factor} \sim a^{-4}\,,\\
\Lambda &= \text{dark energy density or } \textbf{cosmological constant} \simeq \text{const}\,.
\end{aligned}$$

We can then write these in terms of todays values of ρ_m, ρ_r:

$$\rho_m = \rho_{m0} a^{-3}, \tag{142}$$
$$\rho_r = \rho_{r0} a^{-4}. \tag{143}$$

The key equation of motion is the **Friedmann equation**, which is derived from **Einstein's Field Equations** and is, for a flat universe:

$$H^2 = \frac{8\pi G}{3}(\rho_m + \rho_r) + \frac{1}{3}\Lambda c^2, \tag{144}$$

It's convenient to tidy this up and use **dimensionless parameters** for the constants in the equations where we can. To do this we use the **critical density**:

$$\rho_c = \frac{3H_0^2}{8\pi G}, \tag{145}$$

which is the critical density for the universe to re-collapse. Then we divide the Friedmann equation by H_0^2,

$$\left(\frac{H}{H_0}\right)^2 = \frac{8\pi G}{3H_0^2}(\rho_{m0}a^{-3} + \rho_{r0}a^{-4}) + \frac{\Lambda c^2}{3H_0^2}, \tag{146}$$
$$\Rightarrow \quad \left(\frac{H}{H_0}\right)^2 = \Omega_m a^{-3} + \Omega_r a^{-4} + \Omega_\Lambda, \tag{147}$$

where

$$\Omega_m \equiv \frac{\rho_{m0}}{\rho_c}, \quad \Omega_r \equiv \frac{\rho_{r0}}{\rho_c}, \quad \text{and} \quad \Omega_\Lambda \equiv \frac{\Lambda c^2}{3H_0^2}. \tag{148}$$

These are dimensionless density parameters which can be determined by observations: today we have $\Omega_m \simeq 0.3$, $\Omega_\Lambda \simeq 0.7$. The matter density is often quoted as $\Omega_m h^2 \approx 0.143$. The radiation component today is much smaller $\Omega_r \simeq 4.194 \times 10^{-5}\, h^{-2}$. The solution to (146) is found by noting it is a **differential equation** for $a(t)$, and solving it numerically (it can be done analytically in terms of special functions).

Once we know the Hubble rate as a function of scale factor we can find the age using

$$t_0 = \int_0^{t_0} dt = \int_0^1 \frac{dt}{da} da = \int_0^1 \frac{da}{aH}. \tag{149}$$

Cosmological Epochs:

Have a look at Fig. 57. This shows us that as we go backwards in time to smaller a (since the universe is expanding now, things must be closer together in the past!), the density of the different components increases – except for the cosmological constant! Today we are in a phase which is the matter era, transitioning into the Dark Energy era. The important thing for BBN is that the *matter and radiation densities do not increase at the same rate*. The extra redshift factor in the radiation density means that as we compress the universe the radiation starts to gravitationally dominate over the matter, until we see that early enough the universe is radiation dominated.

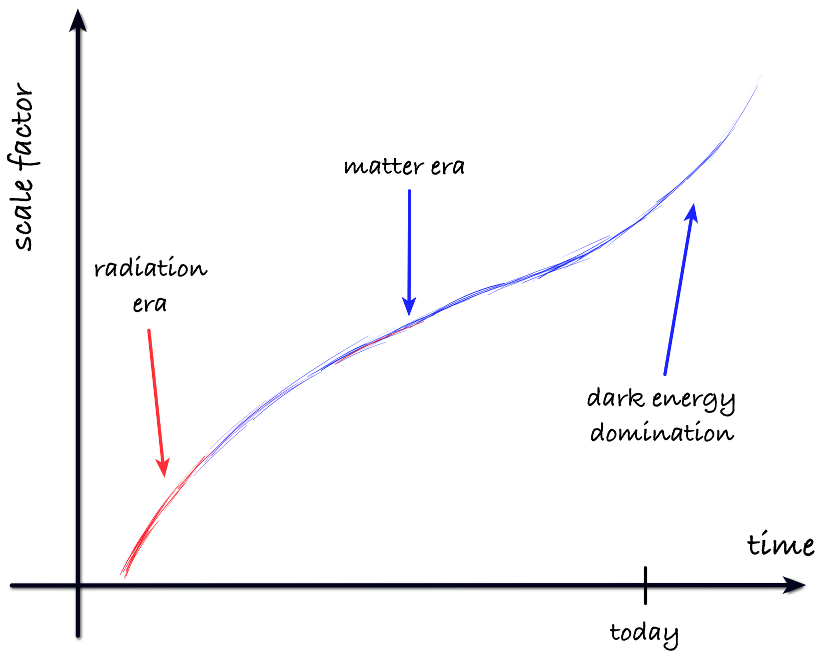


Figure 56: Sketch of the scale factor as a function of time, showing the different cosmological eras. This is the solution to the Friedmann equation for radiation, matter and a cosmological constant.

Therefore there exists a time (t_{eq}) where $\rho_m = \rho_r$:

$$a_{eq} = \frac{\Omega_r}{\Omega_m} \approx 2.94 \times 10^{-4} . \tag{150}$$

(From this we can find $t_{eq} = \int_0^{a_{eq}} \frac{da}{aH}$.) Now, temperature evolves as $T(t) = \frac{T_0}{a}$, where $T_0 = 2.725\,\mathrm{K}$ today (this is the temperature of the **cosmic microwave background** today). Therefore:

$$T_{eq} \approx 6.5 \times 10^4\, \Omega_m h^2\, \mathrm{K} \tag{151}$$

At earlier times, $t < t_{eq}$, at temperatures higher than this, radiation dominates the Friedmann equation:

$$\left(\frac{H}{H_0}\right)^2 \simeq \Omega_r a^{-4} \simeq \Omega_m \left(\frac{T_0}{T_{eq}}\right)\left(\frac{T}{T_0}\right)^4 \tag{152}$$

Now,

$$H = \frac{\dot{a}}{a} = \frac{1}{a}\frac{da}{dt} = \sqrt{\Omega_r} a^{-2} H_0 \tag{153}$$

$$\Rightarrow \quad \frac{da}{dt} = H_0 \sqrt{\Omega_r} a^{-1} \tag{154}$$

$$\Rightarrow \quad \int_0^a a da = \int_0^t H_0 \sqrt{\Omega_r} dt \tag{155}$$

$$\Rightarrow \quad a \propto t^{\frac{1}{2}} \tag{156}$$

$$\text{i.e.,} \quad tT^2 \simeq \text{Const} \tag{157}$$

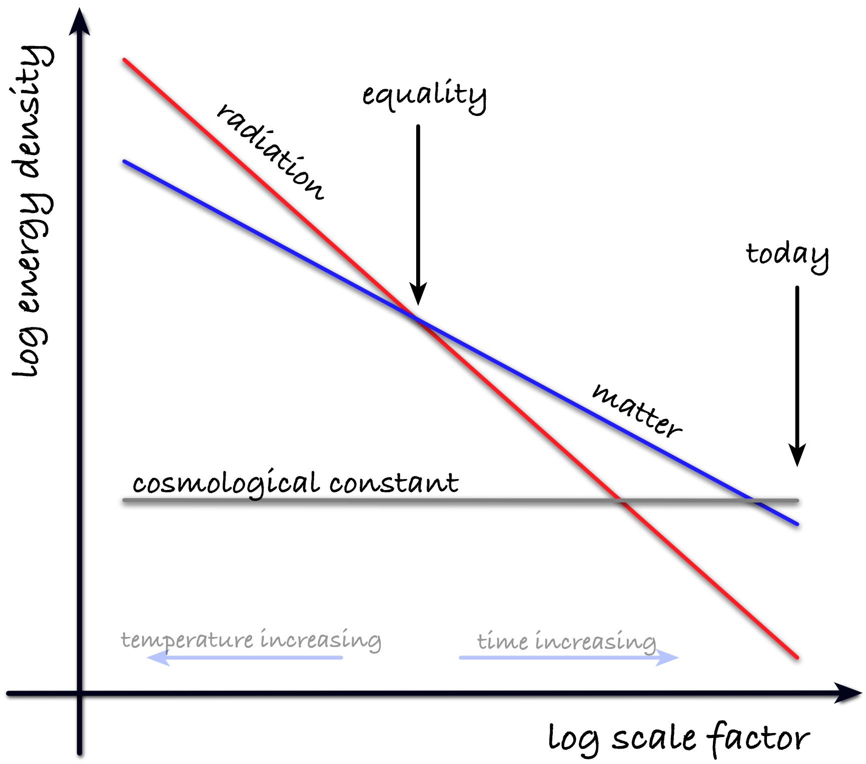


Figure 57: The scaling of the different types of matter in the universe with scale factor. Whichever is largest tells us what is dominating the cosmic dynamics. Note the log scaling whereby *power laws become straight lines*.

i.e. $a^2 = 2\sqrt{\Omega_r}H_0 t \to tT^2 \simeq 6.5 \times 10^{19}\,\mathrm{K}^2\,\mathrm{s}$ or:

$$T(t) = 8.07 \times 10^9\,\mathrm{K}\left(\frac{1\,\mathrm{s}}{t}\right)^{\frac{1}{2}} . \tag{158}$$

The energy equivalent for this is:

$$k_b T \simeq 0.7\,\mathrm{MeV}\left(\frac{1\,\mathrm{s}}{t}\right)^{\frac{1}{2}} . \tag{159}$$

So, as t increases, T decreases $H \sim T^2 \sim \frac{1}{t}$, and density also decreases, $\rho \sim T^4 \sim \frac{1}{t^2}$.

A key consideration for BBN is whether the interaction rate is higher than the expansion rate of the universe. For a mixture of interacting particles the interaction rate is given by $\Gamma = n \times (\sigma v)$, and therefore if:

$\Gamma \gg H \longrightarrow$ Equilibrium $\left(t_c = \dfrac{1}{\Gamma} \ll t_H = \dfrac{1}{H}\right)$

which means particles interact so rapidly they don't feel the expansion of the universe.

$\Gamma \ll H \longrightarrow$ Particles decouple or '**freeze-out**',

which means they are pulled apart faster than they interact.,

Digression: Why is a power-law a straight line on a log-log plot?
Take a relationship like $\rho_m = \rho_{m0} a^{-3}$ which we plot on a log-log scaling. This means we are plotting $\log \rho_m$ on the y-axis against $\log a$ on the x-axis. We have

$$\log \rho_m = \log(\rho_{m0} a^{-3}) = \log(\rho_{m0}) - 3 \log a$$

so its the same as plotting the function

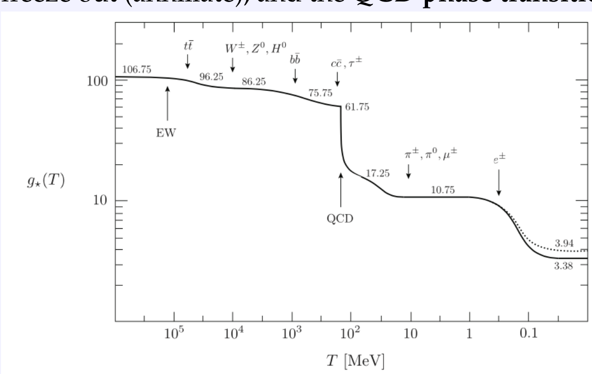
$$y = \text{const.} - 3x$$

which is a straight line with slope -3.

Digression: The actual equation for the temperature evolution is a bit more complicated than in (159). The actual result is approximately

$$k_b T g_*(T)^{1/4} \simeq 1.5\,\text{MeV} \left(\frac{1\,\text{s}}{t}\right)^{\frac{1}{2}} . \tag{160}$$

Here, $g_*(T)$ is the **number of relativistic degrees of freedom**, and drops as various particle species freeze out (anhiliate), and the **QCD phase transition** takes place – it looks like this:



Once freeze-out occurs, particles of mass m change from being relativistic ($kT \gg mc^2$), to non-relativistic ($kT \ll mc^2$), The number density and energy density fall exponentially, and particle and antiparticle pairs **annihilate**.

Some key functions for us in equilibrium are:

	$kT \gg mc^2$	$kT \ll mc^2$
Number Density	$n \sim T^3$	$n \sim (mkT)^{\frac{3}{2}} e^{-\frac{mc^2}{kT}}$
Energy Density	$\rho \sim T^4$	$\rho = mn$

We are generally going to calculate ratios of quantities so all the constants involved in these relations are not needed (except in the exponential).

Let us now evolve the universe and see what happens!

Start clock at $t \ll 1\,\text{s}$, $T \gg 1\,\text{MeV}$ ($T \lesssim 10\,\text{MeV}$).

The main particles are n and p, which are the non-relativistic particles (their masses are of order GeV≫MeV) and $e^{\pm}$, ν_e, $\bar{\nu}_e$, γ, which are all relativistic. Neutrons and protons are kept in equilibrium through the following set of **weak** reactions. The relativistic particles interact via

$$e^+ + e^- \leftrightarrow \nu_e + \bar{\nu}_e \tag{161}$$
$$e^+ + e^- \leftrightarrow \gamma + \gamma\,, \tag{162}$$

as well as scattering between electrons and neutrinos. These reactions happening to the right are **annihilation**, and to the left are **pair production**. We also have the reactions keeping n and p in equilibrium:

$$p + e^- \leftrightarrow n + \nu_e \tag{163}$$
$$n + e^+ \leftrightarrow p + \bar{\nu}_e \tag{164}$$
$$n \leftrightarrow p + e^- + \bar{\nu}_e\,. \tag{165}$$

$t \sim 1\,\mathrm{s}$, $T \sim 1\,\mathrm{MeV}$

The reaction rate for the reaction $\nu_e + \bar{\nu}_e \leftrightarrow e^+ + e^-$ is $\Gamma_\nu \propto T^5$ where the proportionality constant involves the **weak coupling constant** – this determines the strength of the weak force (like the gravitational constant tells us how strong gravity is). Once we reach the temperature

$$\frac{\Gamma_\nu}{H} \sim \left(\frac{kT}{1\,\mathrm{MeV}}\right)^3 \tag{166}$$

we find that *neutrinos decouple* around 1 MeV (more accurately, 0.8 MeV). This means that the weak interactions given by (163), (164) happen to the right only, and the interaction (161) stops, meaning that the equilibrium between n and p starts to break down.

Shortly after this $e^+ + e^- \to \gamma + \gamma$ electrons become non-relativistic ($kT \sim m_e c^2 \sim 0.5\,\mathrm{MeV}$). This means that all electron positron pairs annihilate into photons, heating the photons relative to the neutrinos which are now decoupled. After this, there are no more positrons left, and there are the same number of electrons and protons (the universe has no net charge). The equilibrium between n and p has been broken, and the main reactions left are

$$p + e^- \to n + \nu_e \tag{167}$$
$$n \to p + e^- + \bar{\nu}_e\,. \tag{168}$$

BBN can now begin! Protons and neutrons can start to form isotopes of H, He and trace amounts of Li. Nearly all of the neutrons wind up in Helium-4, as any free n will decay. So we need to figure out the ratio of number densities of neutrons to protons, n_n/n_p, before fusion reactions start; then we figure out how long it will take to make helium-4, during which time neutrons decay, which then will tell us the amount of helium produced (roughly speaking!).

Neutron Freeze-out

First then, the initial ratio of number densities of neutrons to protons, n_n/n_p. In equilibrium, $T \gtrsim 1\,\mathrm{MeV}$ - weak interaction keep equilibrium. This means,

$$\left(\frac{n_n}{n_p}\right)_{eq} = \left(\frac{m_n}{m_p}\right)^{\frac{3}{2}} \exp\left(-\frac{(m_n - m_p)c^2}{kT}\right) \simeq \exp\left(-\frac{Q}{kT}\right) \tag{169}$$

Digression: If weak interactions kept going for $T \ll 1\,\text{MeV}$, there would be no n!

where the mass difference between protons and neutrons is $Q = (m_n - m_p)c^2 \approx 1.293\,\text{MeV}$. For $T \lesssim 1\,\text{MeV}$, fraction of n starts to drop...

So, the freeze-out ratio when equilibrium stops is:

$$\frac{n_n}{n_p} \simeq 0.2 \quad \text{at} \quad T = 0.8\,\text{MeV}\,. \tag{170}$$

We define the neutron fraction:

$$X_n = \frac{n_n}{n_n + n_p} = \frac{\frac{n_n}{n_p}}{1 + \frac{n_n}{n_p}} \tag{171}$$

implying

$$X_n(0.8\,\text{MeV}) \simeq 0.17 \simeq \frac{1}{6}\,. \tag{172}$$

Now, free neutrons decay with $\tau_n = 887\,\text{s}$ so:

$$X_n(t) = X_n(0.8\,\text{MeV})e^{-\frac{t}{\tau_n}} \simeq \frac{1}{6}e^{-\frac{t}{\tau_n}}\,. \tag{173}$$

$t \sim 100\,\text{s}$ **Deuterium formation**

$$n + p \to {}^2\text{H} + \gamma$$

Above $T \sim 0.2\,\text{MeV}$ this happens both ways (i.e. the deuterium is broken up), but at lower temperatures deuterium is formed. Now lets find out how much is formed. First, consider

$$\left(\frac{n_D}{n_n n_p}\right)_{eq} \sim \left(\frac{m_D}{m_n m_p}\frac{1}{kT}\right)^{\frac{3}{2}} e^{-\frac{(m_D - m_n - m_p)c^2}{kT}} \tag{174}$$

In the exponential we have the binding energy of deuterium, $B_D = (m_n + m_p - m_D)c^2 = 2.22\,\text{MeV}$, so,

$$\left(\frac{n_D}{n_p}\right)_{eq} \sim n_n^{eq}\left(\frac{1}{m_p T}\right)^{\frac{3}{2}} e^{\frac{B_D}{T}} \tag{175}$$

Now, let $n_n \sim n_{\text{baryons}} \equiv \eta n_\gamma \sim \eta T^3$:

$$\left(\frac{n_D}{n_p}\right)_{eq} \sim \eta\left(\frac{kT}{m_p c^2}\right)^{\frac{3}{2}} e^{\frac{B_D}{T}} \tag{176}$$

where η is the **baryon-photon ratio** which is a crucial number and is measured to be about $\eta \approx 6 \times 10^{-10}$ (Baryons are particles made from 3 quarks.) . It takes a long time to form lots of deuterium since $\left(\frac{kT}{m_p c^2}\right)^{\frac{3}{2}} e^{\frac{B_D}{T}} \sim \frac{1}{\eta}$ for $T \sim 0.06\,\text{MeV}$.

Once there's decent amounts of deuterium around helium formation begins.

$t \sim 300\,\mathrm{s}$ **Helium formation**

Helium-4 starts to form through the reactions:

$$^2\mathrm{H} + p \rightarrow {}^3\mathrm{He} + \gamma$$

then,

$$\begin{aligned} &{}^2\mathrm{H} + {}^3\mathrm{He} \rightarrow {}^4\mathrm{He} + p \\ \text{or} \quad &n + {}^3\mathrm{He} \rightarrow {}^4\mathrm{He} + \gamma \end{aligned}$$

The overall net reaction is $2n + 2p \rightarrow {}^4\mathrm{He}$, but this happens in stages as long as neutrons are available. It needs deuterium first to start the reaction; because this takes a long time to form it's known as the **deuterium bottleneck**.

How many n available to form ^{4}He? This will tell us how much is produced overall. We have,

$$X_n(t_{nuc} \sim 300\,\mathrm{s}) \sim \frac{1}{6} e^{-\frac{300\,\mathrm{s}}{900\,\mathrm{s}}} \sim 0.12 \sim \frac{1}{8} \tag{177}$$

Hence,

$$\frac{n_{He}}{n_p} = \frac{n_{He}}{n_H} = \frac{\frac{1}{2} n_n}{n_p} \simeq \frac{\frac{1}{2} X_n(t_{nuc})}{1 - X_n(t_{nuc})} \sim \frac{1}{16} \tag{178}$$

Thus the mass fraction of He is:

$$\frac{m_{He}}{m_H} = \frac{4 n_{He}}{n_\gamma} \simeq \frac{1}{4} \tag{179}$$

This is a key prediction of Big-Bang Nucleosynthesis! Note the sensitive dependence on τ_n, η, $Q = m_n - m_p$ and the neutrino freeze out temperature, the strength of the weak force and the strength of gravity! All these numbers can be measured elsewhere: for example η is measured in CMB peaks. So this is a real prediction!

5.1.2 Light Element Synthesis

Other light elements form at this time, until about $t \sim 20$ mins. ^{4}He via tritium:

$$\begin{aligned} {}^3\mathrm{He} + n &\rightarrow {}^3\mathrm{H} + p \\ {}^2\mathrm{H} + n &\rightarrow {}^3\mathrm{H} + \gamma \end{aligned}$$

then,

$$\begin{aligned} &{}^3\mathrm{H} + p \rightarrow {}^4\mathrm{He} + \gamma \\ \text{or} \quad &{}^2\mathrm{H} + {}^3\mathrm{H} \rightarrow {}^4\mathrm{He} + n \end{aligned}$$

Competing mechanisms happen at different rates.

Lithium and Berillium

$$^{2}\mathrm{H} + {}^{4}\mathrm{He} \rightarrow {}^{6}\mathrm{Li} + \gamma$$
$$^{3}\mathrm{H} + {}^{4}\mathrm{He} \rightarrow {}^{7}\mathrm{Li} + \gamma$$
$$^{3}\mathrm{He} + {}^{4}\mathrm{He} \rightarrow {}^{7}\mathrm{Be} + \gamma$$
$$^{7}\mathrm{Be} \rightarrow {}^{7}\mathrm{Li} + \nu$$
$$^{7}\mathrm{Be} + n \rightarrow {}^{7}\mathrm{Li} + p$$
$$^{6}\mathrm{Li} + p \rightarrow {}^{4}\mathrm{He} + {}^{3}\mathrm{He}$$
$$^{7}\mathrm{Li} + p \rightarrow 2{}^{4}\mathrm{He}$$

All these processes and cross-sections go into Big Bang Nucleosynthesis codes to calculate abundances. An example of how they evolve is shown in Fig. 58 (left). How the abundances change depending on the parameters can change these a lot – the baryon-photon ratio is particularly important, as in Fig. 58 (right).

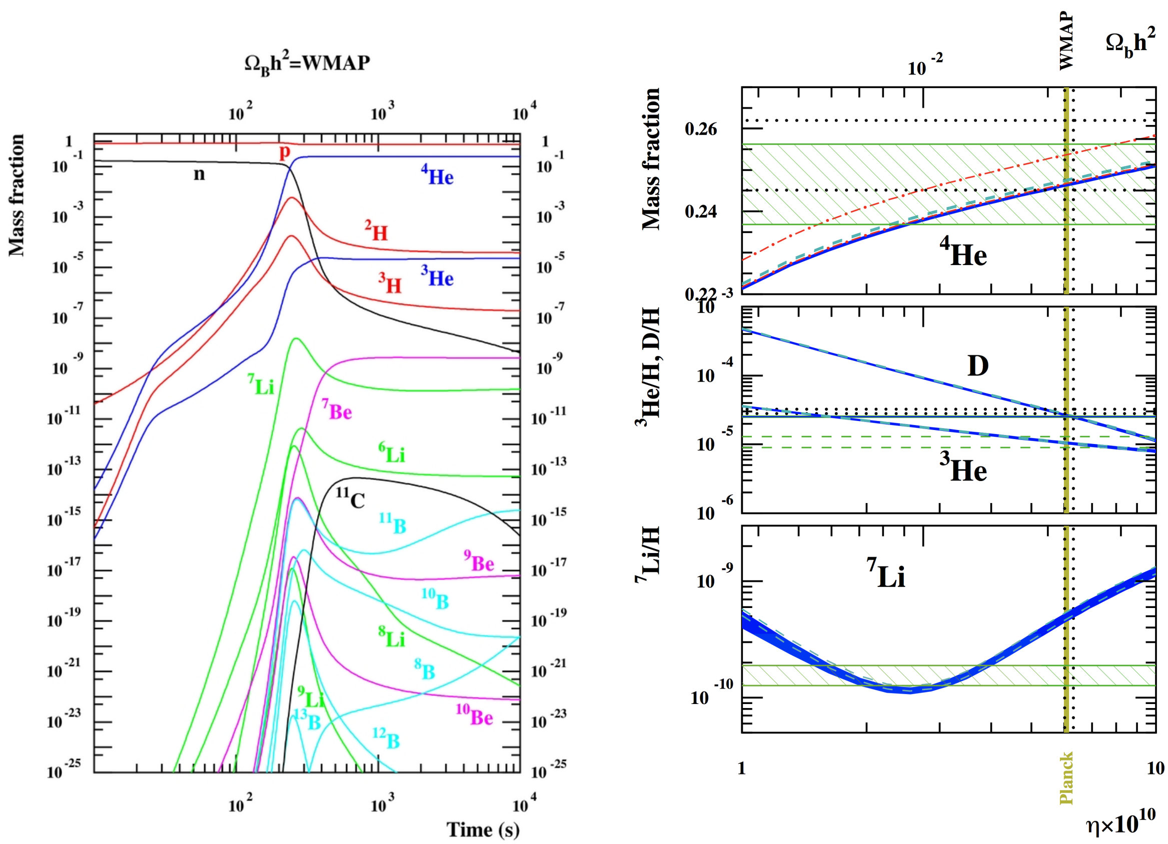


Figure 58: The synthesis of light elements during BBN for a standard cosmological model (left), from Coc et. al., 2011. 10.1088/0004-637X/744/2/158 . Note the 25 order of magnitude spread on the abundances. The plots on the right show the change in calculated abundances as we change η (from Coc, Uzan and Vangioni, JCAP 1410 (2014) 050). The vertical lines show constraints from the CMB (WMAP and Planck refer to recent CMB experiments which measured the fraction of baryons to dark matter and the baryon-photon ratio), the green hatched ares show constraints from other observations – note the mismatch for Lithium-7, an unsolved problem for BBN.

A visualisation of the reactions in the table of nuclides is in Fig. 59.

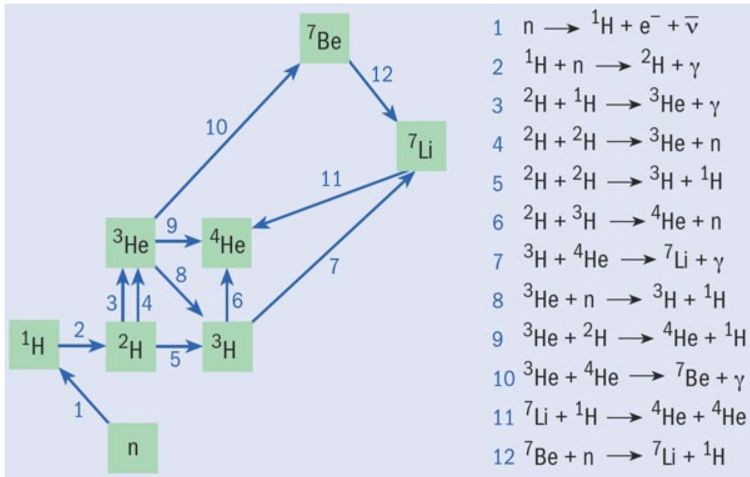


Figure 59: The main reactions in BBN, visualised on a table of nuclides (from Physics World).

For further reading you could look at Primordial Nucleosynthesis in the Precision Cosmology Era by Gary Steigman [Annu. Rev. Nucl. Part. Sci. 2007. 57:463?91, article's doi: 10.1146/annurev.nucl.56.080805.140437]

5.2 Stellar Evolution and Nucleosynthesis

Stars are born from gravitational collapse of dust/gas such as a molecular cloud or a nebulae. The collapse causes the gas to heat up from the gravitation potential energy, and if $M \geq 0.1 M_\odot$ ($1 M_\odot \simeq 2 \times 10^{30}$ kg – Solar Mass) it will get hot enough for fusion to ignite. It will then reach **hydrostatic equilibrium** which is when the thermal radiation pressure balances the gravitational force. For most of its life it is burning hydrogen (by fusion!), which is when the star lives on **main sequence** burning nuclear fuel. The amount of time it stays like this, and its afterlife is determined by the mass of the star.

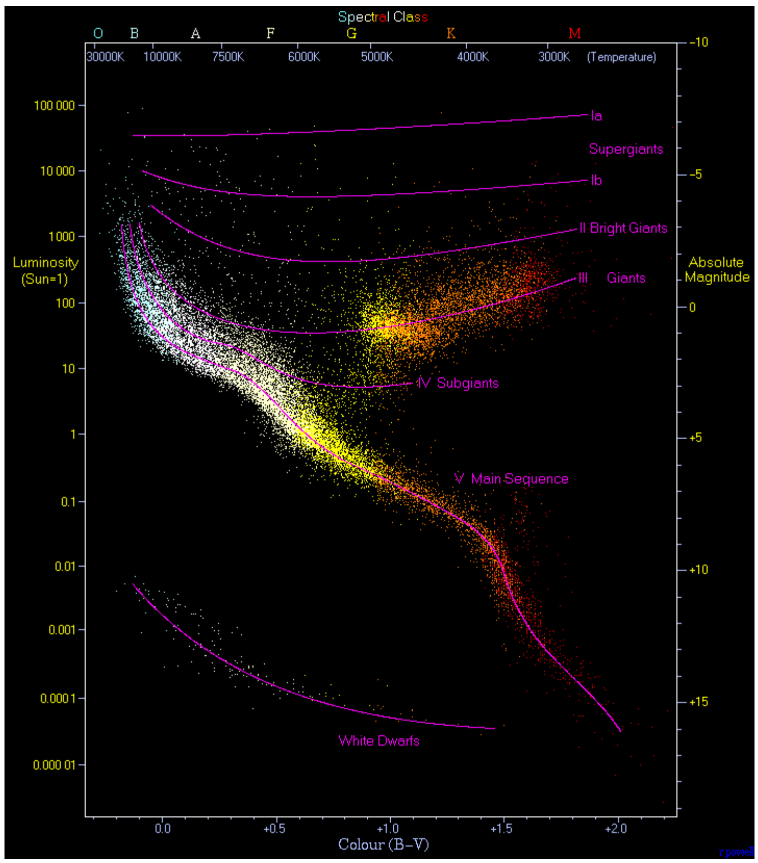


Figure 60: A Hertzprung Russell diagram, which is a plot of surface temperature (on a reversed scale) versus luminosity, which is the energy output per unit time (the solar luminosity is $L_\odot \simeq 3.838 \times 10^{26}$ W). Stars spend most if their life at a point on the main sequence while they burn hydrogen.

It is important to note that the reason stars are hot is actually because the gravitational collapse has heated up an initially diffuse cloud of gas (gravitational potential energy is converted to thermal energy), and the fusion in the core creates the pressure which prevents it collapsing further – which would actually make it hotter. So, fusion acts to create an equilibrium between gravitational collapse and thermal (and radiation) pressure.

5.2.1 Scaling Relationships and Lifetimes

Mass converted to energy via fusion:

$$\text{Lifetime} \sim \frac{\text{total fuel} \sim \text{mass}}{\text{rate of fuel use} \sim \text{luminosity}} \sim \frac{M}{L} \tag{180}$$

A key relation is the mass-luminosity relation:

$$\frac{L}{L_\odot} = \beta \left(\frac{M}{M_\odot}\right)^\alpha \tag{181}$$

where the coefficients vary depending on the mass as

$$\beta \sim 1, \quad \begin{cases} \alpha & \sim 4 \quad \text{for } M \sim M_\odot\,, \\ \alpha & \sim 3.5 \quad \text{for } 2M_\odot \lesssim M \lesssim 20M_\odot\,, \end{cases} \tag{182}$$

$$\beta \sim 3000\,,\ \alpha \sim 1 \quad \text{for } M \gtrsim 20M_\odot\,. \tag{183}$$

(The reason for the big change in power-law behaviour is because the pressure within the star goes from thermal pressure in lower mass stars to radiation pressure in higher mass objects.) Hence,

$$\text{Lifetime} \sim \frac{M}{M^4} \sim \frac{1}{M^3} \tag{184}$$

So, big stars burn **bright** and **fast**. For reference the lifetime of the sun is $\sim 10^{10}$ yr.

Example:
$M = 10M_\odot$ has $L \sim 10^4 L_\odot$, therefore:

$$\text{Lifetime} \sim \frac{10M_\odot}{10^4 L_\odot} \sim 10^{-3} \text{ solar lifetime } \sim 10^7 \text{ yrs}$$

5.3 Core Temperature

In order to understand the fusion processes going on inside a star we need to know the temperature in the core of a star, from which we can figure out the reaction rates for different fusion reactions.

We start form hydrostatic equilibrium – which is the balance of pressure vs gravity:

$$\frac{dP}{dr} = -\frac{Gm(r)\rho(r)}{r^2}, \qquad \frac{dm}{dr} = 4\pi r^2 \rho(r) \tag{185}$$

In this, $\rho(r)$ is a very complicated function, but we know:

$$\text{mean density } \rho \sim \frac{M}{\frac{4}{3}\pi R^3},\ m \to M \tag{186}$$

$$P_{\text{centre}} \sim \frac{GM^2}{R^4} \tag{187}$$

We can then use the ideal gas law $PV = NkT \rightarrow P = \frac{N}{V}kT$ to get:

$$\frac{N}{V} = n = \frac{\rho}{\mu m_H} \tag{188}$$

where μ is the mean atomic weight. Using all this we can get an equation for the core temperature of a star:

$$T_{\text{core}} \sim \frac{GM}{R} \times \mu \times \frac{m_H}{k} \tag{189}$$

$$\sim 1.9 \times 10^7 \left(\frac{M}{M_\odot}\right)\left(\frac{R}{R_\odot}\right)^{-1}\left(\frac{\mu}{0.85}\right) \text{K} \tag{190}$$

For reactions we convert to MeV:

$$kT_{\text{core}} \sim 1.7 \times 10^{-3} \left(\frac{M}{M_\odot}\right)\left(\frac{R}{R_\odot}\right)^{-1} \text{MeV} \tag{191}$$

Note This relation is $M \sim T_{\text{core}}R$. Recall fusion reaction rate:

$$R \sim n^2\langle\sigma v\rangle \sim \frac{1}{m^{\frac{1}{2}}}\frac{1}{(kT)^{\frac{3}{2}}}\int_0^\infty dE e^{(-\sqrt{\frac{E_G}{E}}-\frac{E}{kT})} \tag{192}$$

See fig. 53. The Gamow peak where most fusion is taking place is at

$$E_0 = \frac{1}{4}(E_G(kT)^2)^{\frac{1}{3}} \tag{193}$$

where

$$E_G = mc^2(\pi\alpha)^2 Z^4 \tag{194}$$

for particles of mass m. E.g. proton-proton fusion has $E_G \sim 0.5\,\text{MeV} \gg E_0$ for $T \sim 10^7\,\text{K}$, which means:

$$R \sim \left(\frac{1}{mE_G}\right)^{\frac{1}{2}}\left(\frac{E_G}{kT}\right)^{\frac{2}{3}} e^{-(\frac{E_G}{kT})^{\frac{1}{3}}} \tag{195}$$

which is the lowest temperature reactions occur for $M \lesssim 1M_\odot$. For fusion of heavier nuclei the Gamow energy will be higher, meaning that the temperature must be higher for fusing heave nuclei.

The most important **hydrogen burning** is the PP chain, but at higher temperatures this becomes the CNO cycle. Most of a stars life is spent burning hydrogen in the main sequence. This process conveys hydrogen into helium 'ash'.

5.3.1 Hydrogen Burning I - PP Chains

The fusing of hydrogen is complicated mainly because two protons can't fuse together to form ^{2}He, as this is not a stable nucleus. (Also, free neutrons are not stable, so any which are not fused quickly decay into protons.) So the fusing of 2 protons is complicated, and proceeds via 3 sets of reactions, called PPI, PPII and PPIII. Each of these has a separate reaction rate and cross section but each takes $4p \rightarrow {}^4\text{He} + 2e^+ + 2\nu_e + 26.73\,\text{MeV}$. These reactions consist of a combination of fusion,

β and α decay. Here are the reactions, with branch ratios approximate for the sun. In higher mass stars PPII and PPIII are more common.

$$^1\text{H} + {}^1\text{H} \to {}^2\text{H} + e^+ + \nu_e \tag{196}$$
$$^2\text{H} + {}^1\text{H} \to {}^3\text{He} + \gamma \tag{197}$$

Branching Fraction : 69%

$$^3\text{He} + {}^3\text{He} \to {}^4\text{He} + 2{}^1\text{H} \tag{198}$$

This is the **PPI** chain.

Branching Fraction : 31%

$$^3\text{He} + {}^4\text{He} \to {}^7_4\text{Be} + \gamma \tag{199}$$

Branching Fraction : 99.7%

$$^7\text{Be} + e^- \to {}^7\text{Li} + \nu_e \tag{200}$$
$$^7\text{Li} + {}^1\text{H} \to 2{}^4\text{He} \tag{201}$$

This is the **PPII** chain.

Branching Fraction : 0.3%

$$^7\text{Be} + {}^1\text{H} \to {}^8_5\text{B} + \gamma \tag{202}$$
$$^8_5\text{B}+ \to {}^8_4\text{Be} + e^- + \nu_e \tag{203}$$
$$^8_4\text{Be} \to 2{}^4\text{He} \tag{204}$$

This is the **PPIII** chain.

The reaction $^1\text{H} + {}^1\text{H} \to {}^2\text{H} + e^+ + \nu_e$ is *very* slow, with the average proton waiting 10^9 yr to fuse! (This is similar to the deuterium bottleneck in BBN.) This timescale is determines the main sequence lifetime.

5.3.2 Hydrogen Burning II - CNO Cycle

This process occurs at a higher temperature than pp chains. This means that it requires higher mass stars for it to be significant. In the sun, about 1.7% of the Helium production is via this process whereas in $M \gtrsim 1.6 M_\odot$ it is dominant. The reaction uses Carbon, Oxygen and Nitrogen as catalysts in reaction $4p \to {}^4\text{He} + 2e^+ + 2\nu_e + 26.73\,\text{MeV}$. (2 MeV of this comes from the positrons annihilating with electrons.) These reactions are:

$$^{12}\text{C} + {}^{1}\text{H} \rightarrow {}^{13}\text{N} + \gamma \tag{205}$$

$$^{13}\text{N} \rightarrow {}^{13}\text{C} + e^{+} + \nu_e \quad (t_{\frac{1}{2}} \sim 10\,\text{mins}) \tag{206}$$

$$^{13}\text{C} + {}^{1}\text{H} \rightarrow {}^{14}\text{N} + \gamma \tag{207}$$

$$^{14}\text{N} + {}^{1}\text{H} \rightarrow {}^{15}\text{O} + \gamma \quad \text{Slow!} \tag{208}$$

$$^{15}\text{O} \rightarrow {}^{15}\text{N} + e^{+} + \nu_e \quad (t_{\frac{1}{2}} \sim 2\,\text{mins}) \tag{209}$$

$$^{15}\text{N} + {}^{1}\text{H} \rightarrow {}^{12}\text{C} + {}^{4}\text{He} \tag{210}$$

As you can see this is a cycle because the ^{12}C is regenerated at the end! CNO II, III and IV take place as well but are very rare.

5.3.3 Temperature Dependence

You can find this from $R \sim \left(\frac{E_G}{kT}\right)^{\frac{2}{3}} e^{-\left(\frac{E_G}{kT}\right)^{\frac{1}{3}}}$. Usually the energy output $\sim T^{\text{power}}$ is given by:

$$\epsilon_{pp} \sim \left(\frac{T}{10^6 k}\right)^4, \ \epsilon_{CNO} \sim \left(\frac{T}{10^6 k}\right)^{20} \tag{211}$$

Note that even the powers are approximate. These are the main reactions in the Sun and other main sequence stars.

5.3.4 Helium Burning

After H depleted core contracts to maintain hydrostatic equilibrium, its temperature increases by a further 10 times which causes the helium ash to ignite! This is extremely hot which means the radiation pressure in the core forces outer regions of the star to expand, which gives us a red giant!

Triple-α process

The triple-α process ($3\alpha \rightarrow {}^{12}\text{C} + 7.27\,\text{MeV}$) is done in 2 steps, which are:

$$^{4}\text{He} + {}^{4}\text{He} \rightarrow {}^{8}\text{Be} \tag{212}$$

$$^{8}\text{Be} + {}^{4}\text{He} \rightarrow {}^{12}\text{C} \tag{213}$$

where,

$$\epsilon_{3\alpha} \sim \left(\frac{T}{10^6 k}\right)^{41} \tag{214}$$

Calculated rates actually require ’resonance’ in ^{12}C; this was predicted by Hoyle to account for Carbon-12 abundances. In our R calculation using σ ignores ’resonances’ and excited states with larger cross-sections.

After ^{12}C formed, many heavier elements can be made. The α-ladder is continued by absorbing more α particles, e,g,:

$$ {}^{12}_{6}\mathrm{C} + {}^{4}\mathrm{He} \to {}^{16}_{8}\mathrm{O} \tag{215} $$
$$ {}^{16}_{8}\mathrm{O} + {}^{4}\mathrm{He} \to {}^{20}_{10}\mathrm{Ne} \tag{216} $$

After this it is very rare to happen, because the Coulomb barrier is too high. Other processes at sufficiently high T:

$$ T \sim 6\times 10^{8}\,\mathrm{K}: \quad {}^{12}\mathrm{C} + {}^{12}\mathrm{C} \to \begin{cases} {}^{16}\mathrm{O} + 2\alpha \\ {}^{20}\mathrm{Ne} + \alpha \\ {}^{23}\mathrm{Na} + p \\ {}^{23}\mathrm{Mg} + n \\ {}^{24}\mathrm{Mg} + \gamma \end{cases} \tag{217} $$

$$ T \sim 10^{9}\,\mathrm{K}: \quad {}^{16}\mathrm{O} + {}^{16}\mathrm{O} \to \begin{cases} {}^{24}\mathrm{Mg} + 2\alpha \\ {}^{24}\mathrm{Si} + \alpha \\ ... \\ ... \\ ... \end{cases} \tag{218} $$

And so on, until fusion leaves ^{56}Ni (which β-decays to ^{56}Fe). Fusion of heavier elements stops releasing energy. The main process for $T \gtrsim 10^{9}\,\mathrm{K}$ is α particle capture, where α-particles are sequentially absorbed, stepping along the valley of stability producing nuclei with even Z.

These higher processes don't produce much energy, so they are short lived phases. It only occurs in $M \gtrsim 8M_{\odot}$ stars.

It is in the very final stages of a stars life that most of these processes occur, and most only occur in high mass stars – in these stages, the interiors look like in Fig. 61

5.3.5 Production Of Heavy Elements, $A \gtrsim 60$

Heavier nuclei don't get produced by fusion processes but rather mostly by **neutron capture**, followed by β-decay. Neutron capture moves to the right on a table of nuclides, whereas β-decay moves up and left diagonally. These processes populate the isotopes below the valley of stability.

If neutron capture is slow enough for β-decay to occur, we have an **s-process** which zig zags up the table of isotopes creating stable elements up to ^{209}Bi – see Fig. 62.

If neutron capture is very fast, i.e., much faster than β-decay half life, we have an **r-process** (r for rapid), whereby a nucleus keeps absorbing more and more neutrons. On a table of nuclides, this keeps moving to the right. This happens if there is a huge flux of neutrons, like in a supernovae or neutron star merger. What stops this process from continuing? Neutron rich nuclei are unstable to β^{-} decay, particularly when the number of neutrons is a magic number +1. So, a typical r-process will 'pause' on the table of nuclides while it sequentially absorbs a neutron then undergoes β^{-}

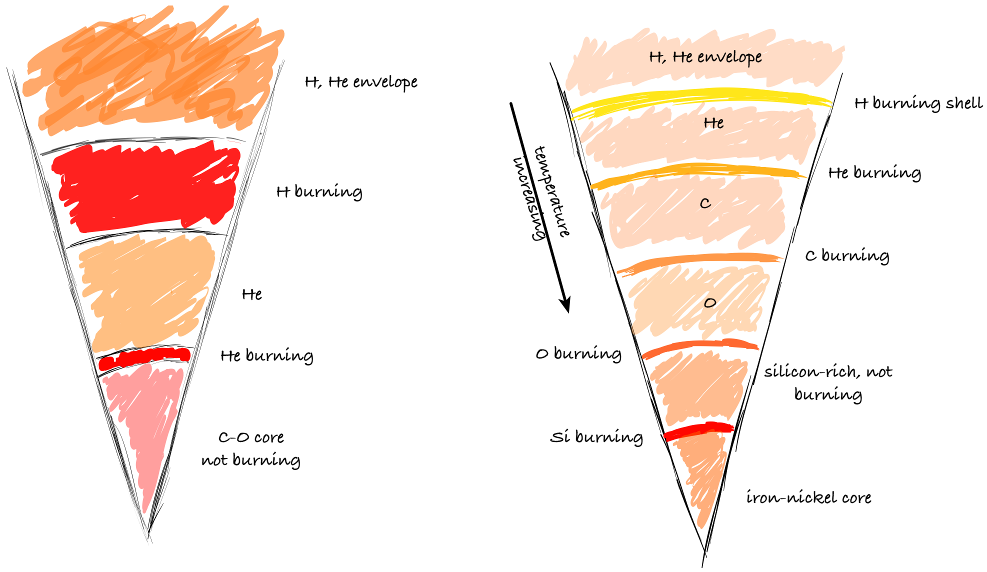


Figure 61: Cartoon of a stellar core of a low mass $M \sim 5M_\odot$ star (left) and high mass $M \gtrsim 10M_\odot$ stars (right). It consists of thin burning shells moving outwards leaving an ash of heavier nuclei behind.

decay. This then zig-zags up towards the valley of stability until adding another neutron doesn't produce a very unstable isotope; and the r-process continues. Note these processes take place in supernovae and neutron star mergers over time-scales of order a second (and is where nearly all heavier nuclei are produced!).

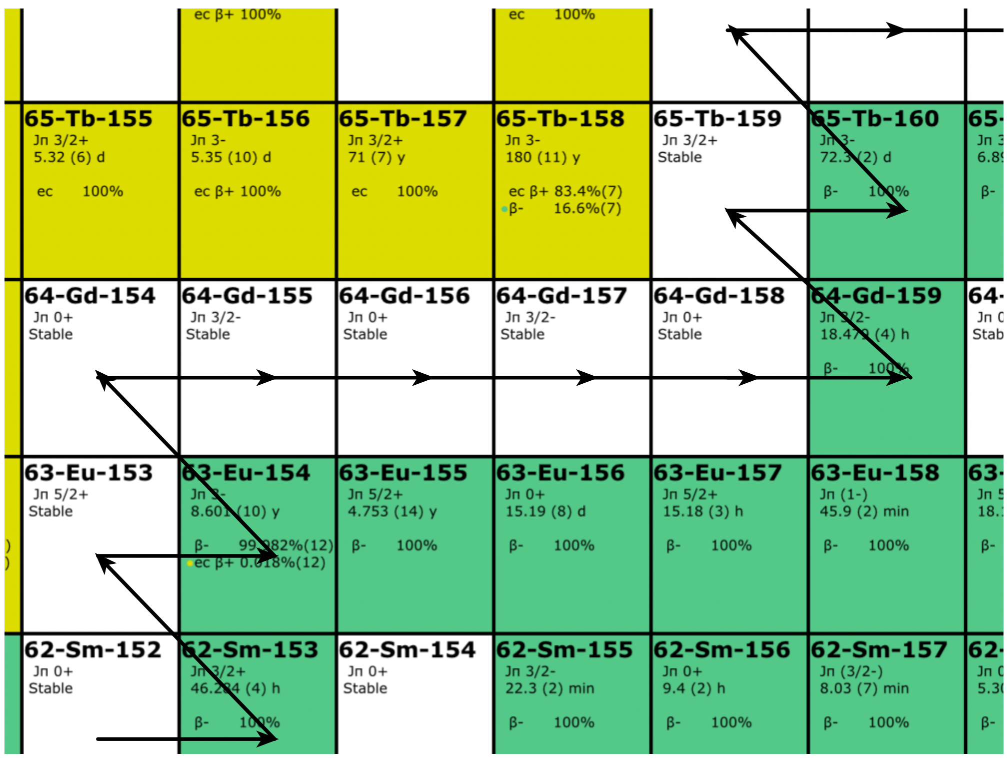


Figure 62: An example of an s-process path. Nuclei absorb neutrons, and moves to the right on a table of nuclides, along stable isotopes until an unstable one is reached. This then β^- decays until a stable nucleus is reached, and the process continues as long as neutrons are available.

5.4 Stellar Remnants

When a star runs out of fuel, the core collapses in different ways depending on their mass:

- $M \lesssim 10M_\odot$ - The outer parts of the core blast of leaving a planetary nebula, and sitting in the centre there is a **white dwarf**.
- $M \lesssim 30M_\odot$ - There is a supernovae explosion, and then a **neutron star** is left.
- $M \gtrsim 30M_\odot$ - There is a supernovae explosion, and then a **black hole** is left.

5.4.1 White Dwarf

These are typically $0.2M_\odot \lesssim M < 1.4M_\odot$, and $R \sim 10^{-2}R_\odot$. There is no fusion in a white dwarf, but instead is held up bu the e^- degenerecy pressure (usually C-O core, or O-Ne-Mg).

Degeneracy Pressure: Fermions can't occupy the same state, therefore the energy levels fill up, and the hight levels create pressure from high kinetic energy.

Chandrasekhar Limit

To derive the Chandrasekhar limit we use a scaling argument. If we take a piece of unit mass:

$$E_{kin} \sim E_{grav} \sim \frac{GM}{R} \tag{219}$$

and we can define E_{kin} as:

$$E_{kin} = N\frac{p^2}{2m_e} \text{ for non relativistic or } E_{kin} = Npc \text{ for relativistic,} \tag{220}$$

where N is the number of electrons in the piece. We know that the electrons are degenerate – 'touching', ($p \sim \Delta p$) We can then use the Uncertainty principle ($\Delta p \Delta x \sim \hbar$) to get:

$$\Delta x \sim \text{ distance between } e^- \tag{221}$$

$$\Delta x \sim \frac{1}{n_e^{\frac{1}{3}}} \tag{222}$$

$$\Delta x \sim N\frac{M}{R^3} \tag{223}$$

Where n_e is the number of electrons per unit volume. Therefore, for non relativistic particles we can get:

$$E_{kin} \sim \frac{N(\Delta p)^2}{2m_e} \tag{224}$$

$$E_{kin} \sim \frac{N\hbar^2 n^{\frac{2}{3}}}{2m_e} \tag{225}$$

$$E_{kin} \sim \frac{M^{\frac{2}{3}} N^{\frac{5}{3}} \hbar^2}{2m_e R^2} \sim E_{grav} \tag{226}$$

$$\therefore R \sim \frac{N^{\frac{5}{3}} \hbar^2}{2m_e G} \frac{1}{M^{\frac{1}{3}}} \tag{227}$$

We can also do this for the relativistic case which gives us:

$$E_{kin} \sim \frac{M^{\frac{1}{3}} N^{\frac{4}{3}} \hbar c}{R} \tag{228}$$

$$\therefore M \sim N^2 \left(\frac{\hbar c}{G}\right)^{\frac{3}{2}} \tag{229}$$

********IMAGE*********

The mass limit as the speed of the electron reaches C is the Chandrasekhar limit:

$$M_{Ch} \simeq 1.4 M_\odot \tag{230}$$

5.4.2 Neutron Stars

These are typically $01.4M_\odot \lesssim M \lesssim 3M_\odot$, and are formed from **** star or accretion onto a a white dwarf. This results in supernovae Ia explosion. The electron degenerecy pressure is not enough to stop the star collapsing, and when $T > 5 \times 10^9$ K the nuclei break up ($e^- + p \rightarrow n + \nu$) only neutrons are left. The neutron degenerecy pressure then halts contaction and we are left with a giant nucleus which has a density of 10^{17} kg/m^3 and a radius of $\sim \frac{1}{M^{\frac{1}{3}}} \sim 10$ km!

Escape Velocity

$$v_{\text{escape}} = \sqrt{\frac{2GM}{R}} \tag{231}$$

For a neutron star the escape velocity is $\sim 0.5c$! What if we add mass?

$$v_{\text{escape}} \sim \sqrt{\frac{M}{R}} \sim \sqrt{M \times M^{\frac{1}{3}}} \sim M^{\frac{2}{3}} \tag{232}$$

If $v_{\text{escape}} = c$, then neutron degeneracy pressure can't stop collapse, and therefore we get a **black hole**! These objects are so dense that nothing can escape!