Introduction.

1. Will talk about 2D Yang Mills theory for $SU(N) / U(N)$ gauge group on Σ_G.

$$S = \int_{\Sigma_G} d^2\sigma \sqrt{g} \left(F_{\mu\nu} F_{\mu\nu} \right)$$

- This theory is invariant under area-preserving diffeomorphisms

- $$Z = \int DA \, e^{-S}$$

only depends on area & genus of $\Sigma_G(A)$.

$$f = *F \qquad \hookrightarrow \quad F = f\mu$$

$$F^a_{ij} = \sqrt{\det G_{ij}}\ \epsilon_{ij} f^a .$$

$$\int F \wedge *F = \int \mu f^2 .$$

$$\int_{\Sigma_G} tr(F \wedge *F) = \int_{\Sigma_G} \mu\ tr(f^2)$$

$$S_{YM} = \frac{1}{4g^2} \int_{\Sigma_G} d^2x \sqrt{\det G_{ij}}\ Tr(f^2) .$$

$$= \left(\frac{1}{4g^2}\right) \int_{\Sigma_G} \mu\ Tr(f^2)$$

- $Tr f^2$ is a scalar
  - invariant under diffeos
  - As long as $\int\mu$ is invariant, the ~~action~~ is invariant.

- gauge field in 2D: No local d.o.f.
- The exact solution is known:

For partition function

$$Z(G, A)$$

$$= \sum_R (\mathrm{Dim}\, R)^{2-2G} e^{-g^2 A C_2(R)}$$

Will write this as

$$= \sum_R (\mathrm{Dim}\, R)^{2-2G} e^{-A C_2(R)}$$

Redefined: $g^2 A \to A$.

- Gross & Taylor (1992-1993) + others showed that the $\left(\frac{1}{N}\right)$ expansion of

$$Z(G, A)$$

$$= \sum_h \frac{1}{\cancel{N}} N^{2-2h} Z_{string}(\Sigma_h \to \Sigma_G)$$

- $Z_{string}(\Sigma_h \to \Sigma_G)$ is a counting function of holomorphic Maps

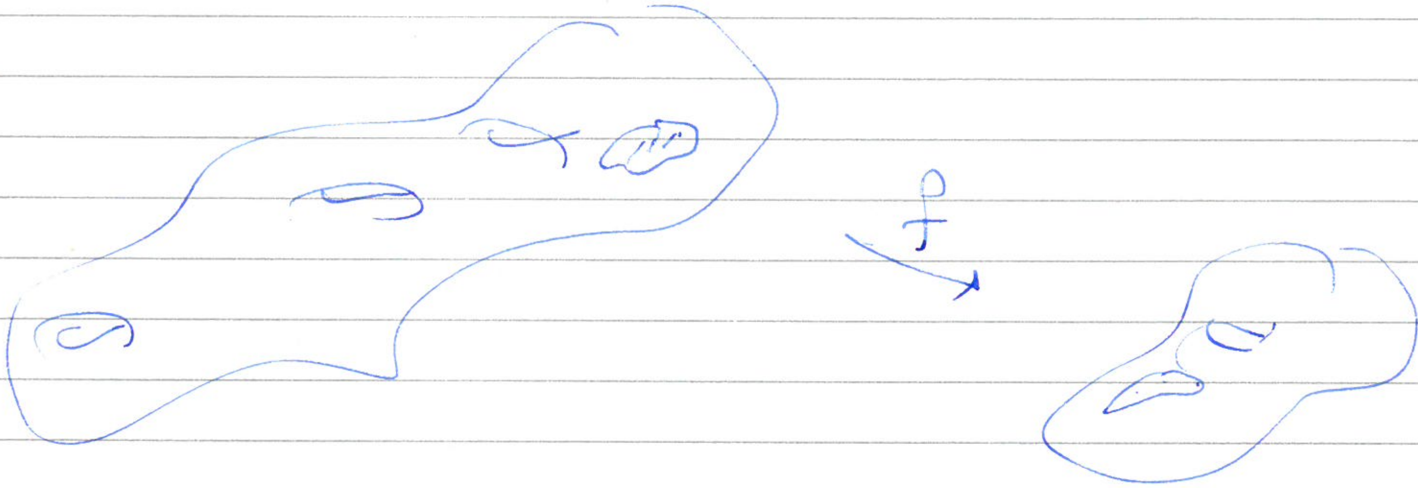
$$\Sigma_h \longrightarrow \Sigma_G(A).$$

→ How ?

– What is a holomorphic map from

$$\Sigma_h \longrightarrow \Sigma_G \quad ?$$

– What does it look like ?



Local patch

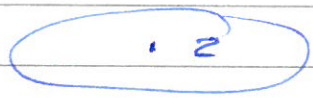
$z, \bar{z}$: $f(z, \bar{z})$

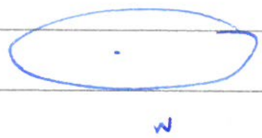
$$\partial_{\bar{z}} f = 0$$

Introducing local coordinate

z at P

& w at $f(P)$





$\mapsto$ $w = z$ is a holo. map.

$w = z^n$ for $n \in \{1, 2, \dots\}$

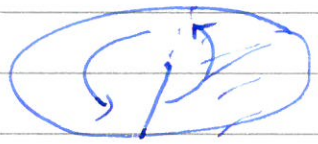
is a holo. map.

$\mapsto$ $w = \sqrt{z}$, $\log z$ Not a hol' map.

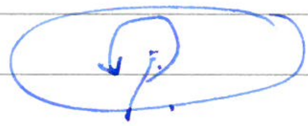
– single-valued.

locally $(p, f(p))$

we have $\underline{\underline{k}}$

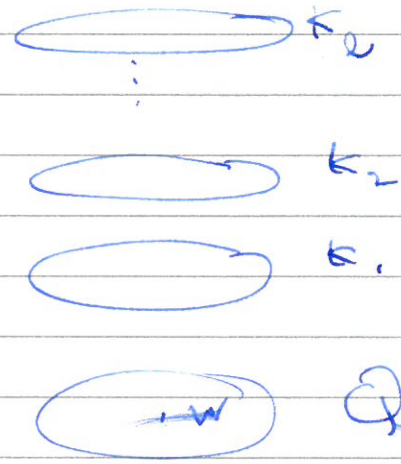


z^2



2-fold cover:

Fixing a point $\underline{\underline{q}}$:



⊗ degree:

· $k_1 + k_2 + \dots k_\ell = n$

= Degree is constant as we move across Σ_g

7

- Maps of a degree n are classified by partitions of n:

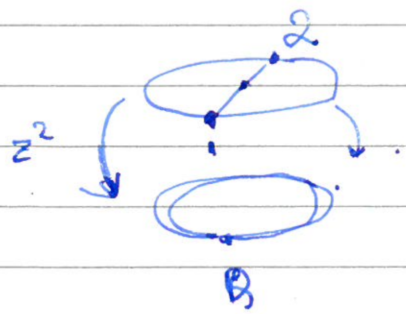
$n = 3$,

$3 = 3$

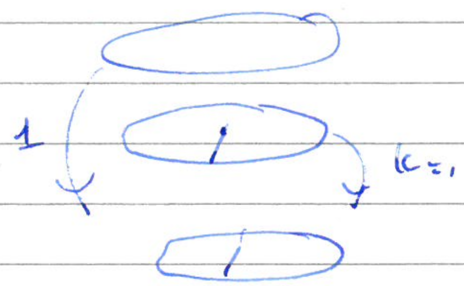
$= 2 + 1$

$= 1 + 1 + 1$.

- $\boxed{\text{Locally:}}$ $w = z^2$.



$1 \to 2$
$2 \to 1$



$1 \to 2$
$2 \to 2$.

8.

In general for degree n

maps: to a dorc,

we can construct a permutation

$\sigma : \{1 \ldots n\} \longrightarrow \{1, \ldots n\}$

3.

$$\begin{matrix} 1 & 2 & 3 \\ & \downarrow & \\ 2 & 3 & 1 \end{matrix} = (1\,2\,3)$$

$$\begin{matrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{matrix} = (12)(3)$$

$$= (12)$$

$$\begin{matrix} 1 & 2 & 3 \\ & \downarrow & \\ 1 & 2 & 3 \end{matrix} = (1)(2)(3)$$

$$= ()$$

→ (123) & (132)

- describe the same map.

- choice of how you label the points

→ $(123) = (23)(123)(23)$

$\sigma = \gamma \sigma \gamma^{-1}$

→ locally hol. maps correspond to perm.

→ Conjugacy classes of perms.

globally : Later

Outline:

1. – Exact answer. (Migdal)

   – Using heat kernel action

2. – How we go from $U(N)$ Representation sums to sums involving permutations – which count holomorphic maps

   – $\mathbb{C}(S_n)$ – group algebra.

   – Centre of the group algebra.

   – Schur-Weyl duality relation between $U(N)$, S_n.

3. – Implications of S.W. for

   . Dim R

   . $C_2(R)$

③ Large N Expansion of 2d YM

(- Chiral Exp.)

- Writing it in a form app. for interp.

④ - Precise mathematical statement of the corr. between perms & holo maps

- How the large N expansion formulae can be seen to be counting maps.

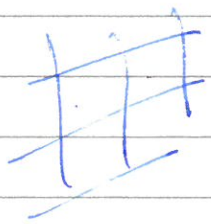
I.1

$\boxed{\text{Part I}}$: Deriving the exact answer for $Z_G(A)$:

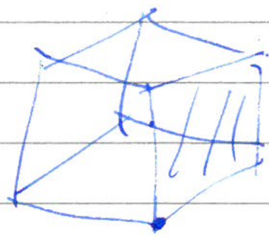
⊗ Lattice gauge theory:

$$\int \prod_E dU_E \;\; \prod_P Z(U_P)$$

- Cubic lattice:



2D



- plaquette.
- 2-cell.

4D: ⊗ d

$x_1 x_2 x_3 x_4$.

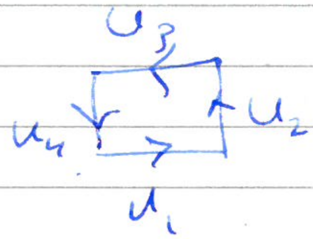
(ij) : $i<j$

$(x_1 x_2)$
$(x_1 x_3)$
$(x_1 x_4)$
$(x_2 x_3)$

Ideal

$$\rightarrow \quad z(U_p) = e^{-\frac{\mathrm{tr}(2 - U_p - U_p^\dagger)}{2g^2}}$$

$$U_p = (U_1 U_2 U_3^\dagger U_4)$$



→ Wilson action:

→ one shows: $\begin{cases} \bullet\ a \to 0 & \text{(naive continuum limit)} \\ \bullet\ \text{discretized Y.M.} & \int \mathrm{tr}\, F^2 \end{cases}$

Also: - heat kernel action

- shown to be equivalent to Wils. at weak coupling

$g \to 0$

$$Z_H(U, A) = \sum_R e^{-g^2 A C_2(R)} \, \mathrm{Dim}\, R \; \chi_R(U)$$

- Sum is over all irreps. of the gauge group

- Dim R. — Dimension of the irrep

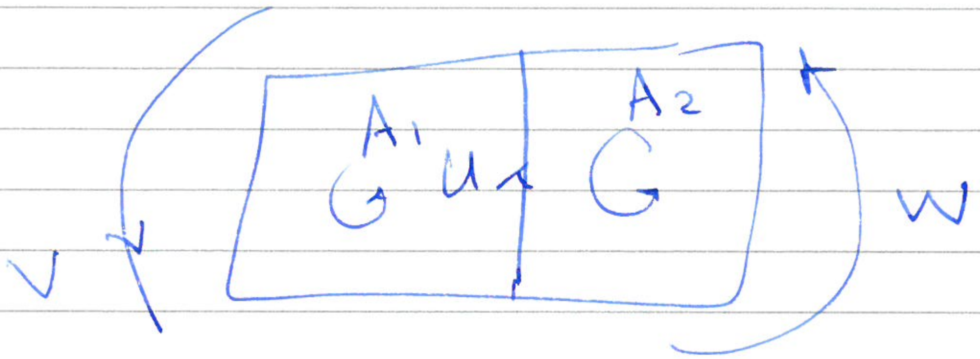
- $C_2(R)$. Quadratic Casimir.

- $\chi_R(U)$:

  Character of U

  $\mathrm{tr}\left(D^R(U)\right)$

- This lattice partition function is R.G. invariant



$$\sum_{R_1, R_2} \int \chi_{R_1}(VU^+)\, \chi_{R_2}(U^+W)\, (\mathrm{Dim}\, R_1)\, (\mathrm{Dim}\, R_2)\, e^{-A_1 C_2(R_1)}\, e^{-A_2 C_2(R_2)}$$

$$= \sum_{R_1, R_2} \delta_{R_1, R_2} \frac{\chi_{R_1}(VW)}{\mathrm{Dim}\, R_1}\, \mathrm{Dim}\, R_1\, \mathrm{Dim}\, R_2\, e^{-A_1 C_2(R_1)}\, e^{-A_2 C_2(R_2)}$$

$$= \sum_{R} (\mathrm{Dim}\, R)\, e^{-A C_2(R)}\, \chi_R(VW)$$

$A = (A_1 + A_2)$

~~$Z_p(A, \ldots)$~~

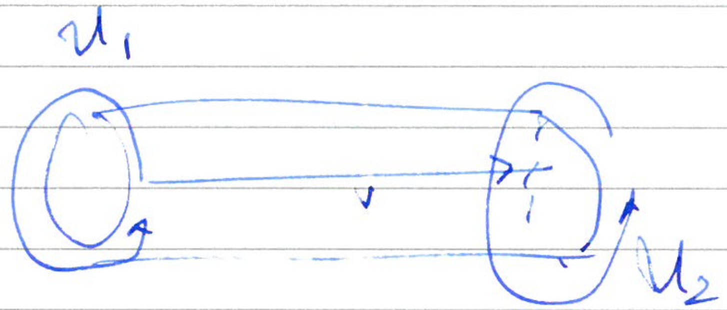
$$\mapsto \int dU \, Z_p(A_1, VU) \, Z_p(A_2, U^{+}W)$$

$$= Z_p(A_1 + A_2, VW)$$

$\therefore$ We can use the coarsest possible discretization — will agree with a finer discretization — i.e. with continuum limit.

I.6

Cylinder:



$$\sum_R \int dV \, \chi_R(U_1 V U_2 V^+) \, (\mathrm{Dim}\, R) \, e^{-A C_2(R)}$$

$$= \sum_R \frac{\chi_R(U_1)\,\chi_R(U_2)}{(\mathrm{Dim}\, R)} \, \mathrm{Dim}\, R \; e^{-A C_2(R)}$$

$$= \sum_R \chi_R(U_1)\,\chi_R(U_2)\, e^{-A C_2(R)}$$

* $\int dU$
  → Haar Measure
  → G-invariant group integ.
  → normalized to 1.

* $$\left\{ \begin{array}{l} \int dU \; U_{ij} \, U^{\dagger}_{kl} = \delta_{il}\delta_{jk} \\ \cancel{\int dU \; \delta_{ij} \, U_{k} \, \delta_{kl} =} \\ \int dU = 1. \end{array} \right\}$$

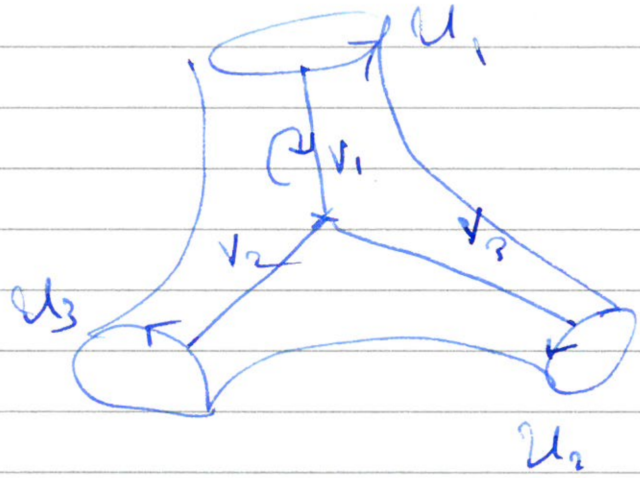
* $$Z(T^2) = \int Z_{cyl.}(U_1, U_2 = U_1^{\dagger}) \, dU_1$$

$$= \sum_R \int \chi_R(U_1) \, \chi_R(U_1^{\dagger}) \, e^{-A\, C_2(R)}$$

$$= \sum_R e^{-A\, C_2(R)}$$

$Z_{3\text{-holed sphere}}$: (U_1, U_2, U_3)

Ex.: →



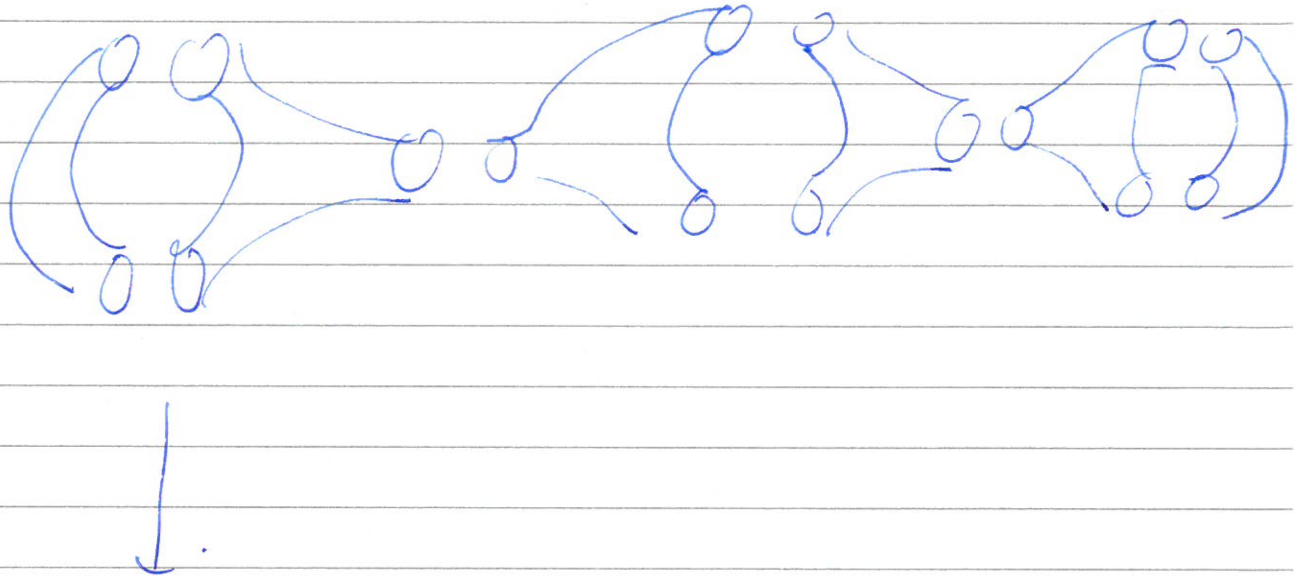
→ 2-cell with identif.

$$\sum_R \int \prod dV \; e^{-A\, C_2(R)} \; \chi_R\left(V_1 U_1 V_1^\dagger V_2 U_2 V_2^\dagger V_3 U_3 V_3^\dagger\right)$$

$$\mathrm{Dim}\, R.$$

$$\hookrightarrow \sum_R e^{-A\, C_2(R)} \left(\mathrm{Dim}\, R\right)^{-1} \chi_R(U_1)\, \chi_R(U_2)\, \chi_R(U_3)$$

genus 2:



derive:

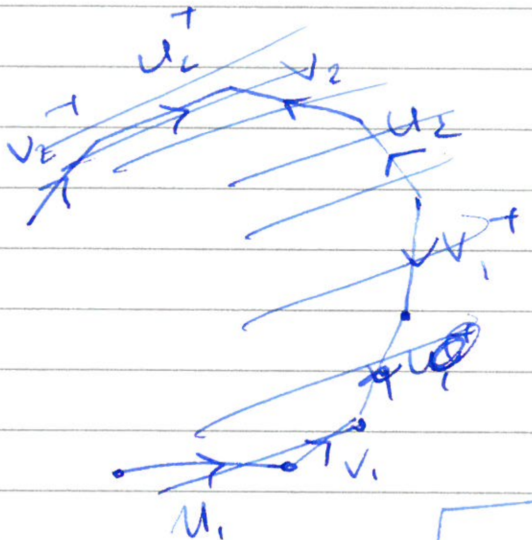
$$Z(G=2, A) = \sum_R (\mathrm{Dim}\, R)^{-2} e^{-A\, C_2(R)}$$

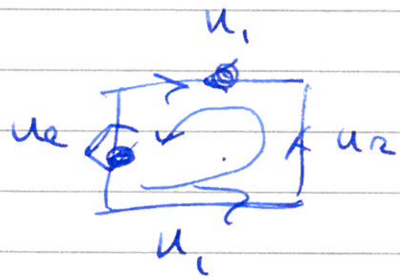
For genus G:

$$Z(G, A) = \sum_R (\mathrm{Dim}\, R)^{2-2G} e^{-A\, C_2(R)}$$

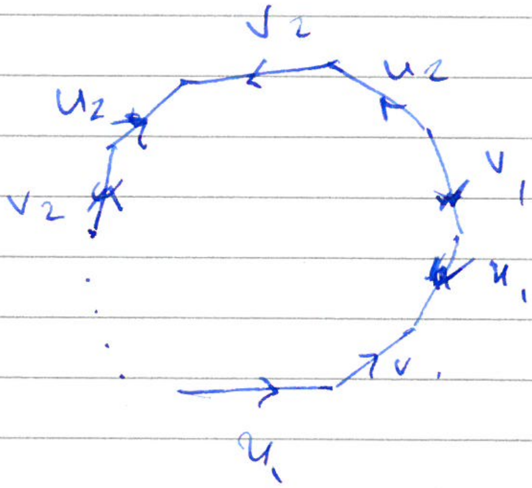
Another way:

- A genus of G surface:





$$\boxed{[U,V] = UVU^\dagger V^\dagger}$$



$$\sum_R (\mathrm{Dim}\,R)^{2+}\, \chi_R\big([U_1,V_1]\,[U_2,V_2]\cdots[U_G,V_G]\big)$$

$$e^{-A C_2(R)} \longrightarrow \text{same answer.}$$

(II):

$\mapsto$ group algebras . $\mathbb{C}(S_n)$

$\mapsto$ Centre . $Z(\mathbb{C}(S_n))$

$\mapsto$. Conj. classes &

. Projectors

$\mapsto$ Schur. Weyl duality

& (Dim R_ν)

II.1

$\mapsto$ S_n :

- group $n!$

- all re-arrangements :

1 2 3
$\downarrow$
1 2 3

(1) (2) (3)

Cycle notation

- $\sigma \in S_n$:

$\mathbb{C}(S_n)$: — is an algebra

— A vector space :

- can scale

- can add.

$$a = \sum_{\sigma \in S_n} a_\sigma \sigma$$

$a \in \mathbb{C}(S_n)$

Dim. $= n!$

Dim $= |G|$

$$b = \sum_{\sigma \in S_n} b_\sigma \sigma$$

$$a \cdot b = \sum_{\sigma, \tau} a_\sigma b_\tau \underbrace{\sigma . \tau}$$

$$= \Sigma$$

. Associative product.

II.3.

→ $$\delta(\sigma) = 1 \quad \text{if } \sigma = id.$$
$$= 0 \quad \text{if } \sigma \neq id.$$

→ Regular Rep:

$$|V^{reg.}| = n!$$

$$\mapsto \quad \delta(\sigma) = \frac{1}{n!} \sum_R d_R \chi_R(\sigma)$$

- R is an irrep.
- d_R is the dimension of the irrep.

II Pf.

→ Centre:

given σ.

$\gamma \sigma \gamma^{-1}$ is said to be conjugate

S_3

$\left[(1)\ (2)\ (3) \right.$

$\left[\begin{array}{l} (12)\ (3) \\ (23)\ (1) \\ (12)\ (3) \end{array} \right.$

$\left[\begin{array}{l} (123) \\ (132) \end{array} \right.$

3. conj. classes

given $\sigma_1, \sigma_2 \in K$.

$\exists\ \gamma \in S_n$ s.t. $\gamma \sigma_2 \gamma^{-1} = \sigma_1$

II.5:

→ Centre: of $\mathbb{C}(S_n)$

$$= Z(\mathbb{C}(S_n))$$

⊕ - Vector subspace

- Any el. of v.s. commute with all $\mathbb{C}(S_n)$.

⊕ Sums over conj. class.

$$\hat{K} = \left(\sum_{\sigma \in K} \sigma \right)$$

Ex. $\hat{K}$ is central element.

⊕ $$\bar{\sigma} = \sum_{\gamma \in S_n} \gamma \sigma \gamma^{-1}$$

is a central element.

II.6

- dim $Z(\mathbb{C}S_n)$

$= p(n)$

\# of partitions of n.

- irreducible characters are in 1-1 corr. with conj. classes

$\chi^R(\sigma) = \chi^R(\tau\sigma\tau^{-1})$

$= \chi^R_K$.

- \# of R = \# of K
- \# of irreps of S_n = ~~\# of~~ $p(n)$.

→ ⊗ For each irrep —

→ There is a central element

→ in fact —

There is a projector

→ $Z(\mathbb{C}(S_n))$ spanned by

$$\{ P_R \mid R \vdash n \}$$

$$\boxed{P_R = \frac{d_R}{n!} \sum_{\sigma \in S_n} \chi_R(\sigma)\, \sigma}$$

⊗ $$P_R^2 = P_R$$

$$P_R P_S = \delta_{RS} P_R$$

$$\to \quad \ell = \sum_K c_K K$$

$$= \sum_R \tilde{c}_R P_R$$

$$\ell\sigma = \sigma\ell \qquad \forall \sigma.$$

$$\mapsto \quad \boxed{\chi_R(\ell\tau) = \frac{\chi_R(\ell)\,\chi_R(\tau)}{d_R}}$$

A

Ex: Follows from Schur's Lemma.

$$\mapsto \quad \boxed{D^R(\ell) = c\, D^R(1)}$$

Ex: Use this – prove above.

S.W. duality:

↦ $V_N = \text{Span}\{e_1, e_2, \dots e_N\}$.

N-dim. fundamental Rep. of $U(N)$

$$\text{Span}\left\{ \left(e_{i_1}\otimes e_{i_2}\otimes e_{i_3}\otimes \cdots \otimes e_{i_n}\right) \right\}$$

$$= V_N^{\otimes n}$$

↦ How do we decompose $V_N^{\otimes n}$ into irreps?

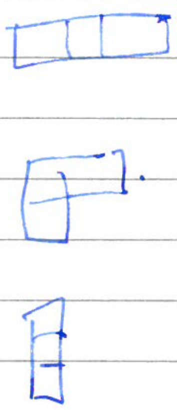
$V_N^{\otimes 2}$: $\frac{1}{2}\left(e_{i_1}\otimes e_{i_2} + e_{i_2}\otimes e_{i_1}\right)$ 

$\frac{1}{2}\left(e_{i_1}\otimes e_{i_2} - e_{i_2}\otimes e_{i_1}\right)$ .

- linear combination of def. symm. type are irreps:

$V_N^{\otimes 3}$:

Has 3 symm. types



$(e_{i_1} \otimes e_{i_2}) \otimes e_{i_3}$

$(e_{i_1} \otimes e_{i_3}) \otimes e_{i_2}$

2 distinct copies of same rep.

2 : $\longrightarrow$ is the dim. of the non-trivial irrep of S_3

$$3! = 1 + 1 + 4 = \underline{6} = \sum_R d_R^2$$

(I.11)

→ The general connection:

$$V_N^{\otimes n} = \bigoplus_{\substack{R \\ \ell(R) \leq N}} V_R^{U(N)} \otimes V_R^{(S_n)}$$

↦ ⓐ $\underline{U(N)}$ & $\mathbb{C}(\underline{S_n})$

— commute with each other in $V_N^{\otimes n}$.

— ~~can be seen~~

ⓑ The full comm of $\mathbb{C}(S_n)$ is $U(N)$

— (image of env. alg of $u(N)$)

ⓒ irreps of $\underline{S_n}$ are labelled by Y.D with $\underline{n}$ boxes

II.12

(a) $\to$ the decomp.

$\mapsto$ R : . Dim R

. d_R .

$$\mathrm{Dim}\, R = \frac{\prod_{\square} (N + c_{\square})}{\prod \text{hooks}}$$

$$d_R = \frac{n!}{\prod \text{hooks}} .$$

II 43

$$\hookrightarrow \operatorname{tr}_{V_N^{\otimes n}} (1 \otimes P_R)$$

$$= \sum_\sigma \frac{d_R \chi_R(\sigma)}{n!} \operatorname{tr}_{V_N^{\otimes n}} (\sigma)$$

$$= \frac{d_R}{n!} \sum_\sigma \chi_R(\sigma) \, N^{C_\sigma}$$

$$= d_R \cdot \operatorname{Dim} R.$$

$$\Rightarrow \boxed{(\operatorname{Dim} R) = \left(\frac{N^n}{n!}\right) \chi_R(\Omega)}$$

$$\Omega = \sum_\sigma \sigma \, N^{C_\sigma - n}$$

$$\Omega \in Z(\mathbb{C}(S_n))$$

II.14.

$\rightarrow \ \mathrm{tr}_{V_N^{\otimes n}}\left(U \otimes P_R\right)$

$= d_R \, \chi_R(U)$

$= \dfrac{d_R}{n!} \sum_\sigma \chi_R(\sigma) \, \mathrm{tr}\left(U^{\otimes n} \sigma\right)$

$= \dfrac{d_R}{n!} \sum_\sigma \chi_R(\sigma) \underbrace{\prod_i \left(\mathrm{tr}\, U^i\right)^{C_i(\sigma)}}$

$\chi_R(U) = \dfrac{1}{n!} \sum_\sigma \chi_R(\sigma) \underbrace{P_\sigma\left(e^{i\vec{\theta}}\right)}$

Symm. poly.

Rmk: $\rightarrow$ II.13 & II.14 :

should be explained slowly

$\mapsto \chi_R(\mathcal{C}\,\tau) = \dfrac{\chi_R(\mathcal{C})\,\chi_R(\tau)}{d_R}$

Exp. slowly ;

$\mapsto$ Thy. of Sym. functions ;

II.15.

III : Large N expansion:

Z_{chir} v/s $Z_{non.chir}$

$$Z_{chir} = \sum_{n=0}^{\infty} \sum_{R \vdash n} (\mathrm{Dim}\, R)^{2-2G}$$

$$= \sum_{n=0}^{\infty} \sum_{R \vdash n} N^{n(2-2G)} \frac{(\chi_R(\Omega))^{2-2G}}{(n!)^{2-2G}}$$

$G = 0$:

$$\frac{N^{2n}}{(n!)^2} (\chi_R(\Omega))^2$$

$$= \frac{N^{2n}}{(n!)^2} \sum_R d_R \chi_R(\Omega^2)$$

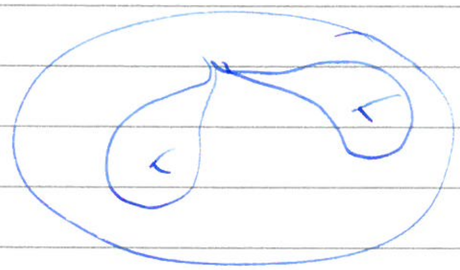
$$Z = \frac{N^{2n}}{(n!)^2} \sum_R d_R \chi_R(\Omega^2)$$

$$= \frac{N^{2n}}{(n!)} \delta(\Omega^2)$$

$$= \frac{N^{2n}}{n!} \sum_{\sigma_1, \sigma_2} \delta(\sigma_1 \sigma_2) N^{(C_{\sigma_1}-1)+(C_{\sigma_2}-1)}$$

Counting pairs of permutations

⊗ Counting branched covers of S^2 with 2 branch points



⊗ Take S^2 w/ 2 branch points:

- Label inv. image of gen. pt get σ_1, σ_2.

- perm. for whole should be trivial.

- Formal way:

– Branched covers of S^2 with k branch pts are counted by ~~counting k-tuples of perm.~~

– Think about

$$\mathrm{Hom}\left(\pi_1\left(S^2 \setminus \{k \text{ pts}\}\right) \to S_n\right).$$

$\mapsto$ For generic case:

$$\Omega = \left(1 + \frac{\sigma}{N}\right) \qquad \underline{\underline{S_2}}.$$

$$\Omega^{-1} = \left(1 + \frac{\sigma}{N}\right)^{-1}$$

Expand. (Binomial)

$\downarrow$ use $\sigma^2 = 1$.

$$= \frac{1}{(1 - 1/N^2)} + \frac{1}{N}\frac{\sigma}{(1 - 1/N^2)}$$

$$\cancel{\Omega^{-1} = 1 - N^{-1}\sigma + \Sigma N^{-1} \cdot N^{-1} \sigma_1 \sigma_2}$$

$$\Omega = \sum_{\sigma} N^{\sigma - n} \sigma$$

$$= 1 + \sum_{\sigma}{}' N^{\sigma - n} \sigma$$

$$\Omega^{-1} = 1 - \sum_{\sigma}{}' N^{\sigma - n} \sigma$$

$$+ \sum_{\sigma_1 , \sigma_2} N^{\sigma_1 - n} \cdot N^{\sigma_2 - n} \sigma_1 \sigma_2$$

$$+ \dots$$

$$+ \cancel{B}(G, d)$$

$$\frac{\chi_R(\Omega \Omega^{-1})}{d_R} = 1$$

$$\Rightarrow \frac{\chi_R(\Omega)}{d_R} \cdot \frac{\chi_R(\Omega^{-1})}{d_R} = 1 \Rightarrow \frac{\chi_R(\Omega^{-1})}{d_R} = \frac{d_R}{\chi_R(\Omega)}$$

$$\sum_R \frac{N^{n(2-2G)}}{(n!)^{2-2G}} \delta \left(\chi_R(\Omega)\right)^{2-2G}$$

$$= \sum_R \frac{N^{n(2-2G)}}{(n!)^{2-2G}} \left(\left(\chi_R(\Omega)\right)^{-1}\right)^{2G-2}$$

$$= \sum_R \frac{N^{n(2-2G)}}{(n!)^{2-2G}} \left(\frac{\chi_R(\Omega^{-1})}{d_R^2}\right)^{2G-2}.$$

$$= \sum_R \frac{N^{n(2-2G)}}{(n!)^{2-2G}} d_R^{2-2G} \left(\frac{\chi_R(\Omega^{-1})}{d_R}\right)^{2G-2}$$

$$= \sum_R N^{n(2-2G)} \left(\frac{d_R}{n!}\right)^2 \left(\frac{n!}{d_R}\right)^{2G} \cdot \frac{\chi_R}{d_R}\left(\Omega^{2-2G}\right)$$

$$\longrightarrow \boxed{\frac{(n!)^2}{d_R^2} = \sum_{s,t} \frac{\chi_R}{d_R}(s t s^{-1} t^{-1})}$$

$$\text{RHS} = \sum_{s,t} \frac{\chi_R}{d_R}(s t s^{-1}) \frac{\chi_R}{d_R}(t^{-1})$$

$$= (n!) \cdot \sum_{t} \frac{\chi_R(t)}{d_R} \frac{\chi_R(t^{-1})}{d_R}$$

$$= \frac{(n!)^2}{(d_R^2)}.$$

$$e = \sum_{s,t} s t s^{-1} t^{-1}$$

$$\sum_{R} \left(\frac{\chi_R}{d_R} (t) \right)^{\cancel{2}G} \frac{\chi_R}{d_R} \left(\Omega^{2-2G} \right) \frac{d_R^2}{(n!)^2} N^{n(2-2G)}$$

$$= \sum_{R} \frac{\chi_R}{d_R} \left(t^{G} \cdot \Omega^{2-2G} \right) \frac{d_R^2}{(n!)^2} N^{n(2-2G)}$$

$$= \frac{1}{n!} \delta \left(t^{G} \cdot \Omega^{2-2G} \right) N^{n(2-2G)}$$

$$= \frac{1}{n!} \sum_{\substack{s_1 t_1 \dots \\ s_G t_G}} \delta\left(s_1 t_1 s_1^{-1} t_1^{-1} \dots s_G t_G s_G^{-1} t_G^{-1} \Omega^{2-2G} \right)$$

$$N^{n(2-2G)}$$

(#) : $\Omega = 1$

- Counting un-branched covers
- $Hom(\pi_1(\Sigma) \to S_n)$
- $\underline{\underline{N}}^x$.

(#) : When expand:

$$\frac{1}{n!} \sum_{\substack{s_1 t_1 \dots s_G t_G \\ \sigma_1 \dots \sigma_L}} N^{n(2-2G) - \sum_i b_i} \delta\left(\prod [s_i, t_i] \, \sigma_1 \dots \sigma_L \right)$$

$\chi(\Sigma_h)$

$d(G, L)$

$\rightsquigarrow \chi(\mathcal{C}_{G,L})$