Holographic Combinatorics

Lecture 2

1. Schur-Weyl Duality
   - $U(N) \longleftrightarrow \mathbb{C}(S_n)$; $Z(\mathbb{C}S_n)$
   - Dim R
   - $\chi_R(U)$

2. Complex Matrix Models
   - Orthogonal Bases
   - Applications to AdS/CFT

3. 2-Matrix Systems
   - $U(N)$
   - Centralizer of a subgroup
   - Fourier transform

Schur-Weyl Duality :

$\otimes$. Consider the group $U(N)$.

$$\mathcal{U}\,\mathcal{U}^{\dagger} = \mathbb{1}$$

$$\mathcal{U} \;:\; N \times N \text{ matrix}$$

$$V_N \;:\; \text{fundamental Rep}$$

$$V_N = \text{Span}\{e_1, e_2, \ldots, e_N\}$$

$$\mathcal{U}\, e_i = \mathcal{U}_{ji}\, e_j$$

$V_N^{\otimes n}$:

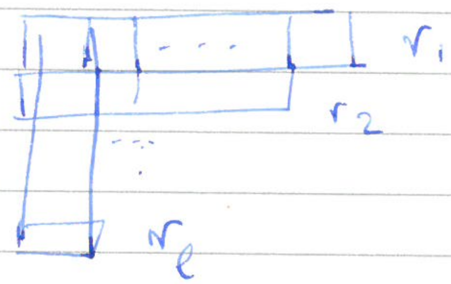
$$e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_N}.$$

$$\mathrm{Dim}\left(V_N^{\otimes n}\right) = N^n.$$

↦ Decomposing into irreps of $U(N)$

. These irreps are classified by

<u>Young diagrams</u>:



$$r_1 + r_2 + \dots + r_\ell = n$$

↦ $\ell(R) \leq N$.

~~Set of~~

So:

- All irreps of $U(N)$
  corr. to all $\underline{\underline{Y.D.}}$
  - all possible n
  - $\ell(R) \leq N$
- All irreps of $\underline{\underline{S_n}}$.
  - Correspond to Y.D. of $\underline{\underline{S_n}}$.

From the $\underline{\text{diagram}}$:

can get $\text{Dim}_N R =$

$=$ Polynomial in N, of degree $\underline{\underline{n}}$.

$\underline{\underline{d_R}} : \frac{n!}{\prod(\text{hooks})}$

$$V_N^{\otimes n} = \bigoplus_R V_R^{U(N)} \otimes V_R^{S_n}$$

$$|i_1 \ldots i_N\rangle$$

$$|R, M_R\rangle \otimes |R, m_R\rangle$$

or $|R, M_R, m_R\rangle$

$$(\mathrm{Dim}\, R \cdot d_R)$$

$$N^n = \sum_{\substack{R \vdash n \\ l(R) \leq N}} \mathrm{Dim}\, R \, d_R$$

$\mapsto$. $U(N)$:

$\otimes$ - $U(N)$ commutes with S_n

$\otimes$ - Anything ~~~~ in $End(V^{\otimes n})$ which commutes with $U(N)$ is in $(\mathbb{C}(S_n))$

$\otimes$. Anything which commutes with S_n is in $U(N)$

$\mapsto$ ~~[illegible]~~

$\otimes$ generalization: $V^{\otimes n}$.

$O(N)$, $Sp(N)$,

$\otimes$

$\mapsto$ Even more generally:

- W is any rep of G;

{ What is the full comm. of G in $\mathrm{End}(W)$?

$\mathrm{Comm}(G, W) = \mathrm{Com}$

- $$\boxed{W = \bigoplus_R V_R^{G} \otimes V_R^{\mathrm{Com}}}$$

. Double Commutant Theorem

. Double Centralizer Theorem.

$\mapsto$. Consider a matrix:

$$\mathcal{U}^{\otimes n} \cdot \sigma \, | i_1 \ldots i_n \rangle$$

$$= \mathcal{U}^{\otimes n} \, | i_{\sigma(1)}, \ldots, i_{\sigma(n)} \rangle$$

$$= \mathcal{U}_{j_1 i_{\sigma(1)}} \mathcal{U}_{j_2 i_{\sigma(2)}} \cdots \mathcal{U}_{j_n i_{\sigma(n)}} \, | j_1 \, j_2 \ldots j_n \rangle$$

c.f.

$$\sigma \cdot \mathcal{U}^{\otimes n} \, | i_1 \ldots i_n \rangle$$

$$\cancel{= \mathcal{U}_{j_1 i_1} | j_1}$$

$$= \mathcal{U}_{j_1 i_1} \mathcal{U}_{j_2 i_2} \cdots \mathcal{U}_{j_n i_n} \, \sigma \, | j_1 \ldots j_n \rangle$$

$$= \mathcal{U}_{j_1 i_1} \mathcal{U}_{j_2 i_2} \cdots \mathcal{U}_{j_n i_n} \, | j_{\sigma(1)} \ldots j_{\sigma(n)} \rangle$$

$j_1 \dots j_n \to j_{\sigma^{-1}(1)} \dots j_{\sigma^{-1}(n)}$

$$= U_{j_{\sigma^{-1}(1)} i_1} U_{j_{\sigma^{-1}(2)} i_2} \cdots U_{j_{\sigma^{-1}(n)} i_n} |j_1 \dots j_n\rangle$$

$$= U_{j_1 i_{\sigma(1)}} U_{j_2 i_{\sigma(2)}} \cdots U_{j_n i_{\sigma(n)}} |j_1 \dots j_n\rangle$$

$U_{i_1 j_1} U_{i_2 j_2}$ commute.

$$= U^{\otimes n} \sigma |i_1 \dots i_n\rangle$$

So $U^{\otimes n}$ & σ commute

$$= \mathrm{tr}_{V^{\otimes n}}(\sigma U^{\otimes n}) = \cancel{\mathrm{tr}_{V^{\otimes n}}(\sigma U)} \ \underline{\underline{\mathrm{tr}_n(\sigma U)}}$$

$$\longrightarrow \langle i_1 \dots i_n | U^{\otimes n} \sigma | i_1 \dots i_n \rangle$$

$$= U_{j_1 i_{\sigma(1)}} \dots U_{j_n i_{\sigma(n)}}$$

$$\delta_{i_1 j_1} \dots \delta_{i_n j_n}$$

$$= U_{i_1 i_{\sigma(1)}} U_{i_2 i_{\sigma(2)}} \dots U_{i_n i_{\sigma(n)}}$$

$$= \underbrace{(\mathrm{tr}\, U)^{c_1^\sigma} (\mathrm{tr}\, U^2)^{c_2^\sigma} \dots (\mathrm{tr}\, U^n)^{c_n^\sigma}}$$

$$= (u_1 + u_2 \dots)^{c_1^\sigma} (u_1^2 + u_2^2 \dots)^{c_2^\sigma} \dots (u_1^n + u_2^n + \dots u_N^n)^{c_n^\sigma}.$$

<u>Symmetric</u>

$$(u_1^k + u_2^k \dots u_N^k)$$

- Symmetric under S_N exchange (Weyl group)
- Power sum sym. <u>functions</u>

$$\mapsto \; P_R = \sum_{\sigma \in S_n} \frac{d_R}{n!} \chi_R(\sigma) \quad \sigma$$

$$tr_n(P_R \, \mathcal{U})$$

$$= \sum_{\sigma \in S_n} \frac{d_R}{n!} \chi_R(\sigma) \underbrace{tr_n(\sigma \mathcal{U})}$$

$$= \qquad d_R \, \chi_R(\mathcal{U})$$

$$\Rightarrow \underbrace{\chi_R(\mathcal{U})}_{\text{Schur Polynomials}} = \underbrace{\sum_{\sigma \in S_n} \left(\frac{1}{n!}\right) \chi_R(\sigma)}_{\text{char. of } S_n} \underbrace{tr_n(\sigma \mathcal{U})}_{\text{Power Sum Sym. Poly}}$$

$\nearrow$ Ratio of determinants

gen. basis

gen. basis change

$\chi_R(\sigma)$ give the ~~position~~ change of basis from power sum sym. poly to Sch.

$$\chi_R(U) = \frac{\det(u_i^{r_j - j + N})}{\det(u_i^{r_j - j \cancel{+N}})}$$

$$= \frac{\det(u_i^{r_j - j + N})}{\det(u_i^{r_j - j})}$$

$$\cancel{\chi_{\square}}\ \chi_{\square}(U) = \sum_i u_i$$

$$\chi_{\square\square}(U) = \frac{1}{2}\left(\sum_i u_i^2 + \left(\sum_i u_i\right)^2\right)$$

$$\chi_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}(U) = \frac{1}{2}\left(-\sum_i u_i^2 + \left(\sum_i u_i\right)^2\right)$$

from $\sum \chi_R(\sigma)\, \mathrm{tr}(\sigma U)$

- This works for complex Z:

  - $\underbrace{\chi_R(Z)} = \frac{1}{n!} \sum_{\sigma \in S_n} \chi_R(\sigma) \, tr(\sigma Z)$

  - Extend $x_i \to z_i$

- Connect with last λ.

$$\chi_R(1) = \mathrm{Dim}_N(R)$$

$$= \sum_{(\sigma \in S_n)} \left(\frac{1}{n!}\right) \chi_R(\sigma) \, N^{C_\sigma}.$$

2. Consider Gaussian Matrix Model

— free field theory in 4D.

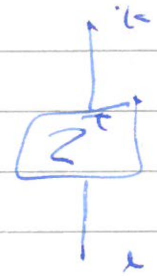
$$\int \mathcal{D}Z \; e^{-\mathrm{tr}(ZZ^\dagger)}$$

$$\int \prod_{i,j} dz_{ij}\, d\bar{z}_{ij} \; e^{-\sum z_{ij}\bar{z}_{ij}}$$

$$\mapsto \int dZ \; e^{-\mathrm{tr}(ZZ^\dagger)} \; Z^i_j \, Z^{\dagger\, k}_{\ \ l}$$

$$= \delta^k_j \, \delta^i_l$$

$\mapsto \quad Z^i_j =$



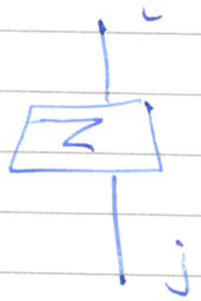
$= (Z^\dagger)^k_l$

$$\mapsto \int dZ\, e^{-\mathrm{tr}(ZZ^\dagger)} \; Z^{i_1}_{j_1}\cdots Z^{i_n}_{j_n}\, (Z^\dagger)^{k_1}_{l_1}\cdots (Z^\dagger)^{k_n}_{l_n}$$

$$= \sum_{\sigma} \delta^{i_1}_{l_{\sigma(1)}}\cdots \delta^{i_n}_{l_{\sigma(n)}}\, \delta^{k_{\sigma(1)}}_{j_1}\cdots \delta^{k_{\sigma(n)}}_{j_n}.$$

$\mapsto$  Next Step?

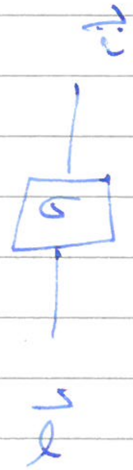
Diagrammatics

z^i_j = 

$$\langle i_1 \cdots i_n \,|\, \sigma \,|\, l_1 \cdots l_n \rangle$$

$$= \langle i_1 \cdots i_n \,|\, l_{\sigma(1)}, \cdots, l_{\sigma(n)} \rangle$$

$$= \delta^{i_1}_{l_{\sigma(1)}} \cdots \delta^{i_n}_{l_{\sigma(n)}} .$$



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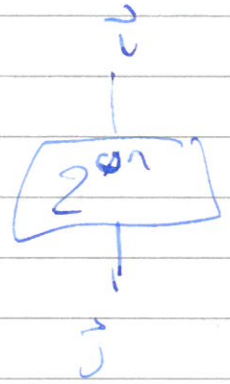
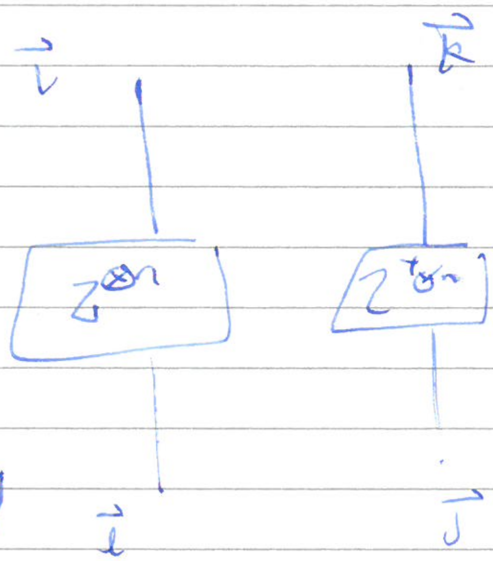
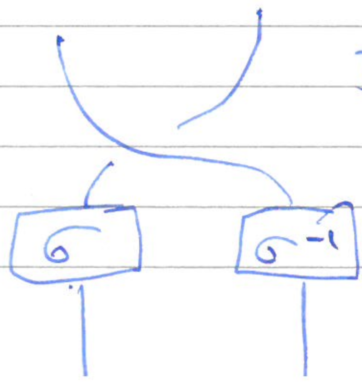


$$= Z^{i_1}_{j_1} \cdots Z^{i_n}_{j_n}$$

$$\rightarrow \int dZ \, e^{-\mathrm{tr}\, Z Z^\dagger}$$

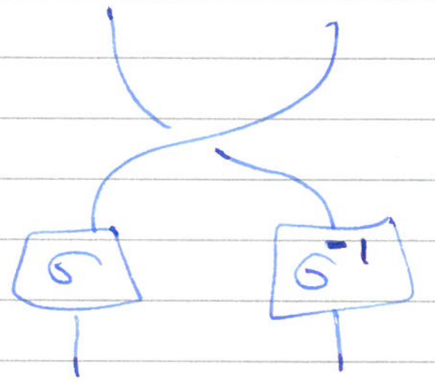


$$= \sum_{\sigma \in S_n}$$



$$\left\langle Z^{\otimes n} \quad (Z^{\dagger})^{\otimes n} \right\rangle$$

$$= \sum_{\sigma \in S_n}$$

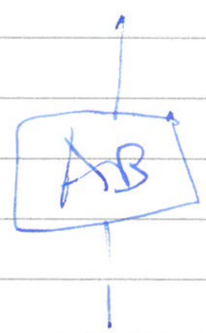


$\mapsto$ Note

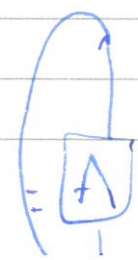
$$(A\ B)^i_j = A^i_k\, B^k_j$$

$=$

$=$



$\mapsto$ Note. $\mathrm{tr}\, A =$

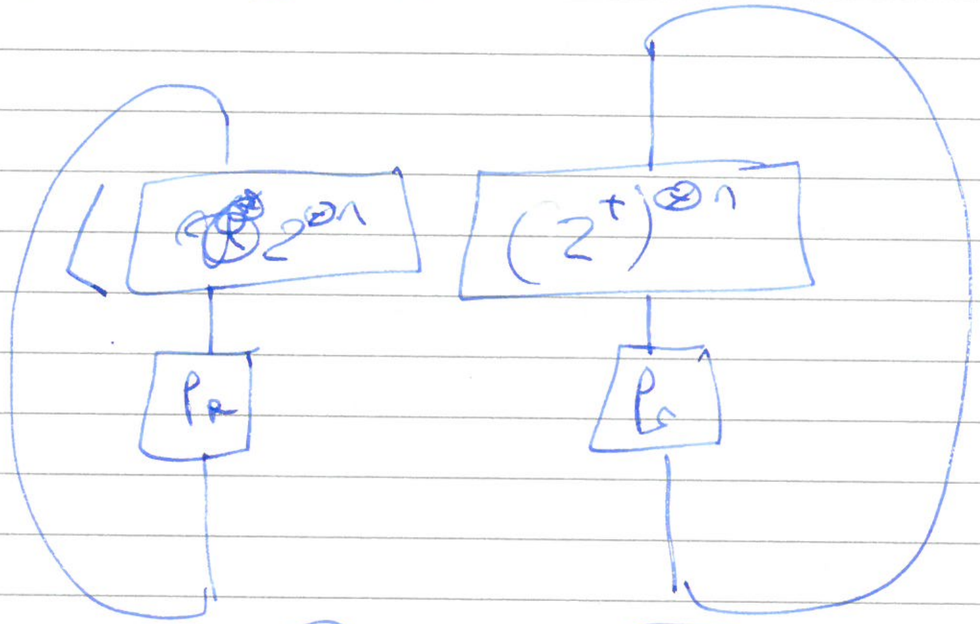


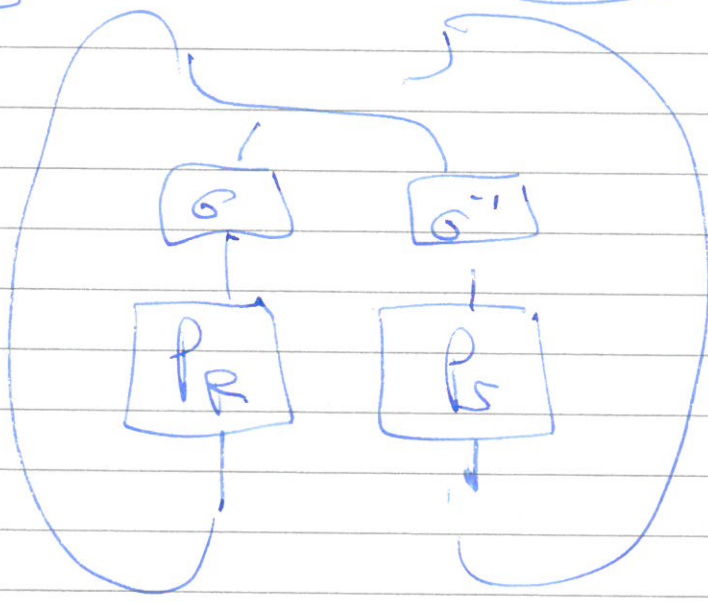
$= \boxed{A}$

$\mapsto$

$$\chi_R(Z) = \frac{1}{d_R} \operatorname{tr}_n \left(P_R Z^{\otimes n} \right)$$

$$\left\langle \chi_R(Z) \; \chi_S(Z^+) \right\rangle$$

$= \frac{1}{d_R d_S}$ 

$= \frac{1}{d_R d_S} \sum_\sigma$ 

$$= \frac{1}{d_R d_S} \sum_\sigma \operatorname{tr} \left(\sigma P_R \sigma^{-1} P_S \right)$$

$$= \frac{1}{d_R d_S} (n!) \, d_R \, \delta_{RS} \, \mathrm{tr}(P_R)$$

$$= \frac{n!}{d_R d_S} \cdot \delta_{RS} \cdot (\mathrm{Dim}\, R)(d_R) .$$

$$= \delta_{RS} \cdot \left(\frac{n! \, (\mathrm{Dim}\, R)}{d_R} \right) .$$

Ref :

Corley Jevicki ; Ramgoolam 2001

Corley ~~Je~~ , Ramgoolam 2002.

⊗ Same calculation works in

4D:

$$\cancel{\langle \phi_k(z) \phi_j(z^+) \rangle}$$

$$\left\langle Z^i_j(x)\, Z^{+k}_l(y) \right\rangle \qquad \begin{matrix} x \\ y \end{matrix} \in \mathbb{R}^4$$

$$= \frac{1}{|x-y|^2} \cdot \delta^k_j\, \delta^i_l$$

$$\to \left\langle \frac{\operatorname{tr} Z^2}{N_2} \frac{\operatorname{tr} Z^2}{N_2} \frac{(\operatorname{tr} Z^{+2})^2}{N_2^2} \right\rangle$$

$\sim \mathcal{O}(1)$

$$\left\langle \operatorname{tr} Z^2 \operatorname{tr} Z^2 \operatorname{tr} Z^{+4} \right\rangle$$

$\sim$ subleading

$\mapsto \quad \underbrace{\mathrm{tr}\, Z^{l}} \longrightarrow \alpha_l^{\dagger} |0\rangle$

local ops. in Radial quantization.

$\mapsto$ Orthogonal states

$\Rightarrow \quad \alpha_{l_1}^{\dagger} \alpha_{l_2}^{\dagger} \cdots |0\rangle$

$\longleftrightarrow \quad \mathrm{tr}\, Z^{l_1}\, \mathrm{tr}\, Z^{l_2} \cdots |0\rangle$

make sense

⊗ in $N=4$ SYM :

⊗ $\operatorname{tr} Z^\ell$

- annihilated by $8/16$ supercharges ~~out of~~ Q

- And the $\underline{\underline{S}}$ sup.

⊗ Multiplet of graviton

Half-BPS

⊗. So single & multi-graviton

corr. trace structures

$\mapsto$ But this orthogonality breaks down.

when $J \sim \sqrt{M}$.

$\langle \underbrace{b\ 2^J}_{\text{Norm.}}\ \underbrace{\sim 2^J}_{\text{Norm}}\ \underbrace{d\ 2^{+2J}}_{\text{Norm}} \rangle$

not subleading.

$\mapsto$ So what is a good basis for

$J > \sqrt{N}$

esp. $J \sim M$.

BBNS : $\longrightarrow$ sub-determinants

CJR :

- Subdeterm.

special case of

X_R

$R =$

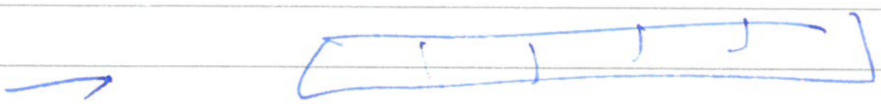
- sphere grants

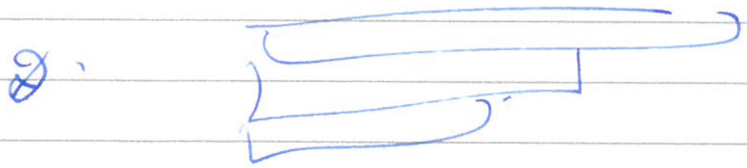
Multiple

Ads grant :-

$

Laptops





Multi Ads

Tests : by company

3-pt. firms

– 1103.4079

Bissi, Kristjansen, Young, Zoubos

– 1209.6624

H. Lin

"giant graviton & Co."

Part III

$\mapsto$ 2-Matrix problem:

$$\mathrm{tr}\, XYXY$$

$$\mathrm{tr}\, X^2 Y^2$$

m copies of X

n copies of Y.

$$\left(X^{\otimes m} \otimes Y^{\otimes n} \right).$$

$$\mapsto \Psi_\sigma \left(X^{\otimes m} \otimes Y^{\otimes n} \right)$$

$$= \mathrm{tr}_{m+n} \left(X^{\otimes m} \otimes Y^{\otimes n} \; \sigma \right)$$

$$= \mathrm{tr}_{m+n} \left(\gamma \; X^{\otimes m} \otimes Y^{\otimes n} \; \gamma^{-1} \; \sigma \right)$$

③ $\cdot$ 2-Matrix:

X, Y:

$$X = (X_1 + i X_2)$$

$$Y = (X_3 + i X_4)$$

$\mapsto$ $tr(\sigma\, \mathcal{U}) \longrightarrow \mathcal{U}^{i_1}_{i_{\sigma(1)}} \cdots \mathcal{U}^{i_n}_{i_{\sigma(n)}}$

$\longrightarrow Z^{i_1}_{i_{\sigma(1)}} \cdots Z^{i_n}_{i_{\sigma(n)}}$

$\mapsto$ $\sigma \sim \gamma \sigma \gamma^{-1}$

gives the same observable.

$$\mapsto \quad Z^{i_1}_{i_{\sigma(1)}} \cdots\cdots Z^{i_n}_{i_{\sigma(n)}}$$

$$= Z^{i_1}_{j_1} \cdots\cdots Z^{i_n}_{j_n} \, \delta^{j_1}_{i_{\sigma(1)}} \cdots \delta^{j_n}_{i_{\sigma(n)}}$$

$$= Z^{i_{\gamma(1)}}_{j_{\gamma(1)}} \cdots\cdots Z^{i_{\gamma(n)}}_{j_{\gamma(n)}} \, \delta^{j_1}_{i_{\sigma(1)}} \cdots \delta^{j_n}_{i_{\sigma(n)}}$$

$$= Z^{i_{\gamma(1)}}_{i_{\sigma(\gamma(1))}} \cdots\cdots Z^{i_{\gamma(n)}}_{i_{\sigma(\gamma(n))}} \qquad \left[\begin{array}{l} j_1 \to i_{\sigma(1)} \\ j_{\gamma(1)} \to i_{\sigma(\gamma(1))} \end{array} \right.$$

$$\cancel{=} \quad \underbrace{\cancel{Z^{i_1}_{i_{\sigma(1)}} \cdots\cdots Z^{i_n}_{i_{\sigma(n)}}}}$$

$$= Z^{i_{\gamma(1)}}_{\gamma\sigma(1)} \cdots\cdots Z^{i_{\gamma(n)}}_{i_{\gamma\sigma(n)}}$$

$$1 \to \gamma^{-1}(1)$$

$$= Z^{i_1}_{\gamma\sigma(\gamma^{-1}(1))} \cdots\cdots Z^{i_n}_{i_{\gamma\sigma(\gamma^{-1}(n))}}$$

$$= Z^{i_1}_{\gamma^{-1}\sigma\gamma(1)} \cdots\cdots Z^{i_n}_{\gamma^{-1}\sigma\gamma(n)}$$

$\mapsto \quad \vartheta_\sigma(z) = \vartheta_{\gamma\sigma\gamma^{-1}}(z)$

Conj. classes $\longleftrightarrow$ $\chi_R(\sigma)$

take $\vartheta_\sigma(z)$

$$\vartheta_R(z) = \sum_\sigma \chi_{R^\vee}(\sigma)\, \vartheta_\sigma(z)$$

$= \cancel{\sigma}$ $\quad \forall \, \gamma \in \underline{S_m \times S_n}$.

$= \Theta_\sigma(X, Y)$

$= \Theta_{\gamma \sigma \gamma^{-1}}(X, Y)$

$\Rightarrow$ Equiv classes in

S_{m+n}

subject to conjugacy by a subgroup

$\longmapsto$ Subgroup conjugacy classes

(X)

a. c. classes

$\chi^{R}(G)$

subgroup. conj. classes :

What functions

on $\mathbb{C}(S_{n+1})$

will form a complete set of fns on H - conj. classes.

⊗ Complete set of f.

$$\underbrace{\mathbb{C}(S_{n+1})}$$

$$- \quad D^{R}_{IJ}(\sigma)$$

- Conj. inv $\Rightarrow$ trace

$$\chi^{R}(\sigma)$$

→ So we must do a

Sub-group trace

$\rightarrow D^{R}_{IJ}(\sigma)$

$\rightarrow \langle R, I | \sigma | R, J \rangle$

Decompose

$\langle R, I |$

$= \sum_{r_1, r_2, m_1, m_2, \nu_1} \langle R, I | r_1, r_2, m_1, m_2, \nu_1 \rangle \langle r_1, r_2, m_1, m_2, \nu_1 |$

Branching coeffs

→ Now we have exposed some subgroup indices

$$\sum_{\sigma \in S_{n+m}} \langle R, I | \sigma | R, J \rangle \langle R, J | r_1, r_2, m_1, m_2, \nu_1 \rangle \langle r_1, r_2, m_1, m_2, \nu_2 | R, I \rangle$$

$$\sum_{m_1, m_2} \text{trace}$$

$$= \underbrace{\chi^{R}_{R_1 R_2 \nu_1 \nu_2}(\sigma)}$$

Restricted Character

$$1 \leq n_1 \leq g(R_1, R_2, R)$$

$\Rightarrow$ Number of ops : (m, n)

$$= \sum_{\substack{R_1 \vdash m \\ R_2 \vdash n \\ R \vdash m+n}} \left(g(R_1, R_2, R) \right)^2$$

<u>Finite</u> $\underline{N}$

$$l(R) \leq N$$

$$\mathcal{O}_\sigma(X, Y)$$

$$\mathcal{O}^{R}_{R_1 R_2 \nu_1 \nu_2}(X, Y)$$

$$= \sum_{\sigma \in S_{m+n}} \chi^{R}_{R_1 R_2 \nu_1 \nu_2}(\sigma) \, \mathcal{O}_\sigma(X, Y)$$

form an orthogonal basis for Gaussian 2-pt fn of Multi-Matrix systems

- B. H. R
- Kimura, Ramgoolam
- Bhattacharyya, Collins, deMk

2007/2008